Bulk Universality for Wigner Matrices

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1. Wigner Matrices and the Local Semicircle Law

Hermitian Wigner Matrices: $N \times N$ matrices $H = (h_{kj})_{1 \leq k,j \leq N}$ such that $H^* = H$ and

$$h_{kj} = \frac{1}{\sqrt{N}} \left(x_{kj} + i y_{kj} \right) \qquad \text{for all } 1 \leq k < j \leq N$$

$$h_{kk} = \frac{2}{\sqrt{N}} x_{kk} \qquad \text{for all } 1 \leq k \leq N$$

where x_{kj}, y_{kj} and x_{kk} $(1 \leq k \leq N)$ are iid with

$$\mathbb{E}\, x_{jk} = 0 \qquad \text{and} \qquad \mathbb{E}\, x_{jk}^2 = \frac{1}{2} \qquad \left(\Rightarrow \quad \mathbb{E}\, |h_{jk}|^2 = \frac{1}{N} \right)$$

Remark: scaling so that eigenvalues remain bounded as $N \to \infty$.

$$\mathbb{E} \sum_{\alpha=1}^{N} \lambda_\alpha^2 = \mathbb{E}\, \text{Tr}\, H^2 = \mathbb{E} \sum_{j,k=1}^{N} |h_{jk}|^2 = N^2\, \mathbb{E}\, |h_{jk}|^2$$

$$\Rightarrow \quad \mathbb{E}|h_{jk}|^2 = O(N^{-1})$$

Gaussian Unitary Ensemble (GUE): simplest example of hermitian Wigner ensemble. Probability density given by

$$P(H)\mathrm{d}H = \mathrm{const}\cdot e^{-\frac{N}{2}\mathrm{Tr}\,(H^2)}\mathrm{d}H$$

Big advantage: joint eigenvalue distribution is explicit

$$p(\lambda_1,\ldots,\lambda_N) = \mathrm{const}\cdot\prod_{i<j}^{N}(\lambda_i-\lambda_j)^2\,e^{-\frac{N}{2}\sum_{j=1}^{N}\lambda_j^2}\,.$$

Dyson's sine-kernel distribution for GUE: using the explicit formula for density, local eigenvalue statistics can be computed in limit $N\to\infty$. Let

$$p^{(k)}(\lambda_1,\ldots,\lambda_k) = \int \mathrm{d}\lambda_{k+1}\ldots\mathrm{d}\lambda_N\,p(\lambda_1,\ldots,\lambda_N)$$

be the k-point correlation function. Then

$$\frac{1}{\varrho_{sc}^k(E)}\,p^{(k)}\Big(E+\frac{x_1}{N\varrho_{sc}(E)},..,E+\frac{x_k}{N\varrho_{sc}(E)}\Big)\to\det\left(\frac{\sin(\pi(x_i-x_j))}{\pi(x_i-x_j)}\right)_{i,j\le k}$$

Semicircle Law (Wigner, 1955): for any $\delta > 0$,

$$\lim_{\eta \to 0} \lim_{N \to \infty} \mathbb{P} \left(\left| \frac{\mathcal{N}[E - \frac{\eta}{2}; E + \frac{\eta}{2}]}{N\eta} - \rho_{\text{sc}}(E) \right| \geq \delta \right) = 0$$

where

$$\mathcal{N}[I] = \text{number of eigenvalues in interval } I$$
$$\rho_{\text{sc}}(E) = \frac{1}{2\pi}\sqrt{1 - E^2/4}.$$

Remark 1: semicircle independent of distribution of entries.

Remark 2: Wigner result concerns the macroscopic density, that is the density in intervals containing order N eigenvalues.

What about density of states in smaller intervals?

Theorem [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu |x_{ij}|} < \infty$ for some $\nu > 0$, and fix $|E| < 2$. Then, for any $\delta > 0$,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathbb{P}\left(\left| \frac{\mathcal{N}\left[E - \frac{K}{2N}; E + \frac{K}{2N}\right]}{K} - \rho_{\text{sc}}(E) \right| \geq \delta \right) = 0$$

More precisely, we show that

$$\mathbb{P}\left(\left| \frac{\mathcal{N}\left[E - \frac{K}{2N}; E + \frac{K}{2N}\right]}{K} - \rho_{\text{sc}}(E) \right| \geq \delta \right) \leq C e^{-c\delta\sqrt{K}}$$

for all $K > 0$, uniformly in $N > N_0(\delta)$.

Intermediate scales: if $\eta(N) \to 0$ such that $N\eta(N) \to \infty$, we have

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \frac{\mathcal{N}\left[E - \frac{\eta(N)}{2}; E + \frac{\eta(N)}{2}\right]}{N\eta(N)} - \rho_{\text{sc}}(E) \right| \geq \delta \right) = 0$$

Previous results by Khorunzhy, Bai-Miao-Tsay, and Guionnet-Zeitouni (up to scales $\eta(N) \simeq N^{-1/2}$).

Main ingredients of proof: upper bound on density and fixed point equation for Stieltjes transform.

Upper bound: observe that

$$\mathcal{N}[E-\eta/2, E+\eta/2] = \sum_\alpha \mathbf{1}(|\lambda_\alpha - E| \leq \eta)$$
$$\leq \sum_\alpha \frac{\eta^2}{(\lambda_\alpha - E)^2 + \eta^2} = \eta \, \mathrm{Im} \, \sum_\alpha \frac{1}{\lambda_\alpha - E - i\eta}$$

and hence

$$\rho = \frac{\mathcal{N}[E-\eta/2, E+\eta/2]}{N\eta}$$
$$\leq \frac{1}{N} \mathrm{Im} \, \mathrm{Tr} \, \frac{1}{H - E - i\eta} = \frac{1}{N} \mathrm{Im} \, \sum_{j=1}^N \frac{1}{H - E - i\eta}(j,j)$$

We bound, for example, the $(1,1)$-element of the diagonal.

Decomposing H as

$$H = \begin{pmatrix} h_{11} & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix}$$

we find (Feshbach map)

$$\frac{1}{H-z}(1,1) = \frac{1}{h_{11} - z - \mathbf{a} \cdot (B-z)^{-1}\mathbf{a}} = \frac{1}{h_{11} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{\lambda_\alpha - z}}$$

with

$$\xi_\alpha = N|\mathbf{a} \cdot \mathbf{u}_\alpha|^2 \qquad \Rightarrow \qquad \mathbb{E}\,\xi_\alpha = 1$$

where λ_α and $\mathbf{u}_\alpha$ are eigenvalues and eigenvectors of B.

We conclude that

$$\text{Im}\ \frac{1}{H-E-i\eta}(1,1) \le \frac{1}{\eta + \frac{1}{N}\sum_\alpha \frac{\eta}{(\lambda_\alpha - E)^2 + \eta^2}} \le \frac{N\eta}{\sum_{\alpha : |\lambda_\alpha - E| \le \eta} \xi_\alpha} \le \frac{C}{\rho}$$

Fixed point equation: we consider the Stieltjes transform

$$m_N(z) = \frac{1}{N}\text{Tr}\,\frac{1}{H-z}, \qquad m_{\text{sc}}(z) = \int \text{d}y \frac{\rho_{\text{sc}}(y)}{y-z}$$

Convergence of the density follows if we can prove that

$$m_N(z) \to m_{\text{sc}}(z), \qquad \text{for Im } z = \eta \geq K/N.$$

The Stieltjes transform m_{sc} solves the fixed point equation

$$m_{\text{sc}}(z) + \frac{1}{z + m_{\text{sc}}(z)} = 0$$

It is enough to show that, with high probability,

$$\left| m_N(z) + \frac{1}{z + m_N(z)} \right| \leq \delta$$

To this end, we use again

$$m_N(z) = \frac{1}{N} \sum_j \frac{1}{h_{jj} - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha^{(j)}}{\lambda_\alpha^{(j)} - z}}$$

2. Delocalization of Eigenvectors

Let $\mathbf{v} = (v_1, \dots, v_N)$ be an ℓ_2-normalized vector in $\mathbb{C}^N$. Distinguish two extreme cases:

Complete localization: one large component, for example

$$\mathbf{v} = (1, 0, \dots, 0) \qquad \Rightarrow \qquad \|\mathbf{v}\|_p = 1, \text{ for all } 2 < p \leq \infty$$

Complete delocalization: all components have same size,

$$\mathbf{v} = (N^{-1/2}, \dots, N^{-1/2}) \quad \Rightarrow \quad \|\mathbf{v}\|_p = N^{-1/2+1/p} \ll 1$$

Theorem [Erdős-S.-Yau, 2008]:
Suppose $\mathbb{E}\, e^{\nu|x_{ij}|} < \infty$ for some $\nu > 0$. Fix $\kappa > 0$, $2 < p \leq \infty$. Then

$$\mathbb{P}\Big(\exists \mathbf{v} : H\mathbf{v} = \mu\mathbf{v}, \mu \in [-2+\kappa, 2-\kappa], \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_p \geq MN^{-\frac{1}{2}+\frac{1}{p}}\Big) \leq Ce^{-c\sqrt{M}}$$

for all M, N large enough.

Idea of proof: we write $\mathbf{v} = (v_1, \mathbf{w})$. Hence $H\mathbf{v} = \mu\mathbf{v}$ implies

$$\begin{pmatrix} h-\mu & \mathbf{a}^* \\ \mathbf{a} & B-\mu \end{pmatrix} \begin{pmatrix} v_1 \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix} \quad \Rightarrow \quad \mathbf{w} = v_1(\mu - B)^{-1}\mathbf{a}$$

By normalization

$$1 = v_1^2 + \mathbf{w}^2 \quad \Rightarrow \quad |v_1|^2 = \frac{1}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\mu-\lambda_\alpha)^2}} \qquad (\xi_\alpha = N|\mathbf{a}\cdot\mathbf{u}_\alpha|^2),$$

where λ_α and $\mathbf{u}_\alpha$ are the eigenvalues and the eigenvectors of B.

$$|v_1|^2 \le \frac{1}{\frac{1}{N\eta^2}\sum_{\alpha:|\lambda_\alpha-\mu|\le\eta}\xi_\alpha}$$

Choosing $\eta = K/N$, for a sufficiently large $K > 0$, we find

$$|v_1|^2 \le \frac{K^2}{N}\frac{1}{\sum_{\alpha:|\lambda_\alpha-\mu|\le K/N}\xi_\alpha} \le c\,\frac{K}{N}$$

with high probability, because, by the local semicircle law, there must be order K eigenvalues λ_α with $|\lambda_\alpha - \mu| \le K/N$. □

3. Level Repulsion

Theorem [Erdős-S.-Yau, 2008]: Suppose $\mathbb{E}\, e^{\nu|x_{ij}|} < \infty$ for some $\nu > 0$, fix $|E| < 2$.

Fix $k \geq 1$, and assume that the probability density $h(x) = e^{-g(x)}$ of the matrix entries satisfies the bound

$$\left|\widehat{h}(p)\right| \leq \frac{1}{(1+Cp^2)^{\sigma/2}}, \qquad \left|\widehat{hg''}(p)\right| \leq \frac{1}{(1+Cp^2)^{\sigma/2}} \qquad \text{for } \sigma \geq 5+k^2.$$

Then there exists a constant $C_k > 0$ such that

$$\mathbb{P}\left(\mathcal{N}\left[E-\frac{\varepsilon}{2N}; E+\frac{\varepsilon}{2N}\right] \geq k\right) \leq C_k\, \varepsilon^{k^2}$$

for all N large enough, and all $\varepsilon > 0$.

Remark: for GUE, we have

$$p(\lambda_1,\ldots,\lambda_N) \simeq \prod_{i<j}(\lambda_i-\lambda_j)^2 \quad \Rightarrow \quad \mathbb{P}(\mathcal{N}_\varepsilon \geq k) \simeq \varepsilon^{k^2}$$

4. Universality for Wigner Matrices

Universality: local eigenvalue statistics in the limit $N \to \infty$ is expected to depend only on symmetry, but to be independent of probability law of matrix entries.

Remark: universality at the edges of the spectrum was established by Soshnikov in 1999 using the moment method. Here I will consider universality in the bulk of the spectrum.

In 2001, Johansson established the validity of bulk universality for ensembles of hermitian Wigner matrices with a Gaussian component (result was later extended by Ben Arous-Péché).

Johansson's approach: consider matrices of the form

$$H = H_0 + t^{\frac{1}{2}} V$$

where V is a GUE-matrix, and H_0 is an arbitrary Wigner matrix.

The matrix H can be obtained by letting every entry of H_0 evolve under a Brownian motion up to time t (more prec. t/N).

The distribution of the eigenvalues of the matrix evolves then according to Dyson's Brownian motion

$$\mathrm{d}\lambda_\alpha = \frac{\mathrm{d}B_\alpha}{\sqrt{N}} + \frac{1}{N} \sum_{\beta \neq \alpha} \frac{1}{\lambda_\alpha - \lambda_\beta} \mathrm{d}t, \qquad 1 \leq \alpha \leq N$$

where $\{B_\alpha : 1 \leq \alpha \leq N\}$ is a collection of independent Brownian motion.

The joint probability distribution of the eigenvalues $\mathbf{x} = (x_1, \ldots, x_N)$ of H is

$$p(\mathbf{x}) = \int \mathrm{d}\mathbf{y}\; q_t(\mathbf{x};\mathbf{y})\; p_0(\mathbf{y})$$

where p_0 is the distribution of the eigenvalues $\mathbf{y} = (y_1, \ldots, y_N)$ of H_0 and

$$q_t(\mathbf{x};\mathbf{y}) = \frac{N^{N/2}}{(2\pi t)^{N/2}} \frac{\Delta_N(\mathbf{x})}{\Delta_N(\mathbf{y})}\; \det \left(e^{-N(x_j-y_k)^2/2t}\right)_{j,k=1}^{N},$$

with the Vandermonde determinant

$$\Delta(\mathbf{x}) = \prod_{i<j}^{N} (x_i - x_j) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_N \\ \ldots & \ldots & \ldots & \ldots \\ x_1^N & x_2^N & \ldots & x_N^N \end{pmatrix}$$

This can be proven using the Harish-Chandra/Itzykson-Zuber formula

$$\int_{U(N)} e^{-\frac{N}{2t}\mathrm{Tr}\,(U^* R(\mathbf{x}) U - H_0(\mathbf{y}))^2} \mathrm{d}U = \frac{1}{\Delta(\mathbf{x})\Delta(\mathbf{y})}\; \det \left(e^{-\frac{N}{2t}(x_j-y_i)^2}\right)_{1\le i,j\le N}$$

The k-point correlation function of p is therefore given by

$$p^{(k)}(x_1, \ldots, x_k) = \int q_t^{(k)}(x_1, \ldots, x_k; \mathbf{y})\, p_0(\mathbf{y})\mathrm{d}\mathbf{y}$$

where

$$\begin{aligned} q_t^{(k)}(x_1, \ldots, x_k; \mathbf{y}) &= \int q_t(\mathbf{x}; \mathbf{y})\, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_N \\ &= \frac{(N-k)!}{N!}\ \det\Big(K_{t,N}(x_i, x_j; \mathbf{y})\Big)_{1 \le i,j \le k} \end{aligned}$$

with

$$K_{t,N}(u, v; \mathbf{y}) = \frac{N}{(2\pi i)^2 (v-u) t}$$

$$\times \int_\gamma \mathrm{d}z \int_\Gamma \mathrm{d}w\, (e^{-N(v-u)(w-r)/t} - 1) \prod_{j=1}^N \frac{w - y_j}{z - y_j}$$

$$\times \frac{1}{w-r}\Big(w - r + z - u - \frac{t}{N} \sum_j \frac{y_j - r}{(w-y_j)(z-y_j)}\Big) e^{N(w^2 - 2vw - z^2 + 2uz)/2t}$$

where γ is the union of two horizontal lines and Γ is a vertical line in the $\mathbb{C}$-plane, and $r \in \mathbb{R}$ is arbitrary.

Convergence of k-point correlation follows from

$$\frac{1}{N\varrho(u)} K_{t,N}\left(u+\frac{x_1}{N\varrho(u)}, u+\frac{x_2}{N\varrho(u)}; \mathbf{y}\right) \to \frac{\sin\pi(x_2-x_1)}{\pi(x_2-x_1)} \qquad \text{for a.e. } \mathbf{y}$$

To prove convergence of $K_{t,N}$ to sine-kernel Johansson uses

$$\frac{1}{N\varrho(u)} K_{t,N}\left(u, u+\frac{\tau}{N\varrho}; \mathbf{y}\right) = N\int_\gamma \frac{\mathrm{d}z}{2\pi i}\int_\Gamma \frac{\mathrm{d}w}{2\pi i} h_N(w) g_N(z,w) e^{N(f_N(w)-f_N(z))}$$

with

$$f_N(z) = \frac{1}{2t}(z^2-2uz)+\frac{1}{N}\sum_j \log(z-y_j)$$

$$g_N(z,w) = \frac{1}{t(w-r)}[w-r+z-u] - \frac{1}{N(w-r)}\sum_j \frac{y_j-r}{(w-y_j)(z-y_j)}$$

$$h_N(w) = \frac{1}{\tau}\left(e^{-\tau(w-r)/t\varrho}-1\right)$$

and performs a detailed asymptotic saddle analysis.

Beyond Johansson: what happens if $t = t(N) \to 0$? Consider

$$t = N^{-1+\varepsilon}$$

Similar integral representation but asymptotic analysis is more delicate and requires microscopic convergence to the semicircle.

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Let $p_N^{(k)}$ be the k-point eigenvalue correlation function for the ensemble $H = H_0 + t^{1/2}V$, where H_0 is an arbitrary Wigner matrix, V is an independent GUE matrix, and $t \geq N^{-1+\varepsilon}$. Then

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}^k(E)} p_N^{(k)} \left(E + \frac{x_1}{N\rho_{sc}(E)}, \ldots, E + \frac{x_k}{N\rho_{sc}(E)} \right) = \det \left(\frac{\sin(\pi(x_i - x_j))}{(\pi(x_i - x_j))} \right)_{i,j=1}^k$$

Time reversal to remove Gaussian part: let $h(x)$ be the density of the matrix elements of H_0.

The matrix elements of $H = H_0 + t^{\frac{1}{2}} V$ have density

$$h_t(x) = (e^{tL}h)(x), \qquad \text{with} \qquad L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

Then

$$\int \frac{|h_t(x) - h(x)|^2}{h(x)} dx \leq Ct^2$$

Letting $F = h^{\otimes N^2}$ and $F_t = (e^{tL}h)^{\otimes N^2}$ we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \dots dx_{N^2} \leq CN^2 t^2$$

It is only small for $t \ll N^{-1}$.

Hence $t = N^{-1+\varepsilon}$ is still not enough.

We would like to write

$$h = e^{tL} v_t \qquad \text{with} \qquad v_t = e^{-tL} h$$

But the heat equation cannot be reversed.

$\Rightarrow$ approximate inversion of heat semigroup

Define $v_t = (1 - tL)h$. Then

$$h_t = e^{tL} v_t \simeq h + t^2 L^2 h \qquad \Big(\text{while} \quad e^{tL} h \simeq h + tLh\Big)$$

Therefore

$$\int \frac{|h_t - h|^2}{h} dx \leq C t^4$$

Hence, if $F = h^{\otimes N^2}$ and $F_t = h_t^{\otimes N^2}$, we find

$$\int \frac{|F_t - F|^2}{F} dx_1 \dots dx_{N^2} \leq C N^2 t^4 \ll 1 \qquad \text{for } t = N^{-1+\varepsilon}$$

Theorem [Erdős-Péché-Ramirez-S.-Yau]: Suppose H is a hermitian Wigner matrix, whose entries have law $g = e^{-h}$, for $h \in C^6(\mathbb{R})$. Then,

$$\lim_{N\to\infty} \frac{1}{\rho_{\text{sc}}^2(E)} p_N^{(2)}\left(E + \frac{x_1}{N\rho_{\text{sc}}(E)}, E + \frac{x_2}{N\rho_{\text{sc}}(E)}\right) = \frac{\sin(\pi(x_1 - x_2))}{(\pi(x_1 - x_2))}$$

Shortly after we posted our result, Tao-Vu submitted a paper with same results. Combining two approaches, one can remove all conditions, at least after averaging over the variable u.

Theorem [Erdős-Ramirez-S.-Tao-Vu-Yau, 2009]: Fix $\varepsilon > 0$ and $|u_0| < 2 - \varepsilon$. Fix $k \geq 1$, then

$$\lim_{N\to\infty} \frac{1}{2\varepsilon} \int_{u_0-\varepsilon}^{u_0+\varepsilon} du \frac{1}{[\rho_{sc}(u)]^k} p^{(k)}\left(u + \frac{x_1}{N\rho_{sc}(u)}, \dots, u + \frac{x_k}{N\rho_{sc}(u)}\right)$$

$$= \det\left(\frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)}\right)_{i,j=1}^k$$

5. Universality for Non-Hermitian Ensembles

The local relaxation flow: Dyson Brownian Motion describes evolution of eigenvalues. Equilibrium measure is GUE measure

$$\mu(\mathbf{x})\mathrm{d}\mathbf{x} = \frac{e^{-\mathcal{H}(\mathbf{x})}}{Z}\mathrm{d}\mathbf{x}, \qquad \mathcal{H}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\frac{x_j^2}{2} - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$

The evolution of an initial probability density function $f\mu$ w.r.t DBM is described by the heat equation

$$\partial_t f_t = L f_t,$$

with the generator

$$L = \sum_{i=1}^{N}\frac{1}{2N}\partial_i^2 + 2\sum_{i=1}^{N}\left(-\frac{1}{4}x_i + \frac{1}{2N}\sum_{j\neq i}\frac{1}{x_i - x_j}\right)\partial_i$$

Relaxation time of Dyson's Brownian motion given by

$$\frac{1}{2N}\nabla^2\mathcal{H} \geq O(1) \quad \Rightarrow \quad \text{relaxation on times } O(1)$$

Idea: introduce new flow with shorter relaxation time. Define

$$\widetilde{\mathcal{H}}(\mathbf{x}) = N\left[\sum_{j=1}^{N}\left(\frac{x_j^2}{2} + \frac{1}{2R^2}(x_j-\gamma_j)^2\right) - \frac{2}{N}\sum_{i<j}\log|x_j - x_i|\right]$$

$$= \mathcal{H}(\mathbf{x}) + \frac{N}{2R^2}\sum_{j=1}^{N}(x_j-\gamma_j)^2$$

where γ_j is position of the j-th eigenvalue w.r.t. semicircle law, and $R = N^{-\varepsilon} \ll 1$.

Introduce new equilibrium measure $\omega(\mathbf{x}) = e^{-\widetilde{H}(\mathbf{x})}/\widetilde{Z}$ and new evolution

$$\partial_t g_t = \widetilde{L} g_t \qquad \text{with} \qquad \widetilde{L} = L - \frac{1}{R^2}\sum_{j=1}^{N}(x_j - \gamma_j)\,.$$

Observe that

$$\frac{\nabla^2 \widetilde{\mathcal{H}}(\mathbf{x})}{N} \geq CR^{-2} \geq N^{2\varepsilon} \gg 1 \quad \Rightarrow \quad \text{relaxation on short times}$$

Hence, if $\mathcal{G}_{i,n}(\mathbf{x}) = G\Big(N(x_i - x_{i+1}), \dots, N(x_{i+n-1} - x_{i+n})\Big)$, we find

$$\Big| \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,n} \mathrm{d}\omega - \int \frac{1}{N} \sum_{i \in J} \mathcal{G}_{i,n}\, g\, \mathrm{d}\omega \Big| \le C_n \Big(\frac{D_\omega(\sqrt{g}) R^2}{N} \Big)^{1/2}$$

with the Dirichlet form

$$D_\omega(h) = \frac{1}{N} \sum_{j=1}^{N} \int \left| \partial_{x_j} h \right|^2 \mathrm{d}\omega$$

On other hand, if difference between generators is small, we expect $f_t \mu \simeq \omega = \psi \mu$. In fact, for $t \gg R^2$, we find that

$$D_\omega(\sqrt{f_t/\psi}) \le C N \Lambda \qquad \text{where} \quad \Lambda = \mathbb{E}_t \sum_j |x_j - \gamma_j|^2 .$$

From microscopic semicircle law, we find $\Lambda \le N^{-\varepsilon}$.

This implies universality for ensembles of the form $H_0 + t^{1/2}V$, if $t \geq N^{-\varepsilon}$, for arbitrary symmetry.

Time-reversal argument implies universality for all matrices whose entries have enough regularity.

Combining with the result of Tao-Vu, we find universality for arbitrary ensembles.

Theorem [Erdős, S., Yau, 2009]: For arbitrary hermitian, symmetric or symplectic Wigner matrices with subexponentially fast decaying entries, the weak limit as $N \to \infty$ of the averaged k-particle correlation function

$$\frac{1}{2\varepsilon}\int_{E-\varepsilon}^{E+\varepsilon} \mathrm{d}u \, \frac{1}{\rho_{\mathrm{sc}}^k(u)} \, p_N^{(k)} \left(u + \frac{x_1}{\rho_{\mathrm{sc}}(u)N}, \ldots, u + \frac{x_k}{\rho_{\mathrm{sc}}(u)N} \right)$$

coincides with that of the corresponding Gaussian ensemble (GUE, GOE, or GSE) for all $|E| < 2$ and $\varepsilon > 0$.

6. Open Problems

Random Band Matrices: consider $N \times N$ hermitian matrices H with independent entries h_{ij} with

$$\mathbb{E}h_{ij} = 0, \qquad \mathbb{E}|h_{ij}|^2 = \begin{cases} 1/W & \text{if } |i-j| \leq W \\ 0 & \text{if } |i-j| > W \end{cases}$$

$W \ll \sqrt{N}$: localization, Poisson statistics.

$W \gg \sqrt{N}$: delocalization, Wigner-Dyson's statistics.

Anderson Model: consider the Hamiltonian

$$H = \Delta + \lambda V_\omega(x) \qquad \text{acting on } \ell^2(\mathbb{Z}^d)$$

where $\{V_\omega(x) : x \in \mathbb{Z}^d\}$ is a collection of iid real random variables.

$d \geq 3$, λ small enough: delocalized eigenvectors (extended states), Wigner-Dyson's sine-kernel statistics.

7. Appendices

Why is local semicircle important for asymptotic analysis?

Recall that

$$\frac{1}{N\varrho(u)}K_{t,N}\Big(u, u+\frac{\tau}{N\varrho(u)};\mathbf{y}\Big)$$
$$= N\int_\gamma \frac{\mathrm{d}z}{2\pi i}\int_\Gamma \frac{\mathrm{d}w}{2\pi i} h_N(w) g_N(z,w) e^{N(f_N(w)-f_N(z))}$$

with

$$f_N(z) = \frac{1}{2t}(z^2 - 2uz) + \frac{1}{N}\sum_j \log(z - y_j)$$

Saddles are determined by the equation

$$f_N'(z) = \frac{1}{t}(z-u) + \frac{1}{N}\sum_j \frac{1}{z-y_j} = 0$$

There are two complex conjugated solutions $z = q_N^{\pm}$.

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There are two complex conjugated solutions $z = q_N^{\pm}$.

By the convergence to the semicircle on scales of order $N^{-1+\varepsilon}$, we have, with high probability,

$$q_N^{\pm} = q^{\pm} + O(tN^{-\varepsilon/2})$$

where

$$q^{\pm} = u(1-2t) \pm 2ti\sqrt{1-u^2} + O(tN^{-\varepsilon/2})$$

are the two solutions of

$$\frac{1}{t}(q^{\pm}-u) + \int \frac{\varrho_{sc}(y)}{q^{\pm}-y}\mathrm{d}y = 0$$

The integration paths can be shifted to pass through the saddles.

Only important contribution arises from z, w both close to $q_{N,\pm}$.

Contribution from saddles can be computed through local change of variable which makes the exponent quadratic (Laplace method).

As $N \to \infty$, saddle contribution leads to sine-kernel.

Tao-Vu approach: let H and H' be two Wigner matrices whose entries have distribution x, y; assume that typical distance between eigenvalues is order one ($x, y \simeq \sqrt{N}$).

Assume that

$$\mathbb{E}\, x^m = \mathbb{E}\, y^m \qquad \text{for} \qquad 1 \le m \le 4$$

Fix $k \ge 1$ and consider a nice function $G : \mathbb{R}^k \to \mathbb{R}$. Then

$$\left|\mathbb{E}\, G(\lambda_{\alpha_1}(H), \ldots, \lambda_{\alpha_k}(H)) - \mathbb{E} G(\lambda_{\alpha_1}(H), \ldots, \lambda_{\alpha_k}(H))\right| \to 0$$

as $N \to \infty$.

Idea of proof: change one entry at the time.

$H(z) =$ matrix obtained from H replacing (i, j)-entry with z

$F(z) = G(\lambda_\alpha(H(z)))$ (we take $k = 1$)

$$F(x) = F(0) + xF'(0) + \cdots + \frac{x^5}{5!} F^{(v)}(0) + ..$$

$$F(y) = F(0) + yF'(0) + \cdots + \frac{y^5}{5!} F^{(v)}(0) + ..$$

Therefore

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \le \mathbb{E}\,|x|^5 F^{(v)}(0)$$

Observe

$$\mathbb{E}|x|^5 \simeq N^{5/2} \qquad \text{but} \qquad F^{(m)}(0) \simeq N^{-m}$$

In fact

$$F'(0) = G'(\lambda_\alpha(H)) \cdot \frac{\partial \lambda_\alpha}{\partial h_{ij}} = G'(\lambda_\alpha(H)) \cdot \mathbf{v}_\alpha(i)\mathbf{v}_\alpha(j) \simeq N^{-1}$$

Hence

$$|\mathbb{E}F(x) - \mathbb{E}F(y)| \le CN^{-5/2}$$

Repeating this argument N^2 times, we can replace all entries of H; the total error is $O(N^{-1/2})$.

Universality (Tao-Vu): for given H, find Johansson matrix

$$H_t = e^{-t/2} H_0 + (1 - e^{-t})^{1/2} V$$

such that H and H_t have four matching moments.

This is only possible if entries are supported on at least 3 points.

Universality (Erdős-Ramirez-S.-Tao-Vu-Yau): compare H with the evolved matrix

$$H_t = e^{-t/2} H + (1 - e^{-t})^{1/2} V$$

with $t = N^{-1+\delta}$.

Moments do not match, but they are very close.