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Stability of Planar Switched Systems: The Linear Single Input Case

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Abstract

We study the stability of the origin for the dynamical system $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$, where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbf{R}^2$ and $u(.) : [0, \infty[\to [0, 1]$ is a completely random measurable function. More precisely, we find a (coordinates invariant) necessary and sufficient condition on A and B for the origin to be asymptotically stable for each function $u(.)$. This bidimensional problem assumes particular interest since linear systems of higher dimensions can be reduced to our situation. Two unpublished examples in the (more difficult) case in which both matrices have real eigenvalues are analyzed in details.

1 Introduction

By a switched system we mean a family of continuous–time dynamical systems and a rule that determines at any time which dynamical system is responsible for the time evolution. More precisely let $\{f_u : u \in U\}$ be a (finite or infinite) set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{1}$$

The rule is given assigning the so called switching function $u(.) : [0, \infty[\to U$. Here we consider the situation is which the switching function cannot be predicted a priori; it is given from outside and represents some phenomena (e.g. a disturbance) that it is not possible to control or include in the dynamical system model. In the following we use the notation $u \in U$ to indicate a fixed individual system and $u(.)$ to indicate the switching function.

Suppose now that all the f_u have a given property for every $u \in U$. A typical problem is to study under which conditions this property holds for the system (1) for arbitrary switching functions. For a discussion of various issues related to switched system we refer the reader to [9].

In [1, 8] the case of switched linear systems was considered:

$$\dot{x} = A_u x, \quad x \in \mathbf{R}^n, \quad A_u \in \mathbf{R}^{n\times n}, \quad u \in U, \tag{2}$$

and the problem of the asymptotic stability of the origin for arbitrary switching functions was investigated. Clearly we need the asymptotic stability of each single sub–system $\dot{x} = A_u x$, $u \in U$, in order to have the asymptotic stability of (2) for each switching function (i.e. the eigenvalues of each matrix A_u must have strictly negative real part). This will be assumed to be the case throughout the paper. Notice the important point that in the case of linear systems, the asymptotic stability for arbitrary switching functions is equivalent to the more often quoted property of GUES (global exponential stability, uniform with respect to switching), see for example [2] and references therein.

In [1, 8], it is shown that the structure of the Lie algebra generated by the matrices A_u:

$$\mathbf{g} = \{A_u : u \in U\}_{L.A.},$$

is crucial for the stability of the system (2) (i.e. the interrelation between the systems). The main result of [8] is the following:

Theorem 1 (Hespanha, Morse, Liberzon) *If $\mathbf{g}$ is a solvable Lie algebra, then the switched system (2) is asymptotically stable for each switching function $u(.) : [0, \infty[\to U$.*

In [1] a generalization was given. Let $\mathbf{g} = \mathbf{r} \oplus \mathbf{s}$ be the Levi decomposition of $\mathbf{g}$ in its radical (i.e. the maximal solvable ideal of $\mathbf{g}$) and a semi–simple sub–algebra, where the symbol $\oplus$ indicates the semidirect sum.

Theorem 2 (Agrachev, Liberzon) *If $\mathbf{s}$ is a compact Lie algebra then the system (2) is asymptotically stable for every switching function $u(.) : [0, \infty[\to U$.*

Theorem 2 contains Theorem 1 as a special case. Anyway the converse of Theorem 2 is not true in general: if $\mathbf{s}$ is non compact, the system can be stable or unstable. This case was also investigated. In particular, if $\mathbf{s}$ is non compact, but $\mathbf{g}$ has dimension at most 4 as Lie algebra, the authors were able to reduce the problem of the asymptotic stability of the system (2) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [1] for details. For this reason the bidimensional problem assumes particular interest.

More precisely we study the stability of the origin for the switched system:

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{3}$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbf{R}^2$ and $u(.) : [0, \infty[\to [0, 1]$ is an arbitrary measurable switching function.

It is well know that asymptotic stability for linear switching systems is equivalent to the existence of a common Lyapunov function. In [11] necessary and sufficient conditions were obtained for linear bidimensional systems to share a common *quadratic* Lyapunov function, but there are linear bidimensional systems for which this function may fail to be quadratic (see [7]) so that the problem of finding necessary and sufficient conditions on A and B, for the asymptotic stability of the system 3, was open in general.

Our result is obtained with a direct method without looking for a common Lyapunov function, but analyzing the locus in which the two vector fields are collinear, similarly to what people do in optimal synthesis problems on the plane (see [4, 10]).

Three cases are analyzed separately. In the first case both matrices have complex eigenvalues (in the following **(CC)** case). In the second case one of the two matrices has real and the other has complex eigenvalues (in the following **(RC)** case). In the third case both the matrices have real eigenvalues (in the following **(RR)** case). In the **(RR)** case two examples are also given.

There are only three relevant parameters: one depends on the eigenvalues of A, one on the eigenvalues of B, and the last contains the interrelation among the two systems and it is the cross ratio of the four eigenvectors of A and B in the projective line $\mathbf{C}P^1$.

We find the necessary and sufficient conditions for the asymptotic stability. The result can be obtained quite easily except in one case in which the integration of the vector fields has to be done.

In Section 2 we give the basic definitions, we study the properties of the parameters describing the problem and we state the stability theorem giving the main ideas of the proof. In Section 3 we give two examples for the **(RR)** case. In Section 4 we make some final remarks.

2 Basic Definition and Statement of the Main Results

Let A and B be two diagonalizable 2×2 real matrices with eigenvalues having strictly negative real part. Consider the following property:

($\mathcal{P}$) The dynamical system in $\mathbf{R}^2$: $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$ is asymptotically stable at the origin for each measurable function $u(.) : [0, \infty[\to [0, 1]$.

In this Section we state the necessary and sufficient conditions on A and B under which ($\mathcal{P}$) holds. Moreover we state under which conditions we have at least stability (not asymptotic) for each function $u(.)$.

Set $M(u) := uA + (1 - u)B$, $u \in [0, 1]$. In the class of constant functions the asymptotic stability of the origin of the system 3 occurs iff the matrix $M(u)$ has eigenvalues with strictly negative real part for each $u \in [0, 1]$. So this is a necessary condition. On the other hand it is know that if $[A, B] = 0$ then the system (3) is asymptotically stable for each function $u(.)$. So in the following we will always assume the conditions

H1: Let λ_1, λ_2 (resp. λ_3, λ_4) be the eigenvalues of A (resp. B). Then $Re(\lambda_1)$, $Re(\lambda_2)$, $Re(\lambda_3)$, $Re(\lambda_4) < 0$.

H2: $[A, B] \neq 0$ (that implies that neither A nor B are proportional to the identity).

For simplicity we will also assume:

H3: A and B are diagonalizable (notice that if **H2** and **H3** hold then $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$).

H4: Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbf{C}P^1$ (resp. $\mathbf{V}_3, \mathbf{V}_4 \in \mathbf{C}P^1$) be the eigenvectors of A (resp. B). From **H2** and **H3** we know that they are uniquely defined and $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$. We assume $\mathbf{V}_i \neq \mathbf{V}_j$ for $i \in \{1, 2\}$, $j \in \{3, 4\}$.

The degenerate cases, in which **H1** and **H2** hold, and **H3** or **H4** or both does not, are the following:

- A or B are not diagonalizable. This case (in which ($\mathcal{P}$) can be true or false) can be treated with techniques entirely similar to the ones of this paper.
- A or B are diagonalizable, but one eigenvector of A coincides with one eigenvector of B. In this case, using similar arguments, it is possible to conclude that ($\mathcal{P}$) is true.

Remark 1. One can easily prove that (under the hypothesis **H2** and **H3**), **H4** can be violated only in the **(RR)** case. Moreover hypothesis **H2**, **H3**, **H4** imply that $\mathbf{V}_i \neq \mathbf{V}_j$ for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. This fact permits to define the cross ratio without additional hypothesis (see the definition of cross ratio below).

Theorems 3 below gives necessary and sufficient conditions for the stability of the system (3) in terms of the three (coordinates invariant) parameters defined in Definition 1. Proposition 1 gives some properties of these parameters. Finally Proposition 2 shows the geometric meaning of the parameter containing the interrelation between the two systems.

Definition 1 *Let A and B be two 2×2 real matrices and suppose that* **H1, H2, H3, H4** *hold. Moreover choose the labels $(_1)$ and $(_2)$ (resp. $(_3)$ and $(_4)$) in such a way $|\lambda_2| > |\lambda_1|$ (resp. $|\lambda_3| > |\lambda_4|$) if they are real or $Im(\lambda_2) < 0$ (resp. $Im(\lambda_4) < 0$) if they are complex. Define:*

$$\rho_A := -i\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}; \qquad \rho_B := -i\frac{\lambda_3 + \lambda_4}{\lambda_3 - \lambda_4};$$
$$\mathcal{K} := 2\frac{Tr(AB) - \frac{1}{2}Tr(A)Tr(B)}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}.$$

Moreover define the following function of $\rho_A, \rho_B, \mathcal{K}$:

$$\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1 + \rho_A^2 + \rho_B^2);$$

Notice that $\rho_A \in \mathbf{R}$, $\rho_A > 0$ iff A has complex eigenvalues and $\rho_A \in i\mathbf{R}$, $\rho_A/i > 1$ iff A has real eigenvalues. The same holds for B. Moreover $\mathcal{D} \in \mathbf{R}$. The parameter $\mathcal{K}$ contains important information about the matrices A and B. They are stated in the following Proposition that can be easily proved using a suitable system of coordinates (see also [3]):

Proposition 1 *Let A and B be as in Definition 1. We have the following:*

- *if A and B have both complex eigenvalues then $\mathcal{K} \in \mathbf{R}$ and $|\mathcal{K}| > 1$;*
- *if A and B have both real eigenvalues then $\mathcal{K} \in \mathbf{R} \setminus \{\pm 1\}$;*
- *A and B have one complex and the other real eigenvalues iff $\mathcal{K} \in i\mathbf{R}$;*

Theorem 3 *Let A and B be two real matrices such that* **H1, H2 H3, H4** *hold and define $\rho_A, \rho_B, \mathcal{K}, \mathcal{D}$ as in Definition 1. We have the following stability conditions:*

Case (CC) *If A and B have both complex eigenvalues then:*

Case (CC.1) *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true;*

Case (CC.2) *if $\mathcal{D} > 0$ then:*

Case (CC.2.1) *if $\mathcal{K} < -1$ then $(\mathcal{P})$ is false;*

Case (CC.2.2) *if $\mathcal{K} > 1$ then then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\rho_{CC} := \exp\left[-\rho_A \arctan\left(\frac{-\rho_A\mathcal{K} + \rho_B}{\sqrt{\mathcal{D}}}\right) - \rho_B \arctan\left(\frac{\rho_A - \rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) - \frac{\pi}{2}(\rho_A + \rho_B)\right] \times \sqrt{\frac{(\rho_A\rho_B + \mathcal{K}) + \sqrt{\mathcal{D}}}{(\rho_A\rho_B + \mathcal{K}) - \sqrt{\mathcal{D}}}} < 1 \qquad (4)$$

Case (CC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ is true or false according respectively to the fact that $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC) *If A and B have one complex and the other real eigenvalues, define $\chi := \rho_A\mathcal{K} - \rho_B$ where ρ_A and ρ_B are chosen in such a way $\rho_A \in i\mathbf{R}$, $\rho_B \in \mathbf{R}$. Then:*

Case (RC.1) *if $\mathcal{D} > 0$ then $(\mathcal{P})$ is true;*

Case (RC.2) *if $\mathcal{D} < 0$ then $\chi \neq 0$ and we have:*

Case (RC.2.1) *if $\chi > 0$ then (P) is false. Moreover in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2) *if $\chi < 0$, then:*

Case (RC2.2.A) *if $\mathcal{K}/i \leq 0$ then $(\mathcal{P})$ is true;*

Case (RC2.2.B) *if $\mathcal{K}/i > 0$ then $(\mathcal{P})$ is true iff it holds the following condition:*

$$\rho_{RC} := e^{-\rho_B(\xi^+ - \xi^-)}\sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}} \times \sqrt{\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i+1)}\cos^2\theta^+ + \left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i-1)}\sin^2\theta^+} < 1$$

where:
$$E := \mathcal{K}/i + \sqrt{-\mathcal{K}^2 + 1},$$
$$m^{\pm} := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\mathcal{K}/i},$$
$$\theta^+ := \arctan m^+,$$
$$\xi^{\pm} := \arctan\left(\frac{m^{\pm} - 1}{E(m^{\pm} + 1)}\right),$$
$$\xi^+ \in]\xi^-, \xi^- + \pi[.$$

Case (RC.3) *If $\mathcal{D} = 0$ then $(\mathcal{P})$ is true or false according respectively to the fact that $\chi < 0$ or $\chi > 0$.*

Case (RR) *If A and B have both real eigenvalues then:*

Case (RR.1) *if $\mathcal{D} < 0$ then $(\mathcal{P})$ is true. Moreover we have $|\mathcal{K}| > 1$;*

Case (RR.2) *if $\mathcal{D} > 0$ then $\mathcal{K} \neq -\rho_A\rho_B$ (notice that $-\rho_A\rho_B > 1$) and :*

Case (RR.2.1) *if $\mathcal{K} > -\rho_A\rho_B$ then (P) is false*

Case (RR.2.2) *if $\mathcal{K} < -\rho_A\rho_B$ then:*

Case (RR.2.2.A) *if $\mathcal{K} > -1$ then (P) is true;*

Case (RR.2.2.B) *if $\mathcal{K} < -1$ then (P) is true iff the following condition holds:*

$$\rho_{RR} := -f^{sym}(\rho_A, \rho_B, \mathcal{K})f^{asym}(\rho_A, \rho_B, \mathcal{K}) \times f^{asym}(\rho_B, \rho_A, \mathcal{K}) < 1, \qquad (5)$$

where:

$$f^{sym}(\rho_A,\rho_B,\mathcal{K}) := \frac{1+\rho_A/i+\rho_B/i+\mathcal{K}-\sqrt{\mathcal{D}}}{1+\rho_A/i+\rho_B/i+\mathcal{K}+\sqrt{\mathcal{D}}};$$

$$f^{aym}(\rho_A,\rho_B,\mathcal{K}) := \left(\frac{\rho_B/i-\mathcal{K}\rho_A/i-\sqrt{\mathcal{D}}}{\rho_B/i-\mathcal{K}\rho_A/i+\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}(\rho_A/i-1)}.$$

Case (RR.3) *If $\mathcal{D}=0$ then $(\mathcal{P})$ is true or false according respectively to the fact that $\mathcal{K} < -\rho_A\rho_B$ or $\mathcal{K} > -\rho_A\rho_B$.*

Finally if $(\mathcal{P})$ is false then in case **CC.2.2** *with $\rho_{CC}=1$, case* **(RC.2.2.B)***, with $\rho_{RC}=1$, case* **(RR.2.2.B)***, with $\rho_{RR}=1$, case* **(CC.3)** *with $\mathcal{K}<-1$, case* **(RC.3)** *with $\chi>0$ and case* **(RR.3)** *with $\mathcal{K} > -\rho_A\rho_B$, for every $C>0$, there exists $C'\leq C$ such that if $|\gamma(0)|<C'$ then $|\gamma(t)|<C$ for every $t\in[0,\infty[$ (i.e. we have stability of the origin). In the other cases there exists a trajectory $\gamma(t)$ such that $\lim_{t\to\infty}|\gamma(t)|=\infty$.*

Notice that the expressions (4) (5) are invariant if we exchange ρ_A with ρ_B. The last statement say when we have at least stability (not asymptotic) for every switching function.

Let us study the geometric meaning of $\mathcal{K}$. Let $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4$ belong to the complex projective line $\mathbf{C}P^1$. Suppose $\mathbf{V}_1\neq\mathbf{V}_2\neq\mathbf{V}_3$ and let (v_1,v_1'), (v_2,v_2'), (v_3,v_3'), (v_4,v_4') be the corresponding homogeneous coordinates. The *cross ratio* $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ is defined in the following way. Make a Moebious transformation such that $\mathbf{V}_1,\mathbf{V}_2$ became the fundamental points (i.e. of homogeneous coordinates respectively $(0,1)$ and $(1,0)$) and $\mathbf{V}_3$ the unity point (i.e. of homogeneous coordinates $(1,1)$) and let $(\bar{v}_4,\bar{v}_4')$ be the new homogeneous coordinates of $\mathbf{V}_4$. By definition we have $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4) := \bar{v}_4'/\bar{v}_4$. Now the four eigenvectors of A and B are exactly four direction in $\mathbf{C}^2$ i.e. they can be regarded as four points of $\mathbf{C}P^1$ and under the conditions **H2**, **H3**, **H4**, it make sense to compute their cross ratio (cfr. Remark 1). One can immediately obtain:

Proposition 2 *Let A and B be two 2×2 real matrices such that* **H1**, **H2 H3**, **H4** *hold, and $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4$ be the four point in the space $\mathbf{C}P^1$ corresponding respectively to the four eigenvectors of A and B chosen in such a way they correspond respectively to $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ (see Definition 1). Let $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ be their cross ratio and $\mathcal{K}$ the quantity defined in Definition 1. Then $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ and $\mathcal{K}$ are in the one to one relation from $\mathbf{C}\cup\{\infty\}$ to $\mathbf{C}\cup\{\infty\}$:*

$$\mathcal{K}=\frac{\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)+1}{\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)-1},\quad \beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)=\frac{\mathcal{K}+1}{\mathcal{K}-1}.$$

Notice that $\mathcal{K}\neq\infty$ so that $\beta\neq1$. From Proposition 2 and Definition 1 we have the following expression for the cross ratio of the eigenvectors of A and B:

$$\beta=\frac{Tr(AB)-(\lambda_1\lambda_4+\lambda_2\lambda_3)}{Tr(AB)-(\lambda_1\lambda_3+\lambda_2\lambda_4)}$$

In the following we we describe the main idea of the proof of Theorem 3. All the details can be found in [5].

We build the "worst trajectory" i.e. the trajectory that at each point has the velocity forming the angle, with the (exiting) radial direction, having the smallest absolute value, without taking care of the module of the velocity (figure 1 A). The main idea is that the system (3) is asymptotically stable iff this trajectory tends to the origin. The worst trajectory is constructed in the following way. We study the locus $Q^{-1}(0)$ (where $Q=det(Ax,Bx)$) in which the two vector fields Ax and Bx are collinear. The quantity $\mathcal{D}$, defined in Definition 1, is proportional to the discriminant of the quadratic form Q. We have several cases:

- If $Q^{-1}(0)$ contains only the origin then, in the **(CC)** and **(RC)** case, one vector field points always on the same side of the other and the worst trajectory is a trajectory of the vector field Ax or Bx. In this case the system is asymptotically stable (case **(CC.1)** and **(RC.1)** of Theorem 3). See figure 1, case B. The situation is similar in case **(RR.1)** (the worst trajectory tends to the origin).
- If $Q^{-1}(0)$ does not contain only the origin then it is a couple of straight lines passing from the origin (since Q is a quadratic form). If at each point of $Q^{-1}(0)$ the two vector field have opposite versus then there exists a trajectory going to infinity corresponding to a constant switching function (see figure 1, case C). This correspond to cases **(CC.2.1)**, **(RC.2.1)** and **(RR.2.1)** of of Theorem 3 and it is the situation in which there exists $u\in[0,1]$ such that the matrix $M(u)$ admits an eigenvalue with positive real part. If at each point of $Q^{-1}(0)$, the two vector fields have the same versus then the system is asymptotically stable iff the worst trajectory turns around the origin and after one turn the distance from the origin is increasing (see figure 1, cases D and E). The quantities $\rho_{CC},\rho_{RC},\rho_{RR}$ defined in Theorem 3 (for the three cases **(CC)**, **(RC)**, **(RR)** resp.) represent the distance from the origin of the worst trajectory (that at time zero is at distance 1), after half turn). This correspond to cases **(CC.2.2)**, **(RC.2.2)** and **(RR.2.2)** of Theorem 3.
- Finally **(CC.3)**, **(RC.3)** and **(RR.3)** are the degenerate cases in which the two straight lines coincide.

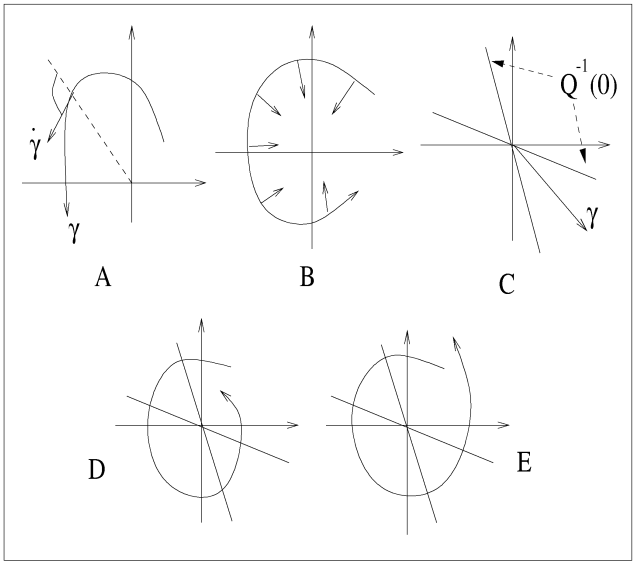

Figure 1

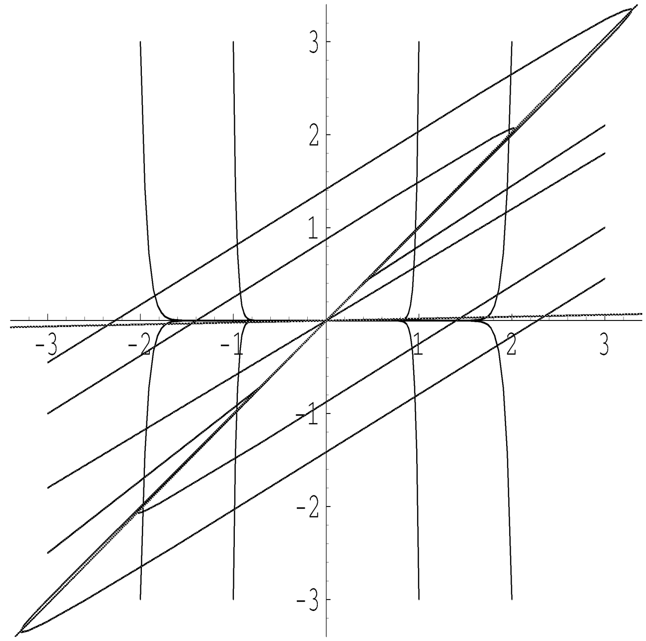

Figure 2

3 Two Example in the (RR) Case

In this section, we show two examples of switching system in which both matrices have real eigenvalues, that are not stable for arbitrary switching functions.

Example 1 (Dynamic Instability)
First we choose a situation in which a trajectory going to infinity, corresponding to a constant switching function, does not exist (Dynamic Instability). In other words the matrix $M(u)$ has negative eigenvalues for every $u \in [0,1]$. Consider the following matrices:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -30 \end{pmatrix}$$
$$B = \begin{pmatrix} -67 & 66 \\ -42 & 41 \end{pmatrix}$$

The eigenvalues of B are -1 and -25. For these two matrices: $\rho_A/i \sim 1.069$, $\rho_B/i \sim 1.083$, $\mathcal{K} = -4.5$, $\mathcal{D} \sim 31.99$, $\rho_{RR} \sim 1.22$. Since $\mathcal{D} > 0$, $\mathcal{K} < -\rho_A\rho_B \sim 1.16$, $\mathcal{K} < -1$, we are in the situation **(RR.2.2.B)**. Since $\rho_{RR} > 1$, the system is not stable for arbitrary switching functions.

In Figure 2, some integral curves of the vector fields A, B, and the locus $\mathcal{D}$ (that are the two straight lines) are plotted. In Figure 3, a "worst" trajectory is plotted. Notice that $\gamma(0) \sim (-2,0)$ and after one turn it reaches the point $P \sim (-2.9, 0)$. It follows $|P|/|\gamma(0)| \sim 1.45 \sim \rho_{RR}^2$. We recall that ρ_{RR} is the distance from the origin of the points reached by the worst trajectory satisfying $|\gamma(0)| = 1$, after half turn.

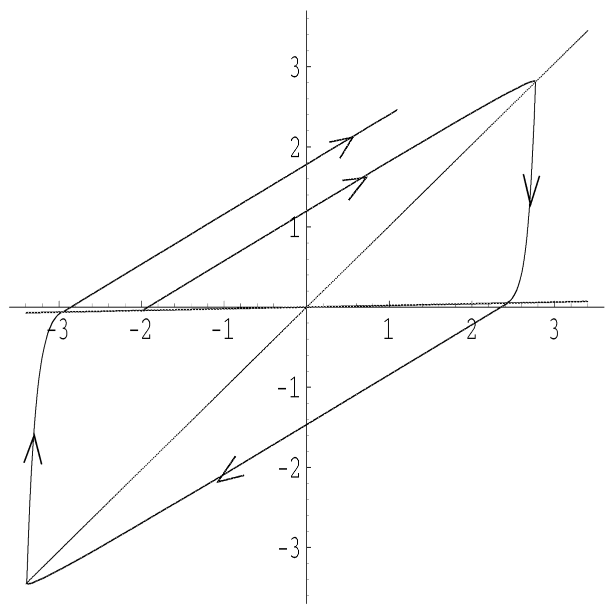

Figure 3

Example 2 (Static Instability)
Consider the following matrices:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -30 \end{pmatrix}$$

$$B = \begin{pmatrix} 41 & -42 \\ 66 & -67 \end{pmatrix}$$

As in Example 1, the eigenvalues of B are -1 and -25, $\rho_A/i \sim 1.069$, $\rho_B/i \sim 1.083$. But now the sign of $\mathcal{K}$ is opposite:

$$\mathcal{K} = +4.5.$$

In this case $\mathcal{D} \sim 11.14$. Since $\mathcal{D} > 0$ and $\mathcal{K} > -\rho_A \rho_B \sim 1.16$, we are in the situation **(RR.2.1)** and the system is not stable. Notice that, in this case, there exists a trajectory going to infinity, corresponding to a constant switching function (Static Instability).

4 Final Remarks

Using the results of [1, 6, 8] and by Theorem 3 we have a complete algorithm to study the asymptotic stability of a switched linear system in any dimension at least in the case in which:

$$A_u = uA^1x + (1-u)A^2x, \quad u \in [0,1], \ A^1, A^2 \in \mathbf{R}^{n\times n},$$

where A and B are diagonalizable and $\dim\{A_1, A_2\}_{L.A.} \leq 4$. The case in which A or B are not diagonalizable can be treated with similar techniques.

Generalization can be done for more complex sets U. One is the following m–input system:

$$\dot{x} = \sum_{i=1}^{m} u_i A^i x, \quad \sum_{i=1}^{m} u_i = 1, \quad u_i \geq 0 \ \ (i = 1, ..., m),$$
$$x \in \mathbf{R}^n, \quad A^1, ..., A^m \in \mathbf{R}^{n\times n}.$$

With exactly the same techniques used in this paper, one can find a coordinates invariant necessary and sufficient condition for the stabilizability of a control system of the kind 2 where all the matrices have eigenvalues with strictly *positive* real part. This problem was also studied in in [12], but not in terms of a minimum number of coordinates–free parameters. We refer to [12] for details.

Some results can be obtained for the non-linear version of the problem treated in this paper:

$$\dot{x} = uF(x) + (1-u)G(x), \qquad (6)$$

where $x \in \mathbf{R}^2$, $F(.)$, $G(.)$ are $\mathcal{C}^\infty$ generic functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $F(0) = 0$, $G(0) = 0$ and the two dynamical systems $\dot{x} = F(x)$, $\dot{x} = G(x)$ are globally asymptotically stable at the origin. We are interested to study under which conditions on $F(.)$ and $G(.)$ the origin of the system 6 is globally asymptotically stable for every measurable function $u(.) : [0, \infty[\to [0,1]$.

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