

Random Matrices and Analyticity of the Planar Limit

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Wick's Formula

Assume that $\mu \sim N(0, C)$ on \mathbb{R}^n .

- $\langle f \rangle$ is the expectation with respect to μ
- $\langle x_i \rangle = 0$, $\langle x_i x_j \rangle = c_{ij}$.
- For any linear functions $f_1, f_2, \dots, f_{2k} : \mathbb{R}^n \rightarrow \mathbb{C}$,

$$\langle f_1 f_2 \dots f_{2k} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \dots \langle f_{p_k} f_{q_k} \rangle$$

$p_1 q_1 p_2 q_2 \dots p_k q_k$ permutation of $\{1, 2, \dots, 2k\}$ with
 $p_1 < p_2 < \dots < p_k$ and $p_1 < q_1, p_2 < q_2, \dots, p_k < q_k$.

For $\mu \sim N(0, 1)$ and $f_1 = f_2 = f_3 = f_4 = x$,

$$\langle x^4 \rangle = \langle f_1 f_2 f_3 f_4 \rangle = \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle = 3.$$

Random Matrices and Moments

$\mathcal{M}_N^{sa}(\mathbb{C})$ are $N \times N$ Hermitian matrices.

$$\text{Tr}M^2 = \sum_i x_{ii}^2 + 2 \sum_{i < j} \text{Re}^2(x_{ij}) + 2 \sum_{i < j} \text{Im}^2(x_{ij})$$

The distribution is

$$\frac{1}{C_N} e^{-\frac{N}{2} \text{Tr}M^2} dM$$

dM the Lebesgue measure on $\mathcal{M}_N^{sa}(\mathbb{C})$.

$$\langle x_{ij} \rangle = 0$$

$$\langle x_{ij} x_{kl} \rangle = \frac{1}{N} \text{ iff } i = l, j = k.$$

$$\text{Tr}(M^{2k}) = \sum x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{2k-1} i_1}$$

$$\langle \text{Tr}(M^{2k}) \rangle = \sum \langle x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_{2k-1} i_1} \rangle$$

Example:

$$\langle N\text{Tr}(M^4) \rangle = N \sum \langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} \rangle$$

Use Wick's formula to compute this.

Example: Couple $i_1 i_2$ with $i_2 i_3$ and $i_3 i_4$ with $i_4 i_1$, thus

$$i_1 = i_3, i_2 = i_2, i_3 = i_1, i_4 = i_4.$$

There are three free indices, thus the contribution from such a choice to $\langle \text{Tr}(M^4) \rangle$ is N^2 .

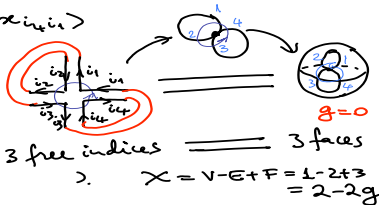
These indices define a "diagram" or a "map" on a sphere (a genus 0 surface).

$$N\langle \text{Tr } M^4 \rangle = N \sum \langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} \rangle$$

Case 1 $i_1 i_2 \sim i_2 i_3 \mid i_1 = i_3$
 $i_3 i_4 \sim i_4 i_1 \mid i_2 = i_4$

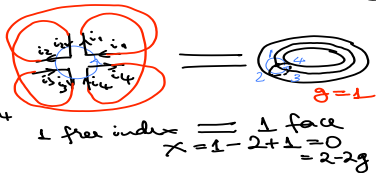
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Contribution to $N\langle \text{Tr } M^4 \rangle$:
 $N \times \frac{1}{N^2} \times N^3 = N^2 = N^{2-2g}$



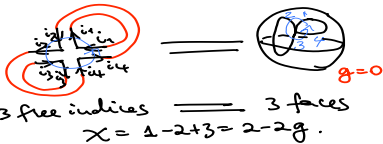
Case 2 $i_1 i_2 \sim i_3 i_4 \mid i_1 = i_4$
 $i_2 i_3 \sim i_4 i_1 \mid i_2 = i_3$
 $i_3 = i_4$
 $i_1 = i_2 = i_3 = i_4$

Contribution to $N\langle \text{Tr } M^4 \rangle$:
 $N \times \frac{1}{N^2} \times N = 1 = N^{2-2g}$



Case 3 $i_1 i_2 \sim i_4 i_2 \mid i_1 = i_1$
 $i_2 i_3 \sim i_3 i_4 \mid i_2 = i_4$
 $i_3 = i_3$

Contribution to $N\langle \text{Tr } M^4 \rangle$:
 $N \times \frac{1}{N^2} \times N^3 = N^2 = N^{2-2g}$



$$N^2(\text{Tr} M^4)^2 = \sum_{\substack{i_1, i_2, i_3, i_4 \\ j_1, j_2, j_3, j_4}} \langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} x_{j_1 j_2} x_{j_2 j_3} x_{j_3 j_4} x_{j_4 j_1} \rangle$$

$$\langle \underbrace{x_{i_1 i_2}}_{\text{---}} \underbrace{x_{i_2 i_3}}_{\text{---}} \underbrace{x_{i_3 i_4}}_{\text{---}} \underbrace{x_{i_4 i_1}}_{\text{---}} \underbrace{x_{j_1 j_2}}_{\text{---}} \underbrace{x_{j_2 j_3}}_{\text{---}} \underbrace{x_{j_3 j_4}}_{\text{---}} \underbrace{x_{j_4 j_1}}_{\text{---}} \rangle$$

$$\begin{aligned} i_1 i_2 &\rightarrow i_2 i_1 \\ i_3 i_4 &\rightarrow i_4 i_3 \\ i_4 i_1 &\rightarrow i_1 i_4 \\ j_1 j_2 &\rightarrow j_2 j_1 \\ j_3 j_4 &\rightarrow j_4 j_3 \end{aligned}$$

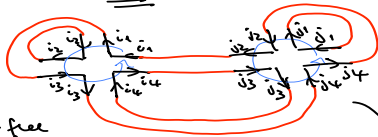
$$\left\{ \begin{array}{l} i_1 = i_3 \\ i_2 = i_4 \\ i_4 = j_3 = j_4 \\ j_1 = j_2 \end{array} \right.$$

4 free indices

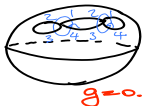
4 faces

Contribution to $N^2(\text{Tr} M^4)^2$:

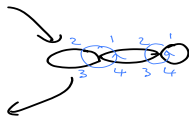
$$N^2 \times \frac{1}{N^4} \times N^4 = N^2 = N^{2-2g}$$



||



$g=0$.



$$\chi = V - E + F = 2 - 4 + 4 = 2 - 2g$$

The Formal Matrix Models: An Example

$$\begin{aligned}\mathcal{U}(t, N) &:= \langle e^{Na_4 \text{Tr} M^4} \rangle \\ &= \sum_{n \geq 1} \frac{(a_4 N)^n}{n!} \langle (\text{Tr} M^4)^n \rangle \\ &= \sum_{n \geq 1} \frac{a_4^n}{n!} \sum N^\chi\end{aligned}$$

where the sum is over all diagrams with n 4-valent vertices. χ is the Euler characteristic of the diagram.

If the diagram is on a surface, then

$$\chi = V - E + F = 2 - 2g$$

where g is the genus of the surface.

$$\begin{aligned}\mathcal{Z}(t, N) &:= \frac{1}{N^2} \log \langle e^{Na_4 \text{Tr} M^4} \rangle \\ &= \sum_{g \geq 0} \mathcal{F}_g(a_4) N^{-2g}\end{aligned}$$

where

$$\mathcal{F}_g(a_4) = \sum_{n \geq 0} \frac{a_4^n}{n!} K_n(g)$$

where $K_n(g)$ is the number of *connected* diagrams with n 4-valent vertices and genus g .

This can be thought of as the generating function of these diagrams.

We will be mainly interested in the first of these, namely the planar limit \mathcal{F}_0 .

Take

$$\mathcal{V}(x) = \sum_{k \geq 1} \frac{a_k x^k}{k}.$$

$$\mathcal{Z}(t, N) := \frac{1}{N^2} \log \langle e^{N \text{Tr} \mathcal{V}(\sqrt{t} M)} \rangle$$

Then

$$\mathcal{Z}(t, N) = \mathcal{F}_0(t) + N^{-2} \mathcal{F}_1(t) + N^{-4} \mathcal{F}_2(t) + \dots$$

where $\mathcal{F}_g(t)$ is the generating function of the numbers $K_n(g)$ of *connected genus g -diagrams with n edges and weights a_k on the vertices of valence k* .

We are interested in the planar limit $\mathcal{F}_0(t)$.

Conjecture (T'Hooft's)

If \mathcal{V} is analytic at 0, then \mathcal{F}_0 is also analytic at 0.

In particular, the numbers $K_n(0)$ grow at most exponentially.
Extreme potentials: $\mathcal{V}(x) = \sum_{k \geq 1} x^{2k} / (2k) = -\frac{1}{2} \log(1 - x^2)$
and $\mathcal{V}(x) = \sum_{k \geq 1} x^k / k = -\log(1 - x)$.

The Analytical Matrix Models

For a potential which grows sufficiently fast at infinity,

$$\begin{aligned} I_V &= - \lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int \exp(-N \text{Tr}(V(M))) dM \\ &= \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \int V(x) \mu(dx) - \iint \log|x-y| \mu(dx) \mu(dy) \right\}, \end{aligned}$$

If $V(x) = x^2/2 - \mathcal{V}(tx)$ and this makes sense, then, formally

$$I_V(t) = 3/4 - \mathcal{F}_0(t)$$

This is one of the ways of computing the planar limit in the physics literature.

The Minimization

$$I_V = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \int V(x) \mu(dx) - \iint \log|x-y| \mu(dx) \mu(dy) \right\}$$

There is a unique equilibrium measure and it has compact support.

Step 1 Find the support of the equilibrium measure.

In the physics literature, for planar maps calculations, this is assumed to be a single interval.

Step 2 For the case of a single interval support of the equilibrium measure, find a “nice” formula for I_V .

Haagerup's Formula

Lemma

For any real $x, y \in [-2, 2]$, $x \neq y$, we have

$$\log|x - y| = - \sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right)$$

where the series here is convergent on $x \neq y$.

If $x > 2$ and $y \in [-2, 2]$, we have

$$\log|x - y| = \log\left|\frac{x + \sqrt{x^2 - 4}}{2}\right| - \sum_{n=1}^{\infty} \frac{2}{n} \left(\frac{x - \sqrt{x^2 - 4}}{2}\right)^n T_n\left(\frac{y}{2}\right)$$

where the series is absolutely convergent.

Here T_n is the n^{th} Chebyshev polynomial: $T_n(\cos x) = \cos(nx)$.

Orthogonal polynomials w.r.t. $\omega(dx) = \mathbb{1}_{[-2,2]}(x) \frac{dx}{\pi \sqrt{4-x^2}}$.

Proof of Haagerup's Formula

$x = 2 \cos u$ and $y = 2 \cos v$ with $u, v \in (0, \pi)$, $u \neq v$

$$x - y = 2(\cos u - \cos v) = 4 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right),$$

$$\begin{aligned} \log|x-y| &= \log\left|2 \sin\left(\frac{u+v}{2}\right)\right| + \log\left|2 \sin\left(\frac{u-v}{2}\right)\right| \\ &= \log|1 - e^{i(u+v)}| + \log|1 - e^{i(u-v)}| \\ &= \operatorname{Re}\left(\log(1 - e^{i(u+v)}) + \log(1 - e^{i(u-v)})\right) \end{aligned}$$

$$\begin{aligned} \log(1-z) &\stackrel{|z|=1, z \neq 1}{=} -\sum_{n=1}^{\infty} \frac{z^n}{n} \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Re}\left(e^{in(u+v)} + e^{in(u-v)}\right) \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} (\cos(n(u+v)) + \cos(n(u-v))) \\ &= -\sum_{n=1}^{\infty} \frac{2}{n} \cos(nu) \cos(nv) = -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right). \end{aligned}$$

For $x > 2$, $y \in [-2, 2]$, write $x = 2 \cosh u = e^u + e^{-u}$, where $u = \log \frac{x + \sqrt{x^2 - 4}}{2}$ and $y = 2 \cos v$

$$\begin{aligned} \log |x - y| &= \log \left(e^u (1 - e^{-u+iv}) (1 - e^{-u-iv}) \right) \\ &= u + \log(1 - e^{-u+iv}) + \log(1 - e^{-u-iv}) \\ &= u - \sum_{n=1}^{\infty} \frac{2}{n} e^{-nu} \cos(nv). \end{aligned}$$

Corollary

If $\omega(dx) = \mathbb{1}_{[-2,2]}(x) \frac{dx}{\pi \sqrt{4-x^2}}$ is the arcsine law of the interval $[-2,2]$, then

$$\int \log |x - y| \omega(dy) = \begin{cases} 0, & |x| \leq 2 \\ \log \frac{|x| + \sqrt{x^2 - 4}}{2}, & |x| > 2. \end{cases}$$

Corollary

The logarithmic potential of a measure on $[-2, 2]$ is given by

$$\int \log|x - y| \mu(dx) = - \sum \frac{2}{n} T_n\left(\frac{x}{2}\right) \int T_n\left(\frac{y}{2}\right) \mu(dy)$$

where this series makes sense pointwise.

$$\iint \log|x - y| \mu(dx) \mu(dy) = - \sum_{n=1}^{\infty} \frac{2}{n} \left(\int T_n\left(\frac{x}{2}\right) \mu(dx) \right)^2.$$

In particular $\iint \log|x - y| \mu(dx) \mu(dy)$ is finite if and only if

$\sum_{n=1}^{\infty} \frac{2}{n} \left(\int T_n\left(\frac{x}{2}\right) \mu(dx) \right)^2$ is finite.

Corollary

If $\mu \in \mathcal{P}([-2, 2])$ and V is a C^3 potential on $[-2, 2]$, then

$$\begin{aligned} I_V(\mu) &= \int V d\mu - \iint \log|x - y| \mu(dx) \mu(dy) \\ &= \beta_0(V) + 2 \sum_{n=1}^{\infty} \left(\beta_n(V) \alpha_n + \frac{\alpha_n^2}{n} \right) \end{aligned}$$

where

$$\alpha_n = \int T_n\left(\frac{x}{2}\right) \mu(dx) \text{ and } \beta_n(V) = \int_{-2}^2 \frac{V(x) T_n\left(\frac{x}{2}\right) dx}{\pi \sqrt{4 - x^2}}.$$

Working with measures on $[-2, 2]$

$$\begin{aligned} I_V(\mu) &= \beta_0(V) - \frac{1}{2} \sum_{n=1}^{\infty} n\beta_n(V)^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_n + \frac{n\beta_n(V)}{2} \right)^2 \\ &\geq \beta_0(V) - \frac{1}{2} \sum_{n=1}^{\infty} n\beta_n(V)^2 \end{aligned}$$

with equality if and only if

$$1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) \geq 0 \quad \text{for any } x \in [-2, 2],$$

in which case

$$\mu(dx) = \left(1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) \right) \frac{dx}{\pi \sqrt{4-x^2}}.$$

Theorem

The equilibrium measure on $[-2, 2]$ has full support if and only if

$$1 - \sum_{n=1}^{\infty} n\beta_n(V)T_n\left(\frac{x}{2}\right) > 0 \quad \text{for } x \text{ on a dense subset of } [-2, 2].$$

in which case

$$\begin{aligned} I_V &= \inf_{\mu \in \mathcal{P}([-2, 2])} I_V(\mu) = \beta_0(V) - \frac{1}{2} \sum_{n=1}^{\infty} n\beta_n(V)^2 \\ &= \int_{-2}^2 \frac{V(x)dx}{\pi \sqrt{4-x^2}} - \int_0^1 s \left[\left(\int_{-2}^2 \frac{xV'(sx)dx}{2\pi \sqrt{4-x^2}} \right)^2 + \left(\int_{-2}^2 \frac{V'(sx)dx}{\pi \sqrt{4-x^2}} \right)^2 \right] ds. \end{aligned}$$

A More Familiar Look

$$1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) = \int_{-2}^2 \frac{V'(x) - V'(y)}{x - y} \frac{dy}{\pi \sqrt{4 - y^2}}.$$

Proof: Linearity + check for $V(x) = T_k\left(\frac{x}{2}\right)$.

$$\beta_0(V) - \frac{1}{2} \sum_{n=1}^{\infty} n\beta_n(V)^2$$

$$= \int_{-2}^2 \frac{V(x)dx}{\pi \sqrt{4-x^2}} - \int_0^1 s \left[\left(\int_{-2}^2 \frac{xV'(sx)dx}{2\pi \sqrt{4-x^2}} \right)^2 + \left(\int_{-2}^2 \frac{V'(sx)dx}{\pi \sqrt{4-x^2}} \right)^2 \right] ds.$$

Proof: Polarization reduces this to

$$\frac{1}{2} \sum_{n \geq 1} n\beta_n(V_1)\beta_n(V_2)$$

$$= \int_0^1 s \left(\int_{-2}^2 xV_1'(sx) \frac{dx}{2\pi \sqrt{4-x^2}} \right) \left(\int_{-2}^2 xV_2'(sx) \frac{dx}{2\pi \sqrt{4-x^2}} \right) ds$$

$$+ \int_0^1 s \left(\int_{-2}^2 V_1'(sx) \frac{dx}{\pi \sqrt{4-x^2}} \right) \left(\int_{-2}^2 V_2'(sx) \frac{dx}{\pi \sqrt{4-x^2}} \right) ds.$$

Check this for $V_1(x) = x^m$, $V_2(x) = x^n$.

The rest of the proof reduces to the following:

Lemma

$$\sum_p p \binom{2m}{m-p} \binom{2n}{n-p} = \frac{mn}{2(m+n)} \binom{2m}{m} \binom{2n}{n}$$

$$\sum_p (2p+1) \binom{2m+1}{m-p} \binom{2n+1}{n-p} = \frac{(2m+1)(2n+1)}{m+n+1} \binom{2m}{m} \binom{2n}{n}$$

with the convention that $\binom{j}{q} = 0$ for $q < 0$ or $q > j$.

Proved using the wzb method implemented for Mathematica.

The real line case

$V \in C^3(\mathbb{R})$. Then the equilibrium measure on \mathbb{R} has support the interval $[-2c + b, 2c + b]$ if and only if (c, b) is the unique absolute maximizer of

$$H(c, b) := \log c - \frac{1}{2} \int_{-2}^2 \frac{V(cx + b) dx}{\pi \sqrt{4 - x^2}}$$

and for x in a dense set of $[-2, 2]$,

$$\psi_{b,c}(x) = \int_{-2}^2 \frac{V'(cx + b) - V'(cy + b)}{x - y} \frac{dy}{\pi \sqrt{4 - y^2}} > 0.$$

In addition,

$$I_V = -\log c + \int_{-2}^2 \frac{V(cx + b) dx}{\pi \sqrt{4 - x^2}} - \int_0^c s \left[\left(\int_{-2}^2 \frac{xV'(sx + b) dx}{2\pi \sqrt{4 - x^2}} \right)^2 + \left(\int_{-2}^2 \frac{V'(sx + b) dx}{\pi \sqrt{4 - x^2}} \right)^2 \right] ds.$$

$$\begin{cases} \int_{-2}^2 cxV'(cx + b) \frac{dx}{\pi \sqrt{4-x^2}} = 2 \\ \int_{-2}^2 V'(cx + b) \frac{dx}{\pi \sqrt{4-x^2}} = 0. \end{cases}$$

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In the physics literature, $R = c^2$, $S = b$.

$$\begin{cases} \int_{-2}^2 \sqrt{R}xV'(\sqrt{R}x + S) \frac{dx}{\pi \sqrt{4-x^2}} = 2 \\ \int_{-2}^2 V'(\sqrt{R}x + S) \frac{dx}{\pi \sqrt{4-x^2}} = 0. \end{cases}$$

Analytic Perturbations

- Let V be analytic on a neighborhood of the one interval support $[b - 2c, b + 2c]$.
- $u \rightarrow V_u$ “analytic” with u in a “good” Banach space B , $V_0 = V$.

Theorem

Near 0,

- *the equilibrium measure of V_u has support one interval $[b_u - 2c_u, b_u + 2c_u]$*
- *$u \rightarrow (b_u, c_u)$ is analytic.*
- *$I(u)$ depends analytically on u .*

Example: $V = x^2/2$ and $V_t(x) = \frac{x^2}{2} + \sum_{n \geq 1} a_n t^n x^n$ at least on a neighborhood of $[b - 2c, b + 2c]$.

Example: $V_u(x) = \frac{x^2}{2} + \sum_{n \geq 1} a_n \rho^n x^n$ with $\rho > 0$ fixed and $u = (a_1, a_2, \dots, a_n, \dots) \in \ell^1$.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{a_n x^n}{n}.$$

\mathcal{R}, \mathcal{S} , uniquely solves as power series in a_1, a_2, \dots

$$\begin{cases} \int_{-2}^2 \sqrt{\mathcal{R}x} \mathcal{V}'(\sqrt{\mathcal{R}x} + \mathcal{S}) \frac{dx}{\pi \sqrt{4-x^2}} = 2 \\ \int_{-2}^2 \mathcal{V}'(\sqrt{\mathcal{R}x} + \mathcal{S}) \frac{dx}{\pi \sqrt{4-x^2}} = 0 \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i). \end{cases}$$

Equivalently,

$$\begin{cases} \mathcal{R} = 1 + \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-2}{2}} \binom{n-1}{j} \binom{n-1-j}{j+1} \mathcal{R}^{j+1} \mathcal{S}^{n-2-2j} \\ \mathcal{S} = \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-1}{2}} \binom{n-1}{j} \binom{n-1-j}{j} \mathcal{R}^j \mathcal{S}^{n-1-2j} \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i) \end{cases}$$

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{a_n x^n}{n}.$$

\mathcal{R}, \mathcal{S} , uniquely solves as power series in a_1, a_2, \dots

$$\begin{cases} \int_{-2}^2 \sqrt{\mathcal{R}x} \mathcal{V}'(\sqrt{\mathcal{R}x} + \mathcal{S}) \frac{dx}{\pi \sqrt{4-x^2}} = 2 \\ \int_{-2}^2 \mathcal{V}'(\sqrt{\mathcal{R}x} + \mathcal{S}) \frac{dx}{\pi \sqrt{4-x^2}} = 0 \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i). \end{cases}$$

Equivalently,

$$\begin{cases} \mathcal{R} = 1 + \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-2}{2}} \binom{n-1}{j} \binom{n-1-j}{j+1} \mathcal{R}^{j+1} \mathcal{S}^{n-2-2j} \\ \mathcal{S} = \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-1}{2}} \binom{n-1}{j} \binom{n-1-j}{j} \mathcal{R}^j \mathcal{S}^{n-1-2j} \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i) \end{cases}$$

It is known (J. Bouttier, P. Di Francesco, E. Guitter in 2002) that \mathcal{R} and \mathcal{S} are the generating functions of connected planar diagrams with one/two distinguished one/two vertices.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{a_n x^n}{n}.$$

Define as a formal power series in a_1, a_2, \dots

$$\begin{aligned} I_{\mathcal{V}} = & -\frac{1}{2} \log \mathcal{R} + \int_{-2}^2 \frac{\mathcal{V}(\sqrt{\mathcal{R}x + \mathcal{S}}) dx}{\pi \sqrt{4 - x^2}} \\ & - \frac{1}{2} \int_0^{\mathcal{R}} \left[\left(\int_{-2}^2 \frac{x \mathcal{V}'(\sqrt{sx + \mathcal{S}}) dx}{2\pi \sqrt{4 - x^2}} \right)^2 + \left(\int_{-2}^2 \frac{\mathcal{V}'(\sqrt{sx + \mathcal{S}}) dx}{\pi \sqrt{4 - x^2}} \right)^2 \right] ds. \end{aligned}$$

Theorem

$$\mathcal{F}_0 = \frac{3}{4} - I_{\mathcal{V}}$$

Counting Planar Diagrams with a Fixed Number of Edges

For

$$\mathcal{V}(x) = -\frac{x^2}{2} + \sum_{n \geq 1} \frac{a_n t^{n/2} x^n}{n},$$

$$\mathcal{F}_0(t) = 3/4 - I(t)$$

$$\mathcal{F}_0(t) = \sum_{n \geq 1} f_n t^n.$$

We are interested in the growth of f_n when n is large.

Theorem

S.G and I.P

$$\mathcal{F}_0(t) = \frac{1}{t} \int_0^t \frac{(t-s)(2\mathcal{R}(s)\mathcal{S}(s)^2 + \mathcal{R}(s)^2 - 1)}{2s} ds,$$

Alternatively

$$(t^2\mathcal{F}'_0(t))' = 2\mathcal{R}(t)\mathcal{S}^2(t) + \mathcal{R}^2(t).$$

Corollary

If \mathcal{V} is a polynomial, then $(t^2\mathcal{F}'_0(t))'$ is an algebraic function. In particular the coefficients f_n of $\mathcal{F}_0(t)$ satisfy a linear recurrence relation with polynomial coefficients in n (f_n are holonomic).

Extreme Potentials, Counting All Diagrams

Counting diagrams with even valent vertices.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{t^n x^{2n}}{2n} = \frac{x^2}{2} + \log(1 - tx^2)$$

$$\mathcal{S}(t) = 0$$

$$\mathcal{R}(t) = \frac{1 + 4t - \sqrt{1 - 8t}}{8t}$$

$$\mathcal{F}_0(t) = \frac{1 - 24t + 72t^2 - (1 - 20t)\sqrt{1 - 8t}}{128t^2}$$

$$-\frac{3}{8} \log \frac{1 - 4t + \sqrt{1 - 8t}}{2}$$

$$= \frac{1}{2} t^2 {}_3F_2(1, 1, 3/2; 2, 4; 8t) = \sum_{n \geq 1} \frac{3(2n-1)! 2^{n-1}}{n!(n+2)!} t^n$$

$$f_n = \frac{3}{4\sqrt{\pi}} \frac{8^n}{n^{7/2}} \left(1 - \frac{25}{8n} + \frac{945}{128n^2} - \frac{16275}{1024n^3} + O\left(\frac{1}{n^4}\right) \right).$$

Extreme Potentials, Counting All Diagrams

Counting planar diagrams with arbitrary valency.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{t^{n/2} x^n}{n} = \frac{x^2}{2} + \log(1 - \sqrt{t}x)$$

$$\mathcal{S}(t) = \frac{1 - \sqrt{1 - 12t}}{6\sqrt{t}}$$

$$\mathcal{R}(t) = \frac{1 + 12t - \sqrt{1 - 12t}}{18t} \quad (1)$$

$$F_0(t) = \frac{1 - 36t + 162t^2 - (1 - 30t)\sqrt{1 - 12t}}{216t^2} \quad (2)$$

$$f_n = \frac{1}{2} \log \frac{1 - 6t + \sqrt{1 - 12t}}{2} - \frac{2}{\sqrt{\pi}} \frac{12^n}{n^{7/2}} \left(1 - \frac{25}{16n} + \frac{945}{256n^2} - \frac{16275}{2048n^3} + O\left(\frac{1}{n^4}\right) \right).$$

t'Hooft's conjecture in sharp form

Theorem

Assume that

$$\mathcal{V}(t) = \frac{x^2}{2} - \sum_{n \geq 1} \frac{a_n x^n}{n}$$

and

$$\alpha(\mathcal{V}) = \sup_{n \geq 1} |a_n|^{1/n}.$$

Then $\mathcal{F}_0(t)$ has radius of convergence at least $\frac{1}{12 \sqrt{\alpha(\mathcal{V})}}$ with equality for the extreme potential $\mathcal{V} = \frac{x^2}{2} - \sum_{n \geq 1} \frac{x^n}{n}$.

Counting planar diagrams with vertices of valence 3 and 4

This corresponds to the potential

$$\mathcal{V}(x) = \frac{x^2}{2} - t^{3/2} \frac{x^3}{3} - t^2 \frac{x^4}{4}$$

The computations are not exact.

$$f_n \sim C \frac{(1/t_0)^n}{n^{7/2}} \left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + O\left(\frac{1}{n^3}\right) \right)$$

where t_0 is the closest root to $1/5$ of

$$6912 - 13824t - 146592t^2 - 239488t^3 - 2602569t^4 - 4300752t^5 + 79091888t^6 + 304167552t^7 + 410284704t^8 - 1349207040t^9 - 7615156224t^{10} - 4603041792t^{11} + 31506516736t^{12}$$

$$1/t_0 = 4.7751036758 \dots$$

$$C = 1.4826787729 \dots$$

$$d_1 = -7.2166440681 \dots$$

$$d_2 = 37.5616277128 \dots$$

Other extreme potentials

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n \geq 1} a_n x^n$$

The growth

$$f_n \sim C(1/t_0)^n$$

t_0 , the only root in $(0, 1)$ of

$$\begin{aligned} & -226492416 + 962592768t + 34574598144t^2 + \\ & 334387408896t^3 + 7450906184352t^4 + 21095006644064t^5 + \\ & 130097822364531t^6 + 55792303752096t^7 + \\ & 67902575063040t^8 + 19100742451200t^9 + 6115295232000t^{10}. \end{aligned}$$

$$1/t_0 \sim 20.18.$$

Already done

- Study the generating functions for the case of counting planar diagrams with a certain number of faces.
- Growth of the number of diagrams with certain particular structure. As in the example with valency 3 or 4.

To think about

- Genus one extension.
- Extend this to multitrace matrix models.
- Multiple matrix models.