SIAM J. CONTROL OPTIM.
Vol. 41, No. 1, pp. 89–112

STABILITY OF PLANAR SWITCHED SYSTEMS: THE LINEAR SINGLE INPUT CASE*

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Abstract. We study the stability of the origin for the dynamical system $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$, where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbf{R}^2$, and $u(.) : [0, \infty[\to [0, 1]$ is a completely random measurable function. More precisely, we find a (coordinates invariant) necessary and sufficient condition on A and B for the origin to be asymptotically stable for each function $u(.)$.

The result is obtained without looking for a common Lyapunov function but studying the locus in which the two vector fields Ax and Bx are collinear. There are only three relevant parameters: the first depends only on the eigenvalues of A, the second depends only on the eigenvalues of B, and the third contains the interrelation among the two systems, and it is the cross ratio of the four eigenvectors of A and B in the projective line $\mathbf{C}P^1$. In the space of these parameters, the shape and the convexity of the region in which there is stability are studied.

This bidimensional problem assumes particular interest since linear systems of higher dimensions can be reduced to our situation.

Key words. stability, planar, random switching function, switched systems

AMS subject classifications. 93D20, 37N35

PII. S0363012900382837

1. Introduction.

By a switched system we mean a family of continuous-time dynamical systems and a rule that determines at any time which dynamical system is responsible for the time evolution. More precisely, let $\{f_u : \; u \in U\}$ be a (finite or infinite) set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{1}$$

The rule is given assigning the so-called switching function $u(.) : [0, \infty[\to U$. Here we consider the situation in which the switching function cannot be predicted a priori; it is given from outside and represents some phenomena (e.g., a disturbance) that it is not possible to control or include in the dynamical system model.

In the following, we use the notation $u \in U$ to label a fixed individual system and $u(.)$ to indicate the switching function.

Suppose now that all of the f_u have a given property for every $u \in U$. A typical problem is to study under which conditions this property holds for the system (1) for arbitrary switching functions. For a discussion of various issues related to switched systems, we refer the reader to [8].

In [1, 7] the case of switched linear systems was considered:

$$\dot{x} = A_u x, \quad x \in \mathbf{R}^n, \quad A_u \in \mathbf{R}^{n \times n}, \quad u \in U, \tag{2}$$

and the problem of the asymptotic stability of the origin for arbitrary switching functions was investigated. Clearly we need the asymptotic stability of each single

*Received by the editors December 22, 2000; accepted for publication (in revised form) October 17, 2001; published electronically April 2, 2002. This work was supported by a TMR fellowship (Non Linear Control Network), contract FMRX-CT97-0137 (DG 12-BDCN) (CNRS CON00P140DR04).
http://www.siam.org/journals/sicon/41-1/38283.html

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subsystem $\dot{x} = A_u x$, $u \in U$, in order to have the asymptotic stability of (2) for each switching function (i.e., the eigenvalues of each matrix A_u must have strictly negative real part). This will be assumed to be the case throughout the paper.

Notice the important point that in the case of linear systems, the asymptotic stability for arbitrary switching functions is equivalent to the more often quoted property of global exponential stability, uniform with respect to switching (GUES); see, for example, [2] and references therein.

In [1, 7], it is shown that the structure of the Lie algebra generated by the matrices A_u,

$$\mathbf{g} = \{A_u : \ u \in U\}_{L.A.},$$

is crucial for the stability of the system (2) (i.e., the interrelation among the systems). The main result of [7] is the following theorem.

THEOREM 1.1 (Hespanha, Morse, Liberzon). *If* $\mathbf{g}$ *is a solvable Lie algebra, then the switched system* (2) *is asymptotically stable for each switching function* $u(.) : [0, \infty[\rightarrow U$.

In [1] a generalization was given. Let $\mathbf{g} = \mathbf{r} \ni \mathbf{s}$ be the Levi decomposition of $\mathbf{g}$ in its radical (i.e., the maximal solvable ideal of $\mathbf{g}$) and a semisimple subalgebra, where the symbol $\ni$ indicates the semidirect sum.

THEOREM 1.2 (Agrachev, Liberzon). *If* $\mathbf{s}$ *is a compact Lie algebra, then the system* (2) *is asymptotically stable for every switching function* $u(.) : [0, \infty[\rightarrow U$.

Theorem 1.2 contains Theorem 1.1 as a special case. Anyway, the converse of Theorem 1.2 is not true in general: if $\mathbf{s}$ is noncompact, the system can be stable or unstable. This case was also investigated. In particular, if $\mathbf{s}$ is noncompact, then it contains as a subalgebra $sl(2, \mathbf{R})$. Due to that, in the case in which $\mathbf{g}$ has dimension at most 4 as Lie algebra, the authors were able to reduce the problem of the asymptotic stability of the system (2) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [1] for details. For this reason, the bidimensional problem assumes particular interest, and in this paper we give the complete description of that case for a single input system.

More precisely, we study the stability of the origin for the switched system

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{3}$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part, $x \in \mathbf{R}^2$, and $u(.) : [0, \infty[\rightarrow [0, 1]$ is an arbitrary measurable switching function.

It is well known that asymptotic stability for linear switching systems is equivalent to the existence of a common Lyapunov function. In [11] necessary and sufficient conditions were obtained for linear bidimensional systems to share a common *quadratic* Lyapunov function, but there are linear bidimensional systems for which this function may fail to be quadratic (see [6]) so that the problem of finding necessary and sufficient conditions on A and B for the asymptotic stability of the system (3) was open in general.

In this paper, we give the solution to this problem. Our result is obtained with a direct method without looking for a common Lyapunov function but analyzing the locus in which the two vector fields are collinear, to build the “worst trajectory,” similarly to what people do in optimal synthesis problems on the plane (see [4, 5, 9, 10]). We also use the concept of feedback. The idea of building the worst trajectory was used also in [6] for analyzing an example.

Three cases are analyzed separately. In the first case, both matrices have complex eigenvalues (in the following **(CC)** case). In the second case, one of the two matrices has real and the other has complex eigenvalues (in the following **(RC)** case). In the third case, both the matrices have real eigenvalues (in the following **(RR)** case).

There are only three relevant parameters: one depends on the eigenvalues of A, one on the eigenvalues of B (we call them, respectively, ρ_A and ρ_B), and the last contains the interrelation among the two systems, and it is the cross ratio of the four eigenvectors of A and B in the projective line $\mathbf{C}P^1$.

The result can be obtained quite easily except in one case in which the integration of the vector fields has to be done. In this case, the computations are not difficult but long, and they are collected in Appendices A and B. In the **(CC)** and **(RR)** cases, we are even able to write the final result in a relatively compact way (see formulas (5) and (7)).

Fixing the value of the cross ratio, we study the region $\mathcal{R}$ in which the system is asymptotically stable for arbitrary switching functions in the space of the parameters ρ_A and ρ_B. In the **(CC)** and **(RR)** cases it is constituted by one or two open unbounded convex regions, while in the **(RC)** case it is an open unbounded region but not always convex.

In section 2 we give the basic definitions, we study the properties of the parameters describing the problem, and we state the stability theorem giving the main ideas of the proof. In section 3 we prove the stability theorem separately for the three cases **(CC)**, **(RC)**, **(RR)**, and we give some examples. In section 4 we study the shape and the convexity of the region $\mathcal{R}$ for fixed values of the cross ratio. In section 5 we make some final remarks.

2. Basic definitions and statement of the main results. Let A and B be two diagonalizable 2×2 real matrices with eigenvalues having strictly negative real part. Consider the following property:

($\mathcal{P}$) The dynamical system in $\mathbf{R}^2$: $\dot{x}(t) = u(t)Ax(t) + (1-u(t))Bx(t)$ is asymptotically stable at the origin for each measurable function $u(.) : [0,\infty[\to [0,1]$.

In this section we state the necessary and sufficient conditions on A and B under which ($\mathcal{P}$) holds. Moreover, we state under which conditions we have at least stability (not asymptotic) for each function $u(.)$.

Set $M(u) := uA + (1-u)B$, $u \in [0,1]$. In the class of constant functions the asymptotic stability of the origin of the system (3) occurs iff the matrix $M(u)$ has eigenvalues with strictly negative real part for each $u \in [0,1]$. So this is a necessary condition. On the other hand, it is known that if $[A,B] = 0$, then the system (3) is asymptotically stable for each function $u(.)$. So in the following we will always assume the following conditions:

H1. Let λ_1, λ_2 (resp., λ_3, λ_4) be the eigenvalues of A (resp., B). Then $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2)$, $\mathrm{Re}(\lambda_3)$, $\mathrm{Re}(\lambda_4) < 0$.

H2. $[A,B] \neq 0$. (That implies that neither A nor B are proportional to the identity.)

For simplicity we will also assume the following.

H3. A and B are diagonalizable. (Notice that if **H2** and **H3** hold, then $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$.)

H4. Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbf{C}P^1$ (resp., $\mathbf{V}_3, \mathbf{V}_4 \in \mathbf{C}P^1$) be the eigenvectors of A (resp., B). From **H2** and **H3** we know that they are uniquely defined, and $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$. We assume $\mathbf{V}_i \neq \mathbf{V}_j$ for $i \in \{1,2\}$, $j \in \{3,4\}$.

The degenerate cases, in which **H1** and **H2** hold and **H3** or **H4** or both do not, are the following:

- A or B is not diagonalizable. This case (in which $(\mathcal{P})$ can be true or false) can be treated with techniques entirely similar to the ones of this paper.
- A or B is diagonalizable, but one eigenvector of A coincides with one eigenvector of B. In this case, using arguments similar to the ones of the next section, it is possible to conclude that $(\mathcal{P})$ is true.

Remark 1. One can easily prove that (under the hypotheses **H2** and **H3**), **H4** can be violated only in the **(RR)** case (see also subsection 3.3). Moreover, hypotheses **H2**, **H3**, and **H4** imply that $\mathbf{V}_i \neq \mathbf{V}_j$ for $i, j \in \{1,2,3,4\}$, $i \neq j$. This fact permits us to define the cross ratio without additional hypotheses (see the definition of cross ratio below).

Theorem 2.3 gives necessary and sufficient conditions for the stability of the system (3) in terms of three (coordinates invariant) parameters defined in Definition 2.1. The first (ρ_A) depends on the eigenvalues of A, the second (ρ_B) depends on the eigenvalues of B, and the third ($\mathcal{K}$) depends on $\mathrm{Tr}(AB)$, which is a standard scalar product in the space of 2×2 matrices. Proposition 2.2 gives some properties of these parameters. Finally, Proposition 2.4 shows the geometrical meaning of $\mathcal{K}$. It is in one-to-one correspondence with the cross ratio of the four points in the projective line $\mathbf{C}P^1$ that corresponds to the four eigenvectors of A and B. This parameter contains the interrelation among the two systems.

Definition 2.1. *Let A and B be two 2×2 real matrices, and suppose that* **H1**, **H2**, **H3**, *and* **H4** *hold. Moreover, choose the labels $(_1)$ and $(_2)$ (resp., $(_3)$ and $(_4)$) in such a way that $|\lambda_2| > |\lambda_1|$ (resp., $|\lambda_3| > |\lambda_4|$) if they are real or $\mathrm{Im}(\lambda_2) < 0$ (resp., $\mathrm{Im}(\lambda_4) < 0$) if they are complex. Define*

$$\rho_A := -i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}; \qquad \rho_B := -i\frac{\lambda_3+\lambda_4}{\lambda_3-\lambda_4}; \qquad \mathcal{K} := 2\frac{\mathrm{Tr}(AB) - \frac{1}{2}\mathrm{Tr}(A)\mathrm{Tr}(B)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}.$$

Moreover, define the following function of $\rho_A, \rho_B, \mathcal{K}$:

$$\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1+\rho_A^2+\rho_B^2). \tag{4}$$

Notice that $\rho_A \in \mathbf{R}$, $\rho_A > 0$, iff A has complex eigenvalues and $\rho_A \in i\mathbf{R}$, $\rho_A/i > 1$, iff A has real eigenvalues. The same holds for B. Moreover, $\mathcal{D} \in \mathbf{R}$. The parameter $\mathcal{K}$ contains important information about the matrices A and B. They are stated in the following proposition, which can be easily proved using the systems of coordinates of the next section (see also [3]).

Proposition 2.2. *Let A and B be as in Definition* 2.1. *We have the following:*

- *if A and B have both complex eigenvalues, then $\mathcal{K} \in \mathbf{R}$ and $|\mathcal{K}| > 1$;*
- *if A and B have both real eigenvalues, then $\mathcal{K} \in \mathbf{R} \setminus \{\pm 1\}$;*
- *A and B have one complex and the other real eigenvalues iff $\mathcal{K} \in i\mathbf{R}$.*

Theorem 2.3. *Let A and B be two real matrices such that* **H1**, **H2**, **H3**, *and* **H4** *hold, and define $\rho_A, \rho_B, \mathcal{K}, \mathcal{D}$ as in Definition* 2.1. *We have the following stability conditions:*

Case (CC) *If A and B have both complex eigenvalues, then:*

Case (CC.1) *if $\mathcal{D} < 0$, then $(\mathcal{P})$ is true;*

Case (CC.2) *if $\mathcal{D} > 0$, then:*

Case (CC.2.1) *if $\mathcal{K} < -1$, then $(\mathcal{P})$ is false;*

Case (CC.2.2) *if $\mathcal{K} > 1$, then $(\mathcal{P})$ is true iff the following condition holds:*

$$\rho_{CC} := \exp\left[-\rho_A \arctan\left(\frac{-\rho_A\mathcal{K}+\rho_B}{\sqrt{\mathcal{D}}}\right) - \rho_B \arctan\left(\frac{\rho_A-\rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) - \frac{\pi}{2}(\rho_A+\rho_B)\right] \times \sqrt{\frac{(\rho_A\rho_B+\mathcal{K})+\sqrt{\mathcal{D}}}{(\rho_A\rho_B+\mathcal{K})-\sqrt{\mathcal{D}}}} < 1. \tag{5}$$

Case (CC.3) *If $\mathcal{D} = 0$, then $(\mathcal{P})$ is true or false according, respectively, to the fact that $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC) *If A and B have one complex and the other real eigenvalues, define $\chi := \rho_A\mathcal{K} - \rho_B$, where ρ_A and ρ_B are chosen in such a way that $\rho_A \in i\mathbf{R}$, $\rho_B \in \mathbf{R}$. Then:*

Case (RC.1) *if $\mathcal{D} > 0$, then $(\mathcal{P})$ is true;*

Case (RC.2) *if $\mathcal{D} < 0$, then $\chi \neq 0$, and we have:*

Case (RC.2.1) *if $\chi > 0$, then $(\mathcal{P})$ is false. Moreover, in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2) *if $\chi < 0$, then:*

Case (RC2.2.A) *if $\mathcal{K}/i \leq 0$, then $(\mathcal{P})$ is true;*

Case (RC2.2.B) *if $\mathcal{K}/i > 0$, then $(\mathcal{P})$ is true iff the following condition holds:*

$$\rho_{RC} := e^{-\rho_B(\xi^+-\xi^-)}\sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}} \times \sqrt{\left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i+1)}\cos^2\theta^+ + \left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i-1)}\sin^2\theta^+} < 1, \tag{6}$$

where:

$$E := \mathcal{K}/i + \sqrt{-\mathcal{K}^2+1},$$

$$m^\pm := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\mathcal{K}/i},$$

$$\theta^+ := \arctan m^+,$$

$$\xi^\pm := \arctan\left(\frac{m^\pm - 1}{E(m^\pm+1)}\right), \quad \xi^+ \in]\xi^-, \xi^- + \pi[.$$

Case (RC.3) *If $\mathcal{D} = 0$, then $(\mathcal{P})$ is true or false according, respectively, to the fact that $\chi < 0$ or $\chi > 0$.*

Case (RR) *If A and B have both real eigenvalues, then:*

Case (RR.1) *if $\mathcal{D} < 0$, then $(\mathcal{P})$ is true. Moreover, we have $|\mathcal{K}| > 1$;*

Case (RR.2) *if $\mathcal{D} > 0$, then $\mathcal{K} \neq -\rho_A\rho_B$ (notice that $-\rho_A\rho_B > 1$) and:*

Case (RR.2.1) *if $\mathcal{K} > -\rho_A\rho_B$, then (P) is false*

Case (RR.2.2) *if $\mathcal{K} < -\rho_A\rho_B$, then:*

Case (RR.2.2.A) *if $\mathcal{K} > -1$, then (P) is true;*

Case (RR.2.2.B) *if $\mathcal{K} < -1$, then (P) is true iff the following condition holds:*

$$\rho_{RR} := -f^{sym}(\rho_A, \rho_B, \mathcal{K}) f^{asym}(\rho_A, \rho_B, \mathcal{K}) \times f^{asym}(\rho_B, \rho_A, \mathcal{K}) < 1, \tag{7}$$

where:

$$f^{sym}(\rho_A,\rho_B,\mathcal{K}) := \frac{1+\rho_A/i+\rho_B/i+\mathcal{K}-\sqrt{\mathcal{D}}}{1+\rho_A/i+\rho_B/i+\mathcal{K}+\sqrt{\mathcal{D}}};$$
$$f^{aym}(\rho_A,\rho_B,\mathcal{K}) := \left(\frac{\rho_B/i-\mathcal{K}\rho_A/i-\sqrt{\mathcal{D}}}{\rho_B/i-\mathcal{K}\rho_A/i+\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}(\rho_A/i-1)}.$$

Case (RR.3) *If $\mathcal{D}=0$, then $(\mathcal{P})$ is true or false according, respectively, to the fact that $\mathcal{K}<-\rho_A\rho_B$ or $\mathcal{K}>-\rho_A\rho_B$.*

Finally, if $(\mathcal{P})$ is false, then in case **(CC.2.2)** *with $\rho_{CC}=1$, case* **(RC.2.2.B)** *with $\rho_{RC}=1$, case* **(RR.2.2.B)** *with $\rho_{RR}=1$, case* **(CC.3)** *with $\mathcal{K}<-1$, case* **(RC.3)** *with $\chi>0$, and case* **(RR.3)** *with $\mathcal{K}>-\rho_A\rho_B$, for every $C>0$, there exists $C'\leq C$ such that if $|\gamma(0)|<C'$, then $|\gamma(t)|<C$ for every $t\in[0,\infty[$ (i.e., we have stability of the origin). In the other cases, there exists a trajectory $\gamma(t)$ such that $\lim_{t\to\infty}|\gamma(t)|=\infty$.*

Notice that the expressions (5) and (7) are invariant if we exchange ρ_A with ρ_B. The last statement says when we have at least stability (not asymptotic) for every switching function.

Let us study the geometric meaning of $\mathcal{K}$. Let $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4$ belong to the complex projective line $\mathbf{C}P^1$. Suppose $\mathbf{V}_1\neq\mathbf{V}_2\neq\mathbf{V}_3$, and let (v_1,v_1'), (v_2,v_2'), (v_3,v_3'), (v_4,v_4') be the corresponding homogeneous coordinates. The *cross ratio* $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ is defined in the following way. Make a Moebius transformation such that $\mathbf{V}_1,\mathbf{V}_2$ become the fundamental points (i.e., of homogeneous coordinates, respectively, $(0,1)$ and $(1,0)$) and $\mathbf{V}_3$ the unity point (i.e., of homogeneous coordinates $(1,1)$), and let $(\bar v_4,\bar v_4')$ be the new homogeneous coordinates of $\mathbf{V}_4$. By definition we have

$$\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4) := \bar v_4'/\bar v_4 = \frac{\begin{vmatrix} v_1 & v_4\\ v_1' & v_4'\end{vmatrix}\begin{vmatrix} v_2 & v_3\\ v_2' & v_3'\end{vmatrix}}{\begin{vmatrix} v_2 & v_4\\ v_2' & v_4'\end{vmatrix}\begin{vmatrix} v_1 & v_3\\ v_1' & v_3'\end{vmatrix}}. \tag{8}$$

Now the four eigenvectors of A and B are exactly four directions in $\mathbf{C}^2$; i.e., they can be regarded as four points of $\mathbf{C}P^1$, and under the conditions **H2**, **H3**, **H4**, it makes sense to compute their cross ratio (cf. Remark 1).

One can immediately obtain (suggestion: use the systems of coordinates of the next section) the following proposition.

Proposition 2.4. *Let A and B be two 2×2 real matrices such that* **H1**, **H2**, **H3**, *and* **H4** *hold, and let $\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4$ be the four points in the space $\mathbf{C}P^1$ corresponding, respectively, to the four eigenvectors of A and B chosen in such a way that they correspond, respectively, to $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ (see Definition 2.1). Let $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ be their cross ratio and $\mathcal{K}$ the quantity defined in Definition 2.1. Then $\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)$ and $\mathcal{K}$ are in the one-to-one relation from $\mathbf{C}\cup\{\infty\}$ to $\mathbf{C}\cup\{\infty\}$:*

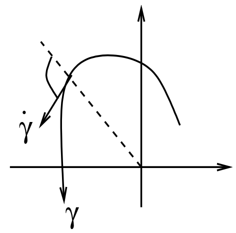
$$\mathcal{K}=\frac{\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)+1}{\beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)-1},\qquad \beta(\mathbf{V}_1,\mathbf{V}_2,\mathbf{V}_3,\mathbf{V}_4)=\frac{\mathcal{K}+1}{\mathcal{K}-1}.$$

Notice that $\mathcal{K} \neq \infty$ so that $\beta \neq 1$. From Proposition 2.4 and Definition 2.1 we have the following expression for the cross ratio of the eigenvectors of A and B:

$$\beta = \frac{\mathrm{Tr}(AB) - (\lambda_1\lambda_4 + \lambda_2\lambda_3)}{\mathrm{Tr}(AB) - (\lambda_1\lambda_3 + \lambda_2\lambda_4)}.$$

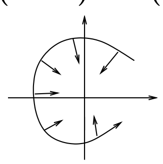
Theorem 2.3 is proved in the next section. Here we describe the main idea of the proof.

We build the "worst trajectory," i.e., the trajectory that at each point has the velocity forming the angle, with the (exiting) radial direction, having the smallest absolute value, without taking care of the module of the velocity.



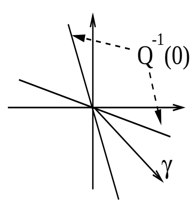
The main idea is that the system (3) is asymptotically stable iff this trajectory tends to the origin. The worst trajectory is constructed in the following way. We study the locus $Q^{-1}(0)$ (the notation is clarified later) in which the two vector fields Ax and Bx are collinear. We have several cases:

- If $Q^{-1}(0)$ contains only the origin, then, in the **(CC)** and **(RC)** cases, one vector field always points on the same side of the other, and the worst trajectory is a trajectory of the vector field Ax or Bx. In this case, the system is asymptotically stable (cases **(CC.1)** and **(RC.1)** of Theorem 2.3).

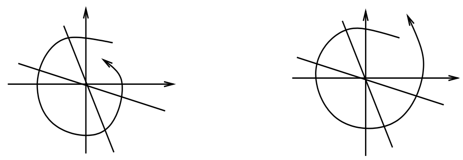


The situation is similar in case **(RR.1).** (The worst trajectory tends to the origin.)

- If $Q^{-1}(0)$ does not contain only the origin, then it is a couple of straight lines passing from the origin (see the next section). If at each point of $Q^{-1}(0)$ the two vector fields have opposite versus, then there exists a trajectory going to infinity corresponding to a constant switching function (see the following figure).



This corresponds to cases **(CC.2.1)**, **(RC.2.1)**, and **(RR.2.1)** of Theorem 2.3, and it is the situation in which there exists $u \in [0, 1]$ such that the matrix $M(u)$ admits an eigenvalue with positive real part. If at each point of Q the two vector fields have the same versus, then the system is asymptotically stable iff the worst trajectory turns around the origin and after one turn the distance from the origin is increasing.



This corresponds to cases **(CC.2.2)**, **(RC.2.2)**, and **(RR.2.2)** of Theorem 2.3.

- Finally, **(CC.3)**, **(RC.3)**, and **(RR.3)** are the degenerate cases in which the two straight lines coincide.

More details are given later.

3. Proof of the stability theorem. In the following, we prove Theorem 2.3 separately for the three cases in which A and B have both complex, one complex and the other real, and both real eigenvalues.

3.1. The case in which A and B have both complex eigenvalues. Let $-\delta_A \pm i\omega_A$ ($\delta_A, \omega_A > 0$) be the eigenvalues of A and $-\delta_B \pm i\omega_B$ ($\delta_B, \omega_B > 0$) be the eigenvalues of B. We have $\rho_A = \delta_A/\omega_A$, $\rho_B = \delta_B/\omega_B$. Choose a system of coordinates in which

$$A = \begin{pmatrix} -\delta_A & -\omega_A/E \\ \omega_A E & -\delta_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\delta_B & -\omega_B \\ \omega_B & -\delta_B \end{pmatrix},$$

where $E \in \mathbf{R}\backslash\{0\}$. This corresponds to put B in the normal form in which its integral curves are "circular spirals" and then, using the invariance of B under rotation, to rotate the coordinates in such a way that the integral curves of A are elliptical spirals with axes corresponding to the x_1 and x_2 directions (see, for example, Figure 3.1). We have

$$[A, B] = \omega_A\omega_B(E - 1/E)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so we assume $E \neq \pm 1$; otherwise, $[A, B] = 0$. In this case we have $\mathcal{K} = \frac{1}{2}(E + \frac{1}{E})$, and without loss of generality we may assume $|E| > 1$.

The locus in which Ax and Bx are collinear is $Q^{-1}(0)$, where

$$\begin{aligned} Q = \det(Ax, Bx) &= x_1^2(-\delta_A\omega_B + \delta_B\omega_A E) \\ &\quad + x_1x_2(\omega_A\omega_B(E - 1/E)) + x_2^2(-\delta_A\omega_B + \delta_B\omega_A/E) \end{aligned}$$

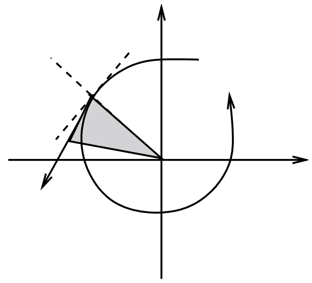
and $x = (x_1, x_2)$. Now let D_{CC} be the discriminant of the quadratic form Q. We have

$$\begin{aligned} D_{CC} &= \omega_A^2\omega_B^2(E - 1/E)^2 - 4(-\delta_A\omega_B + \delta_B\omega_A E)(-\delta_A\omega_B + \delta_B\omega_A/E) \\ &= 4\omega_A^2\omega_B^2\mathcal{D}, \end{aligned} \tag{9}$$

where $\mathcal{D}$ is defined in Definition 2.1.

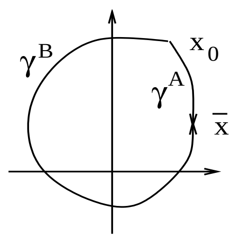
Case 1. If $\mathcal{D} < 0$, then the quadratic form Q has strictly defined sign and $Q^{-1}(0) = \{0\}$. In this case, one vector field always points on the same side of the other. Making a suitable change of coordinates and possibly exchanging the labels $(_A)$ and $(_B)$, we can realize the situation in which Ax always points on the left of Bx for every $x \in \mathbf{R}^2 \setminus \{0\}$. We have two cases.

- Suppose first that $E > 1$. In this case, Ax always points in the grey region of the following picture.



Fix an arbitrary measurable switching function $u(.) : [0,\infty[\to [0,1]$, and let $(x_1(t), x_2(t))$ (resp., $(\rho(t), \theta(t))$) be the Cartesian (resp., polar) coordinates of the solution of $\dot{x}(t) = u(t)Ax(t) + (1-u(t))Bx(t)$, $x(0) = x_0 \in \mathbf{R}^2 \backslash \{0\}$. In this case, we have $\dot{\rho}(t) < 0$ for almost every $t \in [0, +\infty[$ and $(\mathcal{P})$ is true.

- Suppose now that $E < -1$. Fix $x_0 \in \mathbf{R}^2 \setminus \{0\}$, and let γ be a trajectory of the switched system (3) such that $\gamma(0) = x_0$. Let $\gamma^A : [0, t_A] \to \mathbf{R}^2$ (resp., $\gamma^B : [0, t_B] \to \mathbf{R}^2$) be a trajectory of the vector field Ax (resp., Bx) such that $\gamma^A(0) = x_0$ (resp., $\gamma^B(0) = x_0$), and define t_A and t_B in such a way that $\gamma^A(t_A) = \gamma^B(t_B) =: \bar{x}$ is the first intersection point of γ^A and γ^B after x_0.



Let Ω be the simply connected closed set whose border is

$$\partial\Omega = \mathrm{Supp}(\gamma^A|_{[0,t_A]} \cup \gamma^B|_{[0,t_B]}).$$

For every $x \in \partial\Omega$ we have the following. Define $V_u = uAx + (1-u)Bx$. For each $u \in]0,1[$, V_u points inside Ω. Moreover, if $x \notin \{x_0, \bar{x}\}$, V_1 (resp., V_0) points inside Ω or it is tangent to $\partial\Omega$. Fix $\bar{t} > \max\{t_A, t_B\}$. We clearly have $\bar{x} := \gamma(\bar{t}) \in \mathrm{int}(\Omega)$. Using homothety invariance of the system (3), we may easily conclude that $\lim_{t\to\infty} \gamma(t) = 0$ and $(\mathcal{P})$ is true. This proves case **(CC.1)** of Theorem 2.3 (see Example 1 below).

Case 2. If $\mathcal{D} > 0$, then Q has no definite sign and $Q^{-1}(0)$ is a couple of noncoinciding straight lines passing from the origin and forming the following angles with the x_1 axis:

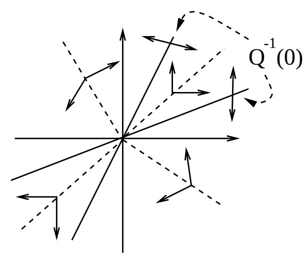
$$\theta^{\pm} = \arctan(m^{\pm}), \text{ where} \tag{10}$$

$$m^{\pm} = \frac{-\omega_A\omega_B(E - 1/E) \pm \sqrt{D_{CC}}}{2(-\delta_A\omega_B + \delta_B\omega_A/E)} = \frac{-(E-1/E) \pm 2\sqrt{\mathcal{D}}}{2(-\rho_A + \rho_B/E)} \text{ if } -\rho_A + \rho_B/E \neq 0, \tag{11}$$

$$m^- = \infty, \quad m^+ = \frac{\delta_A\omega_B - \delta_B\omega_A E}{\omega_A\omega_B(E - 1/E)} = \frac{\rho_A - \rho_B E}{E - 1/E} \text{ if } -\rho_A + \rho_B/E = 0, \tag{12}$$

where we assume that $\theta^- \in [-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\theta^+ \in]\theta^-, \theta^- + \pi[$. Notice that if $E < -1$, then $4(-\delta_A\omega_B + \delta_B\omega_A E)(-\delta_A\omega_B + \delta_B\omega_A/E) > 0$, which implies that $\theta^\pm \in [-\frac{\pi}{2}, 0[$.

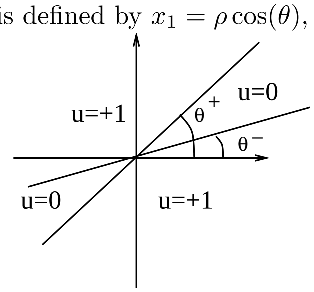
Case 2.1. If $E < -1$ ($\mathcal{K} < -1$), then at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have opposite versus. Consider the four connected components of $\mathbf{R}^2 \setminus Q^{-1}(0)$. In this case, for each point x_0 belonging to two of these regions (see the figure below), it is possible to find $u_0 \in [0,1]$ such that $u_0 A x_0 + (1-u_0) B x_0$ has the exiting radial direction. So the system is not stable for arbitrary switching functions. This situation corresponds to the case in which there exists $u \in [0,1]$ such that $M(u) := uA + (1-u)B$ admits an eigenvalue with positive real part; i.e., there exist trajectories γ corresponding to constant switching functions such that $\lim_{t\to\infty} |\gamma(t)| = \infty$. Case **(CC.2.1)** of Theorem 2.3 is proved (see Example 4 below).



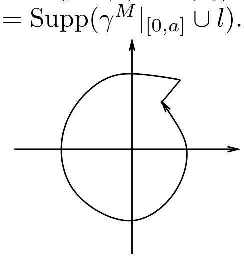
Case 2.2. If $E > 1$ ($\mathcal{K} > 1$), then at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have the same versus (counterclockwise). Fix $x_0 \in \mathbf{R}^2 \setminus \{0\}$, and let $\gamma^M : [0, \infty[\to \mathbf{R}^2$, $\gamma^M(0) = x_0$ be the trajectory corresponding to the feedback

$$u^M(x) = \begin{cases} 0 \text{ if } \theta \in [\theta^-, \theta^+[\text{ or } \theta \in [\theta^- + \pi, \theta^+ + \pi[, \\ +1 \text{ if } \theta \in [\theta^+, \theta^- + \pi[\text{ or } \theta \in [\theta^+ + \pi, \theta^- + 2\pi[, \end{cases} \tag{13}$$

where $\theta \in [\theta^-, \theta^- + 2\pi[$ is defined by $x_1 = \rho\cos(\theta)$, $x_2 = \rho\sin(\theta)$.



Let $(\rho^M(t), \theta^M(t))$ be the polar coordinates of γ^M and a the time defined by $\theta^M(a) = \theta^M(0) + 2\pi$. If $\rho^M(a) < \rho^M(0)$, then let l be the segment joining the points $(\rho^M(0), \theta^M(0))$ with $(\rho^M(a), \theta^M(a))$ and Ω the simply connected region whose border is $\partial\Omega := \mathrm{Supp}(\gamma^M|_{[0,a]} \cup l)$.



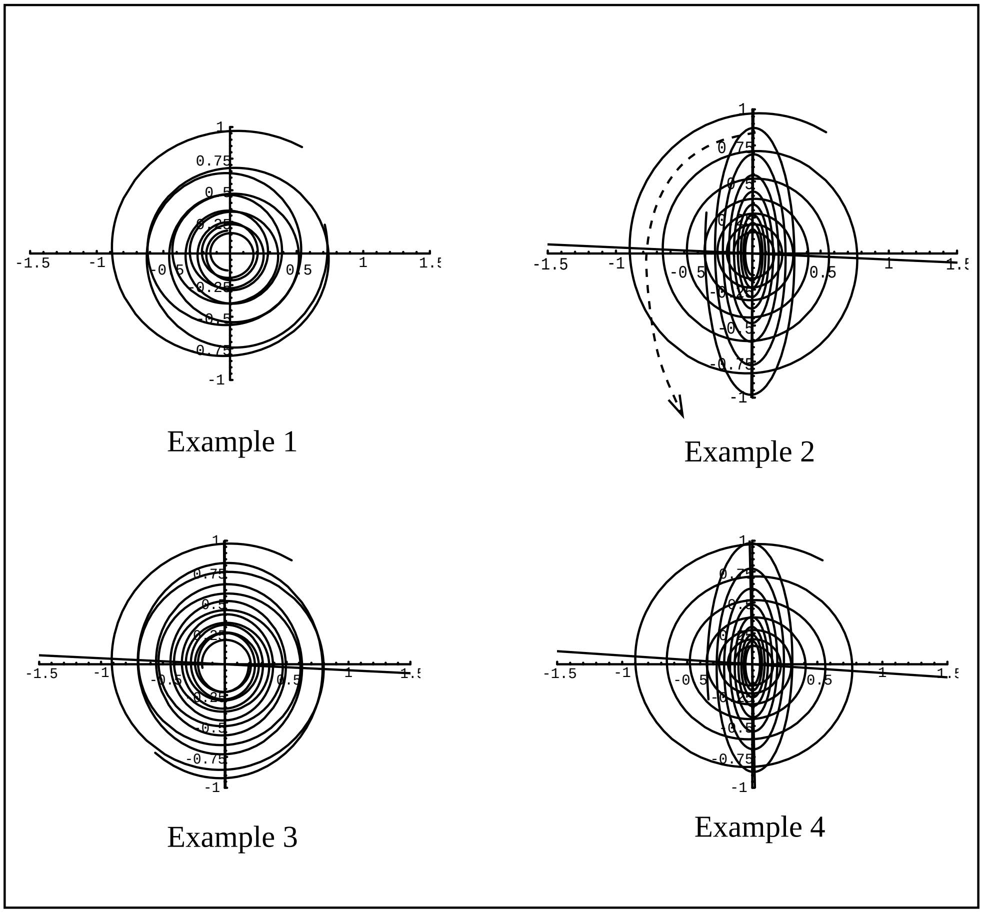


FIG. 3.1. *Examples in the* **(CC)** *case.*

For every $x \in \partial\Omega$, we have the following. Define V_u as in Case 1, $E < -1$. For each $u \in]0,1[$, V_u points inside Ω. Moreover, if $x \notin \{\gamma^M(0), \gamma^M(a)\}$, V_1 (resp., V_0) points inside Ω or is tangent to $\partial\Omega$. Similarly to Case 1 ($E < 1$), we can conclude that ($\mathcal{P}$) is true (see Example 3 below). On the other hand if $\rho^M(a) \geq \rho^M(0)$, then $\gamma^M(t)$ does not tend to the origin and ($\mathcal{P}$) is false (see Example 2 below). The condition $\rho^M(a) < \rho^M(0)$ is satisfied iff condition (5) holds. Formula (5) is obtained in Appendix A . The condition $\rho^M(a) = \rho^M(0)$ (i.e., $\rho_{CC} = 1$) is the case in which we have at least stability (not asymptotic) for every switching function. This concludes the proof of case **(CC.2.2)**.

Case 3. If $\mathcal{D} = 0$, then the two straight lines coincide. If $E > 1$, it is easy to understand that we are in the same situation as that of Case 1. If $E < -1$, then to every $x \in Q$ there exists $u \in [0,1]$ such that $uAx + (1-u)Bx = 0$. In this case, ($\mathcal{P}$) is false, but we have at least stability (not asymptotic). This proves case **(CC.3)** of Theorem 2.3.

Examples. In the following, we give some examples of the various situations in the **(CC)** case. We refer to Figure 3.1.

Example 1. $\rho_A = 0.05$, $\rho_B = 0.06$, $\mathcal{K} = -1.005$. In this case, $\mathcal{D} \sim -0.002$, and $(\mathcal{P})$ is true. In Figure 3.1, two integral curves of the vector fields Ax and Bx are shown. A similar situation but with the two trajectories rotating with the same versus can be obtained with the same values of ρ_A and ρ_B but with $\mathcal{K} = +1.00001$. In this case, $\mathcal{D} \sim -0.00008$ and $(\mathcal{P})$ is true (see case **(CC.1)**).

Example 2. $\rho_A = 0.0375,\ \rho_B = 0.05,\ \mathcal{K} = 1.67$. In this case, $\mathcal{D} \sim 1.79,\ \rho_{CC} \sim 2.62$ and $(\mathcal{P})$ is false. In Figure 3.1, two integral curves of the vector fields Ax and Bx, $\mathcal{D}$ that are the two straight lines (one almost coincides with the x_2 axis) and a trajectory γ such that $\lim_{t\to\infty}|\gamma(t)| = \infty$ (cf. case **(CC.2.2)**) are shown.

Example 3. $\rho_A = 0.0375,\ \rho_B = 0.0425,\ \mathcal{K} = 1.00455$. In this case, $\mathcal{D} \sim 0.0091,\ \rho_{CC} \sim 0.96$, and $(\mathcal{P})$ is true (cf. case **(CC.1)**).

Example 4. Suppose $\rho_A = 0.0375,\ \rho_B = 0.05,\ \mathcal{K} = -1.67$. In this case, $\mathcal{D} \sim 1.77$ and $(\mathcal{P})$ is false (cf. case **(CC.2.1)**).

3.2. The case in which A and B have one complex and the other real eigenvalues. Suppose that A has real eigenvalues $\lambda_1,\ \lambda_2$ $(\lambda_1, \lambda_2 < 0,\ |\lambda_2| > |\lambda_1|)$ and B complex eigenvalues $\lambda_3 = -\delta + i\omega,\ \lambda_4 = -\delta - i\omega$ $(\delta, \omega > 0)$. We have $\rho_A = -i(\lambda_1+\lambda_2)/(\lambda_1-\lambda_2)$ and $\rho_B = \delta/\omega$. We recall that $\rho_A/i > 1,\ \rho_B > 0$. Define

$$R(\varphi) := \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \in SO(2), \tag{14}$$

and choose a system of coordinates in which

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{15}$$

$$\begin{aligned} B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= R^{-1}(\varphi) \begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix} R(\varphi) \\ &= \begin{pmatrix} -\delta - \omega(E - 1/E)\sin(\varphi)\cos(\varphi) & -\omega(E\sin^2(\varphi) + 1/E\cos^2(\varphi)) \\ \omega(E\cos^2(\varphi) + 1/E\sin^2(\varphi)) & -\delta + \omega(E - 1/E)\sin(\varphi)\cos(\varphi) \end{pmatrix}. \end{aligned} \tag{16}$$

We have $\mathcal{K} = i(E - 1/E)\cos(\varphi)\sin(\varphi) \in i\mathbf{R}$, and without loss of generality we may assume that $\varphi \in [0, \pi/2[,\ |E| \geq 1$. Notice that in this case

$$[A, B] = (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \neq 0 \text{ for each } \mathcal{K} \in i\mathbf{R}.$$

Similarly to the previous subsection, the locus in which Ax and Bx are collinear is $Q^{-1}(0)$, where

$$Q = \det(Ax, Bx) = x_1^2(\lambda_1 c) + x_1 x_2 \bar{\chi} + x_2^2(-\lambda_2 b),$$

and by definition $\bar{\chi} := \lambda_1 d - \lambda_2 a = (\lambda_1 + \lambda_2)\omega\mathcal{K}/i - (\lambda_1 - \lambda_2)\delta = (\lambda_1 - \lambda_2)\omega\chi$, where $\chi := \rho_A\mathcal{K} - \rho_B$ (see Theorem 2.3). In this case, the discriminant of the quadratic form Q is

$$D_{RC} = \bar{\chi}^2 + 4\lambda_1\lambda_2 bc = \bar{\chi}^2 - 4\lambda_1\lambda_2\omega^2(-\mathcal{K}^2 + 1) = -\omega^2(\lambda_1 - \lambda_2)^2\mathcal{D}. \tag{17}$$

Notice that $\chi = 0$ implies $\bar{\chi} = 0$, which implies $D_{RC} < 0$, i.e., $\mathcal{D} > 0$. Moreover, $\chi > 0$ implies $\mathcal{K}/i < 0$, which implies $E < -1$. Similarly to the previous subsection, we have the following cases.

Case 1. If $D_{RC} < 0$ $(\mathcal{D} > 0)$, then $(\mathcal{P})$ is true (see Example 1 below).

Case 2. If $D_{RC} > 0$ ($\mathcal{D} < 0$), then $Q^{-1}(0)$ is a couple of noncoinciding straight lines passing from the origin and forming the following angles with the x_1 axis:

$$\theta^{\pm} = \arctan(m^{\pm}), \tag{18}$$
$$m^{\pm} := \frac{-\bar{\chi} \pm \sqrt{D_{RC}}}{2(-\lambda_2 b)} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{2\frac{\lambda_2}{\lambda_1-\lambda_2}(E\sin^2(\varphi) + 1/E\cos^2(\varphi))}.$$

From (17) it follows that $D_{RC} < \bar{\chi}^2$ (i.e., $-\mathcal{D} < \chi^2$) so that in this case we have $\chi \neq 0$ and we may assume

$$\begin{cases} \theta^-, \theta^+ \in]0, \pi/2[\text{ if } \chi \text{ and } E \text{ have the same sign,} \\ \theta^-, \theta^+ \in]-\pi/2, 0[\text{ if } \chi \text{ and } E \text{ have opposite sign.} \end{cases}$$

Case 2.1 If $\chi > 0$, then $\mathcal{K}/i < 0$, which implies $E < -1$, and we have $\theta^-\theta^+ \in]-\pi/2, 0[$. In this case, at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have opposite versus. The same argument of Case 2.1 of section 3.1 shows that $(\mathcal{P})$ is false (see Example 4 below).

Case 2.2 If $\chi < 0$, then in both cases where $E \geq 1$, $E \leq -1$ at each point of $Q^{-1}(0) \setminus \{0\}$ the two vector fields have the same versus.

Case 2.2.A If $E \leq -1$ (which implies $\mathcal{K}/i \leq 0$), then $(\mathcal{P})$ is true because of the following argument.

From $\chi = \rho_A \mathcal{K} - \rho_B < 0$ we have

$$-\frac{\mathcal{K}/i}{\rho_B} < \frac{1}{\rho_A/i} < 1. \tag{19}$$

Now let γ be an integral of the vector field Bx and $(\rho(t), \theta(t))$ its polar coordinates. We have

$$\gamma(t) = R(\varphi) \begin{pmatrix} \rho_0 e^{-\delta t} \cos(\omega t + \varphi_0) \\ \rho_0 E e^{-\delta t} \sin(\omega t + \varphi_0), \end{pmatrix}$$

and $\rho(t) = \rho_0 e^{-\delta t}\sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)}$. Now we prove that the condition (19) implies $\dot{\rho}(t) \leq 0$ for every $t \in Dom(\gamma)$, which clearly implies that $(\mathcal{P})$ is true. We have

$$\dot{\rho}(t) = \rho_0 e^{-\delta t} \left(\frac{(E^2-1)\omega \sin(\omega t + \varphi_0)\cos(\omega t + \varphi_0)}{\sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)}} - \delta \sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)} \right).$$

Therefore, $\dot{\rho}(t) < 0$ iff

$$\frac{(E^2-1)\omega \sin(\omega t + \varphi_0)\cos(\omega t + \varphi_0)}{\sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)}} - \delta \sqrt{\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0)} < 0$$

or, equivalently, iff

$$\cos^2(\omega t + \varphi_0) + E^2 \sin^2(\omega t + \varphi_0) - (E^2-1)\frac{\omega}{\delta} \sin(\omega t + \varphi_0)\cos(\omega t + \varphi_0) > 0. \tag{20}$$

Now if $(E^2-1)\omega/\delta \le -2E$ (that choosing a system of coordinates in which $\varphi = \pi/4$ is equivalent to $-\mathcal{K}/(i\rho_B) \le 1$; see Appendix B), then the condition (20) is satisfied. Hence from (19) we can conclude that $\dot\rho(t) < 0$ for each $t \in Dom(\gamma)$ and $(\mathcal{P})$ is true (see Example 5 below).

Case 2.2.B If $E \ge 1$ (which implies $\mathcal{K}/i \ge 0$), then $(\mathcal{P})$ is true iff condition (6) is satisfied (see Appendix B). Notice that in the case where $\mathcal{K} = 0$ we clearly have that $\rho_{RC} < 1$ and $(\mathcal{P})$ is true (see Examples 2 and 3 below). The case in which $\rho_{RC} = 1$ is the case in which we have at least stability (but not asymptotic) for every switching function.

Case 3. If $\mathcal{D} = 0$, then the two straight lines coincide. If $\chi < 0$, it is easy to understand that we are in the same situation as that of Case 1. If $\chi > 0$, then to every $x \in Q^{-1}(0)$ there exists $u \in [0,1]$ such that $uAx+(1-u)Bx = 0$. In this case, $(\mathcal{P})$ is false, but we have at least stability (not asymptotic). This proves case **(RC.3)** of Theorem 2.3.

This concludes the proof of cases **(RC)**.

Examples. In the following, we give some examples of the various situations in the **(RC)** case. We refer to Figure 3.2.

Example 1. $\rho_A/i = 1.11$, $\rho_B = 0.045$, $\mathcal{K}/i = 0.095$. In this case, $\chi \sim -0.15$, $\mathcal{D} \sim 0.2$, and $(\mathcal{P})$ is true (cf. case **(RC.1)**).

Example 2. $\rho_A/i = 1.11$, $\rho_B = 0.02$, $\mathcal{K}/i = 1.33$. In this case, $\chi \sim -1.49$, $\mathcal{D} \sim -1.62$, $\rho_{RC} \sim 1.4$, and $(\mathcal{P})$ is false (cf. case **(RC.2.2.B)**).

Example 3. $\rho_A/i = 1.11$, $\rho_B = 0.03$, $\mathcal{K}/i = 0.75$. In this case, $\chi \sim -0.85$, $\mathcal{D} \sim -0.37$, $\rho_{RC} \sim 0.98$, and $(\mathcal{P})$ is true (cf. case **(RC.2.2.B)**).

Example 4. $\rho_A/i = 1.11$, $\rho_B = 0.045$, $\mathcal{K}/i = -2.4$. In this case, $\chi \sim 2.6$, $\mathcal{D} \sim -5.3$, and $(\mathcal{P})$ is false (cf. case **(RC.2.1)**).

Example 5. $\rho_A/i = 1.14$, $\rho_B = 1.67$, $\mathcal{K}/i = -0.42$. In this case, $\chi \sim -1.19$, $\mathcal{D} \sim -1.06$, and $(\mathcal{P})$ is true (cf. case **(RC.2.2.A)**).

3.3. The case in which A and B have both real eigenvalues. Let λ_1, λ_2 ($\lambda_1, \lambda_2 < 0$, $|\lambda_2| > |\lambda_1|$) be the eigenvalues of A and λ_3, λ_4 ($\lambda_3, \lambda_4 < 0$, $|\lambda_4| > |\lambda_3|$) be the eigenvalues of B. Choose a system of coordinates such that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{21}$$

$$\begin{aligned} B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= R^{-1}(\pi/4) \begin{pmatrix} \lambda_3 & \alpha(\lambda_4-\lambda_3) \\ 0 & \lambda_4 \end{pmatrix} R(\pi/4) \\ &= \frac{1}{2} \begin{pmatrix} (\lambda_3+\lambda_4) - \alpha(\lambda_4-\lambda_3) & (\lambda_3-\lambda_4) + \alpha(\lambda_4-\lambda_3) \\ (\lambda_3-\lambda_4) - \alpha(\lambda_4-\lambda_3) & (\lambda_3+\lambda_4) + \alpha(\lambda_4-\lambda_3) \end{pmatrix}, \end{aligned} \tag{22}$$

where $R(\varphi)$ is defined as in formula (14) and $\alpha \in \mathbf{R} \setminus \{\pm 1\}$. In this system of coordinates the eigenvectors of A are proportional to $\mathbf{V}_1 = (1,0)$, $\mathbf{V}_2 = (0,1)$ and the eigenvectors of B to $\mathbf{V}_3 = (1,1)$, $\mathbf{V}_4 = ((\alpha-1)/(\alpha+1), 1)$. The geometric meaning of α is the following. Arctan(α) is the angle between the vector $(-1,1)$ and $\mathbf{V}_4$, measured clockwise. We have $\mathcal{K} = \alpha$. Notice that

$$[A,B] = -\frac{1}{2}(\lambda_1-\lambda_2)(\lambda_3-\lambda_4) \begin{pmatrix} 0 & (\alpha-1) \\ (\alpha+1) & 0 \end{pmatrix},$$

so $[A,B] \neq 0$ for every value of α. The case $\alpha = \pm 1$ is excluded (otherwise $\mathbf{V}_4$ is parallel to $\mathbf{V}_2$ or to $\mathbf{V}_1$, respectively, and **(H4)** fails).

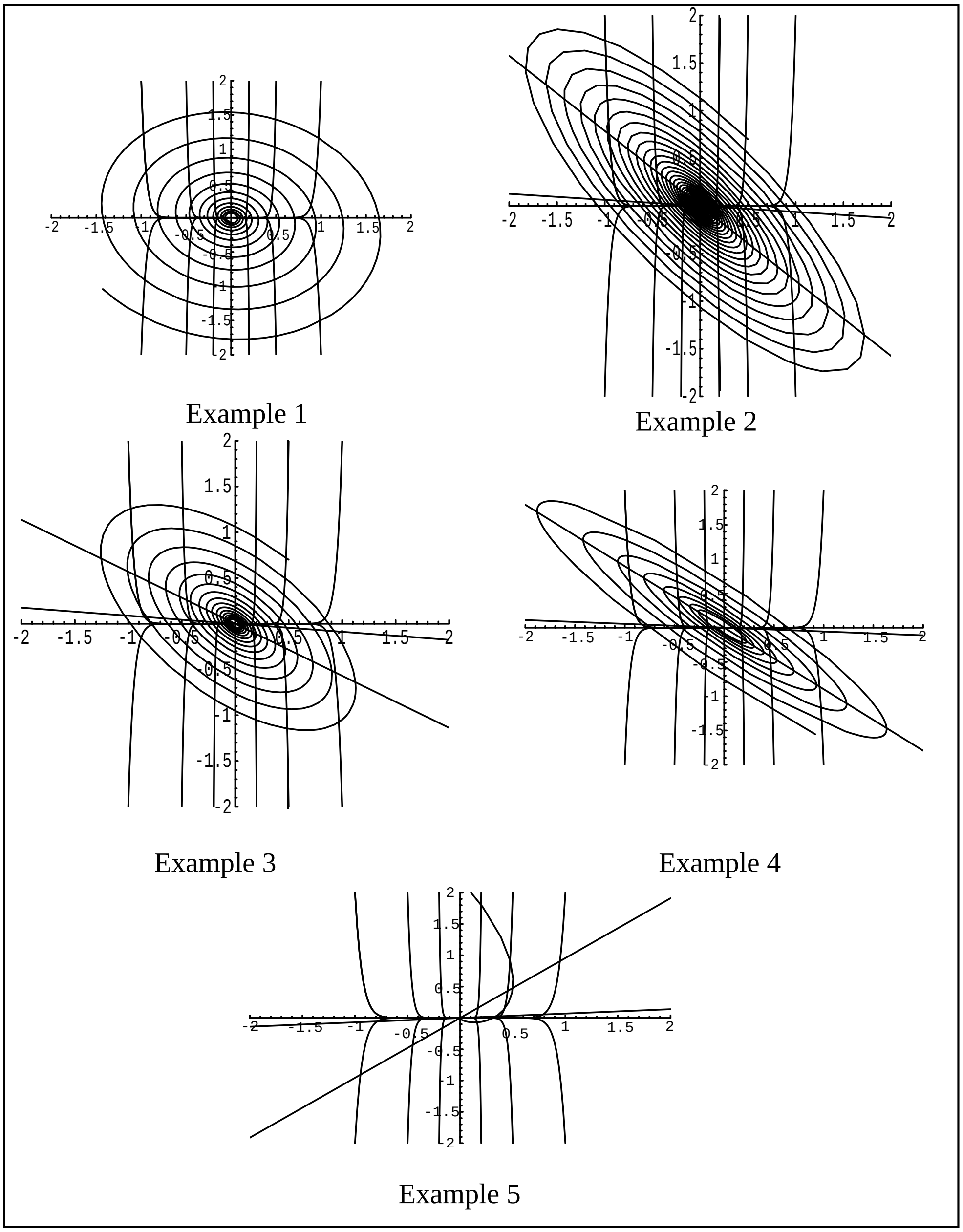


FIG. 3.2. *Examples in the* **(RC)** *case.*

The locus in which Ax and Bx are collinear is $Q^{-1}(0)$, where

$$Q = \det(Ax, Bx) = x_1^2(\lambda_1 c) + x_1 x_2 \bar{\chi} + x_2^2(-\lambda_2 b),$$

and by definition $\bar{\chi} := \lambda_1 d - \lambda_2 a = \frac{1}{2}((\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4) - \mathcal{K}(\lambda_1 + \lambda_2)(\lambda_3 - \lambda_4)) = -\frac{1}{2} i(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\chi$, where $\chi := \rho_A \mathcal{K} - \rho_B \in i\mathbf{R}$. In this case, the discriminant of the quadratic form Q is

$$D_{RR} = \bar{\chi}^2 + 4\lambda_1\lambda_2 bc = \bar{\chi}^2 + \lambda_1\lambda_2(\lambda_3 - \lambda_4)^2(-\mathcal{K}^2 + 1) \\ = \frac{1}{4}(\lambda_1 - \lambda_2)^2(\lambda_3 - \lambda_4)^2\mathcal{D}. \tag{23}$$

Notice that if $\mathcal{K} < 1$, then $\mathcal{D} > 0$. The following lemma states that in the case where $|\mathcal{K}| < 1$ $(\mathcal{P})$ is true.

LEMMA 3.1. *Let A, B be two 2×2 real matrices satisfying* **H1**, **H2**, **H3**, *and* **H4** *and such that their eigenvalues are real. Fix an arbitrary measurable switching function $u(.) : [0, \infty[\to [0, 1]$, and let $(x_1(t), x_2(t))$ (resp., $(\rho(t), \theta(t))$) be the Cartesian (resp., polar) coordinates of the solution of $\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t)$, $x(0) = x_0 \in \mathbf{R}^2 \setminus \{0\}$. If $\mathcal{K} \in]-1, 1[$, we have that $\dot{\rho}(t) < 0$ for almost every $t \in [0, +\infty[$.*

Proof. In this case, it is possible to choose a system of coordinates such that

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

$$B = R^{-1}(\varphi) \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix} R(\varphi)$$

$$= \begin{pmatrix} \cos^2(\varphi)\lambda_3 + \sin^2(\varphi)\lambda_4 & (\lambda_3 - \lambda_4)\sin(\varphi)\cos(\varphi) \\ (\lambda_3 - \lambda_4)\sin(\varphi)\cos(\varphi)(\lambda_3 - \lambda_4) & \sin^2(\varphi)\lambda_3 + \cos^2(\varphi)\lambda_4 \end{pmatrix},$$

where we assume $\varphi \in]0, \pi/2[$. Notice that $\varphi = 0$ is excluded (otherwise $[A, B] = 0$). We have

$$\dot{\rho}(t) = \dot{x}_1(t)\cos\theta(t) + \dot{x}_2(t)\sin\theta(t) \\ = \rho(t)\big(u(t)(\lambda_1\cos^2\theta(t) + \lambda_2\sin^2\theta(t)) \\ + (1 - u(t))(\lambda_3\cos^2(\theta(t) - \varphi) + \lambda_4\sin^2(\theta(t) - \varphi)\big).$$

This means that $\rho(t)$ has the expression

$$\rho(t) = \rho(0)\exp\left(\int_0^t (u(t)f_1(t) + (1 - u(t))f_2(t))\,dt\right),$$

where f_1 and f_2 are analytic functions satisfying $f_1 < \lambda_1$, $f_2 < \lambda_3$. $\square$

If $|\mathcal{K}| > 1$ we have the following cases.

Case 1. If $\mathcal{D} < 0$, then $(\mathcal{P})$ is true.

Case 2. If $\mathcal{D} > 0$, then $Q^{-1}(0)$ is a couple of noncoinciding straight lines passing from the origin and forming the following angles with the x_1 axis:

$$\theta^{\pm} = \arctan(m^{\pm}), \quad m^{\pm} := \frac{-\bar{\chi} \pm \sqrt{D_{RR}}}{2(-\lambda_2 b)} = \frac{-\chi/i \pm \sqrt{\mathcal{D}}}{(\rho_A/i + 1)(1 - \mathcal{K})}. \tag{24}$$

From (23) it follows that $D_{RR} < \bar{\chi}^2$ so that in this case we have $\chi \neq 0$ and we may assume

$$\begin{cases} \theta^-, \theta^+ \in]0, \pi/2[\text{ if } \chi/i \text{ and } \mathcal{K} \text{ have the same sign,} \\ \theta^-, \theta^+ \in]-\pi/2, 0[\text{ if } \chi/i \text{ and } \mathcal{K} \text{ have opposite sign.} \end{cases}$$

We have the following lemma.

LEMMA 3.2. *Let $\mathcal{D} > 0$; then*

(a) $\mathcal{K} > -\rho_A\rho_B$;
(b) *at each point of* $Q^{-1}(0)\setminus\{0\}$, *Ax and Bx have the same (resp., opposite) sign iff* $\mathcal{K} < -\rho_A\rho_B$ *(resp.,* $\mathcal{K} > -\rho_A\rho_B$*).*

Proof. (a) can be checked directly. Let us prove (b). Define $\Lambda^\pm := \mathcal{K} \pm \sqrt{\mathcal{D}} - \rho_A\rho_B$. By direct computation it follows that

- $\Lambda^\pm > 0$ iff $\mathcal{K} > -\rho_A\rho_B$;
- $(\frac{Ax}{\|Ax\|}) = (\frac{Bx}{\|Bx\|})$ for every $x = (h, m^\pm h),\ h \in \mathbf{R}\setminus\{0\},$ iff $\Lambda^\pm < 0$;
- $(\frac{Ax}{\|Ax\|}) = -(\frac{Bx}{\|Bx\|})$ for every $x = (h, m^\pm h),\ h \in \mathbf{R}\setminus\{0\},$ iff $\Lambda^\pm > 0$.

This concludes the proof. □

From Lemma (3.2) we have the following cases (notice that $-\rho_A\rho_B > 1$).

Case 2.1 If $\mathcal{K} > -\rho_A\rho_B$, then $(\mathcal{P})$ is false.

Case 2.2 If $\mathcal{K} < -\rho_A\rho_B$, then:

Case 2.2.A If $\mathcal{K} > 1$, one can easily check that the worst trajectory cannot rotate around the origin and $(\mathcal{P})$ is true.

Case 2.2.B If $\mathcal{K} < -1$, then the worst trajectory rotates around the origin and $(\mathcal{P})$ is true iff condition (7) is satisfied. Condition (7) can be obtained with arguments entirely similar to the ones of Appendices A and B. The case in which $\rho_{RC} = 1$ is the case in which we have at least stability (not asymptotic) for every switching function.

Case 3. If $\mathcal{D} = 0$, then the two straight lines coincide. Similarly to the **(CC)** and **(RC)** cases, if $\mathcal{K} < -\rho_A\rho_B$, then $(\mathcal{P})$ is true. Vice versa, if $\mathcal{K} > -\rho_A\rho_B$, then $(\mathcal{P})$ is false but we have stability (not asymptotic).

4. Asymptotic stability in the space of parameters. Fix a value of the cross ratio, and let $\mathcal{R}$ (resp., $\bar{\mathcal{R}}$) be the region in the (ρ_A, ρ_B) plane in which the system is asymptotically stable (resp., asymptotically stable or only stable) for arbitrary switching functions. In this section, we study the shape and the convexity of $\mathcal{R}$ and $\bar{\mathcal{R}}$.

4.1. The complex-complex case. In Figure 4.1 we show $\mathcal{R}$ for a fixed value of $\mathcal{K}$ in the case in which both A and B have complex eigenvalues.

In the case $\mathcal{K} < -1$, $\mathcal{R}$ is determined by the condition $\mathcal{D} < 0$. The set of values of ρ_A and ρ_B such that $\mathcal{D} = 0$ is the two curved lines of equations $\rho_B = \rho_A\mathcal{K} \pm \sqrt{(\rho_A^2+1)(\mathcal{K}^2-1)}$ of Figure 4.1 (case $\mathcal{K} < -1$). The points of intersection with the two axes are

$$(\rho_A, \rho_B) = (\sqrt{\mathcal{K}^2-1}, 0), \tag{25}$$

$$(\rho_A, \rho_B) = (0, \sqrt{\mathcal{K}^2-1}). \tag{26}$$

In this case, $\mathcal{R}$ is constituted by two connected open convex unbounded regions, while to get $\bar{\mathcal{R}}$ we have to add the points in which $\mathcal{D} = 0$.

In the case where $\mathcal{K} > 1$, $\mathcal{R}$ is determined by the condition $\rho_{CC} < 1$. In Figure 4.1 (case $\mathcal{K} > 1$) the locus $\mathcal{D} = 0$ is drawn with dotted lines, while the locus $\rho_{CC} = 1$ is drawn with a solid line. The points in which the two loci intersect each other and intersect the two axes are given again by formulas (25) and (26). In this case, to study the convexity of $\mathcal{R}$, we have to check if, expressing the locus $\rho_{CC} = 0$ as $\rho_B = f_{\mathcal{K}}(\rho_A)$, we find a convex function. In the following, the label $(_{\mathcal{K}})$ is a parameter, and it will be dropped. Let us indicate the derivative of ρ_{CC} with respect to the first and second

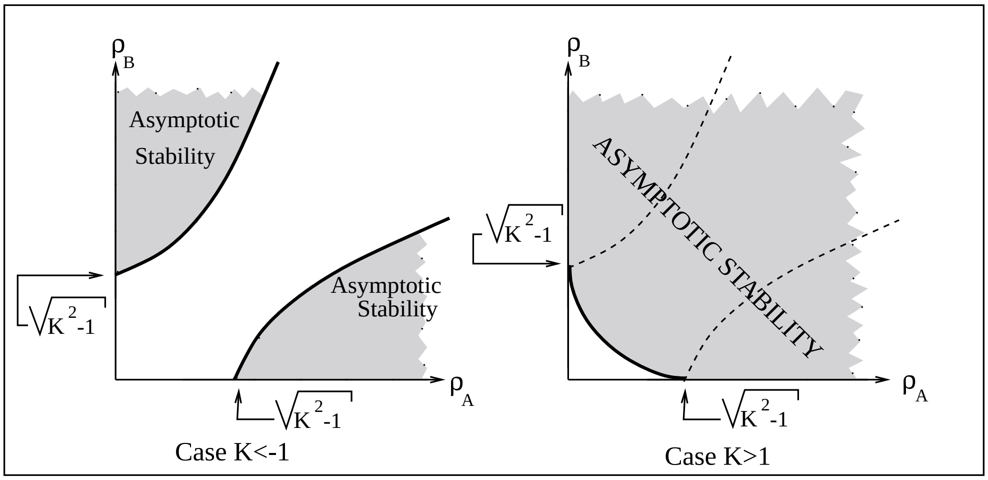


FIG. 4.1. $\mathcal{R}$ *(the grey region) for a fixed value of* $\mathcal{K}$*, in the complex–complex case.*

variable as $\big(\rho_{CC}\big)_1$ and $\big(\rho_{CC}\big)_2$. We have that

$$f'(\rho_A) = F\left(\rho_A, f(\rho_A)\right), \text{ where } F(\rho_A,\rho_B) := -\frac{\big(\rho_{CC}\big)_1}{\big(\rho_{CC}\big)_2} = -\frac{\arctan(\frac{\rho_B-\rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}}) + \frac{\pi}{2}}{\arctan(\frac{\rho_A-\rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}}) + \frac{\pi}{2}},$$

$$f''(\rho_A) = G\left(\rho_A, f(\rho_A)\right), \text{ where}$$

$$\begin{aligned}
G(\rho_A,\rho_B) &= \frac{\partial F(\rho_A,\rho_B)}{\partial \rho_A} + \frac{\partial F(\rho_A,\rho_B)}{\partial \rho_B} F(\rho_A,\rho_B) \\
&= \frac{2}{(1+\rho_A^2)\,(1+\rho_B^2)\,\sqrt{\mathcal{D}}\left(\pi + 2\,\arctan(\frac{\rho_A-\rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}})\right)^3} \\
\times & \quad \Bigg(\left[\rho_A^3\,\rho_B + \rho_A\,\rho_B^3 + 2(1+\rho_A^2+\rho_B^2+\rho_A\rho_B+\rho_A^2\rho_B^2) + \mathcal{K}(2+\rho_A^2+\rho_B^2)\right]\pi^2 \\
& \quad + 4\,\left(1+\rho_B^2\right)\,\left(1+\rho_A^2+\rho_A\,\rho_B+\mathcal{K}\right)\,\pi\,\arctan\left(\frac{\rho_A-\rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right) \\
& \quad + 4\,\left(1+\rho_A^2\right)\,\left(1+\rho_B^2+\rho_B\,\rho_A+\mathcal{K}\right)\,\pi\,\arctan\left(\frac{\rho_B-\rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right) \\
& \quad + 4\,\left(1+\rho_A^2\right)\,(\rho_A\,\rho_B+\mathcal{K})\,\arctan\left(\frac{\rho_B-\rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right)^2 \\
& \quad + 4\,\left(1+\rho_B^2\right)\,(\rho_A\,\rho_B+\mathcal{K})\,\arctan\left(\frac{\rho_A-\rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right)^2 \\
& \quad + 8\,\left(1+\rho_A^2\right)\,\left(1+\rho_B^2\right)\,\arctan\left(\frac{\rho_B-\rho_A\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right)\,\arctan\left(\frac{\rho_A-\rho_B\,\mathcal{K}}{\sqrt{\mathcal{D}}}\right)\Bigg).
\end{aligned}$$

Now the only terms that can be negative are the ones in the third and fourth rows, but it is easy to check numerically that the sum of these two terms with the one in the second row is always bigger than zero. The convexity follows. In this case, $\mathcal{R}$ is a convex open unbounded region, while $\bar{\mathcal{R}}$ is a convex not-open unbounded region (we

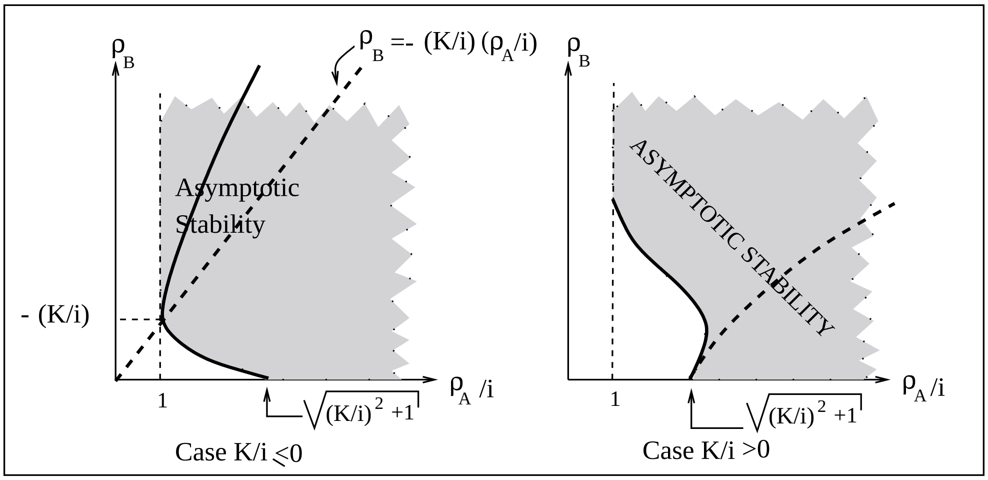


FIG. 4.2. $\mathcal{R}$ *(the grey region) for a fixed value of* $\mathcal{K}$*, in the* **(RC)** *case.*

have to add the points such that $\rho_{CC} = 1$).

4.2. The real-complex case. In the case in which A and B have one complex and the other real eigenvalues, $\mathcal{R}$ is drawn in Figure 4.2. We recall that $\rho_A/i > 1$, $\rho_B > 0$, $\mathcal{K}/i \in \mathbf{R}$.

In the case where $\chi > 0$ (which implies $\mathcal{K}/i < 0$ and $\rho_B < (-\mathcal{K}/i)(\rho_A/i)$), $\mathcal{R}$ is determined by the condition $\mathcal{D} > 0$. The locus $\mathcal{D} = 0$ is the set of points such that $\rho_B = -(\rho_A/i)(\mathcal{K}/i) \pm \sqrt{(-(\rho_A/i)^2+1)(-(\mathcal{K}/i)^2-1)}$. The intersection point with the ρ_A axis is

$$(\rho_A/i, \rho_B) = (\sqrt{(\mathcal{K}/i)^2+1}, 0), \tag{27}$$

and the intersection with the $\rho_A/i = 1$ set is

$$(\rho_A/i, \rho_B) = (1, -(\mathcal{K}/i)).$$

In the case when $\chi < 0$ and $\mathcal{K}/i \leq 0$, we have asymptotic stability. We conclude that in the case when $\mathcal{K}/i \leq 0$, $\mathcal{R}$ is a convex open unbounded region (see Figure 4.2 (case $\mathcal{K}/i \leq 0$)), while to get $\bar{\mathcal{R}}$, we have to add the points in which $\mathcal{D} = 0$.

In the case when $\chi < 0$ and $\mathcal{K}/i > 0$, $\mathcal{R}$ is determined by the condition $\rho_{RC} < 1$. In Figure 4.2 (case $\mathcal{K}/i > 0$), the locus $\mathcal{D} = 0$ is drawn with a dotted line, while the locus $\rho_{CC} = 1$ is drawn with a solid line. The points in which the two loci intersect each other are given by formula (27). In this case, $\mathcal{R}$ is a nonconvex open unbounded region. Again the points in which we have at least stability are the points in which we have asymptotic stability plus the points such that $\rho_{RC} = 1$.

4.3. The real-real case. In the case in which A and B have both real eigenvalues, $\mathcal{R}$ is drawn in Figure 4.3. We recall that $\rho_A/i, \rho_B/i > 1$, $\mathcal{K} \in \mathbf{R} \setminus \{\pm 1\}$. If $\mathcal{K} < -1$, $\mathcal{R}$ is determined by $\rho_{RR} > 0$, while, if $\mathcal{K} > 1$, $\mathcal{R}$ is determined by $\mathcal{D} > 0$. Similarly to the **(CC)** case, we can conclude that $\mathcal{R}$ is a convex open unbounded region, while $\bar{\mathcal{R}}$ is a convex not-open unbounded region. (We have to add the points such that $\rho_{CC} = 1$ and $\mathcal{D} = 0$.)

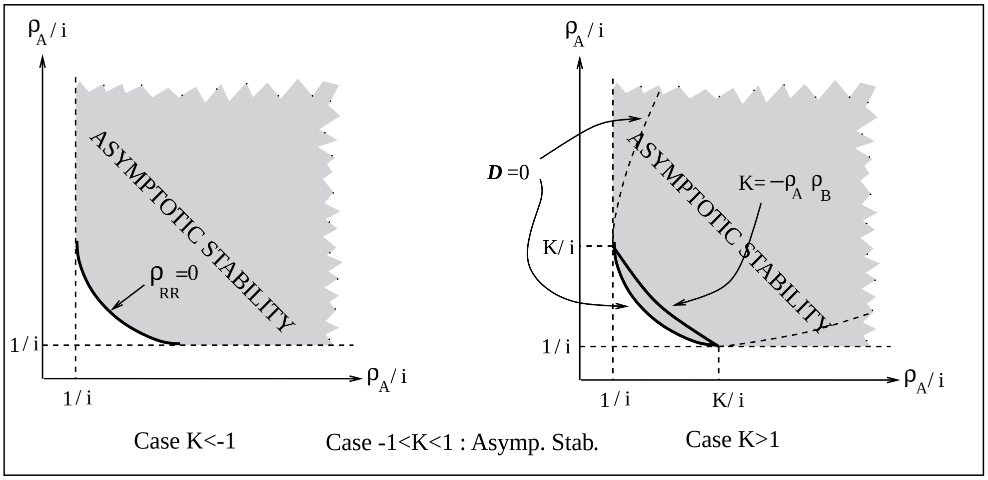


FIG. 4.3. $\mathcal{R}$ *(the grey region) for a fixed value of* $\mathcal{K}$*, in the* **(RR)** *case.*

5. Final remarks. Using the results of [1, 7] and by Theorem 2.3, we have a complete algorithm to study the asymptotic stability of a switched linear system in any dimension at least in the case in which

$$A_u = uA^1x + (1-u)A^2x, \qquad u \in [0,1], \quad A^1, A^2 \in \mathbf{R}^{n\times n},$$

where A and B are diagonalizable and $\dim\{A_1, A_2\}_{L.A.} \leq 4$. The case in which A or B is not diagonalizable can be treated with similar techniques.

Generalization can be done for more complex sets U. One is the following m-input system:

$$\dot{x} = \sum_{i=1}^{m} u_i A^i x, \quad \sum_{i=1}^{m} u_i = 1, \quad u_i \geq 0 \ \ (i = 1, \dots, m), \quad x \in \mathbf{R}^n, \quad A^1, \dots, A^m \in \mathbf{R}^{n\times n}.$$

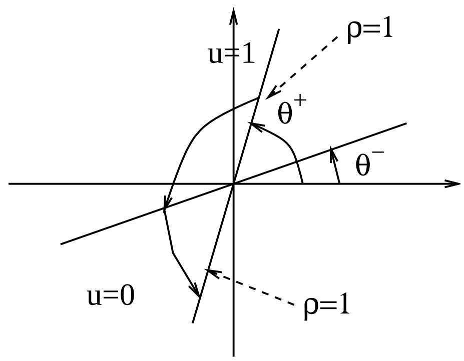
With exactly the same techniques used in this paper, one can find a coordinates invariant necessary and sufficient condition for the stabilizability of a control system of the kind (2), where all the matrices have eigenvalues with strictly *positive* real part. This problem was also studied in [12] but not in terms of a minimum number of coordinate-free parameters. We refer to [12] for details.

Some results can be obtained for the nonlinear version of the problem treated in this paper,

$$\dot{x} = uF(x) + (1-u)G(x), \tag{28}$$

where $x \in \mathbf{R}^2$, $F(.)$, $G(.)$ are $\mathcal{C}^\infty$ generic functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $F(0) = 0$, $G(0) = 0$, and the two dynamical systems $\dot{x} = F(x)$, $\dot{x} = G(x)$ are globally asymptotically stable at the origin. We are interested in studying under which conditions on $F(.)$ and $G(.)$ the origin of the system (28) is globally asymptotically stable for every measurable function $u(.) : [0, \infty[\to [0,1]$.

Appendix A: Proof of formula (5). We refer to the following figure.



Let $\rho(t),\ \theta(t)$ (resp., $x(t), y(t)$) be the polar coordinates (resp., Cartesian) of $\gamma^M(t)$, where we fix the initial condition by setting $\rho(0) = 1,\ \theta(0) = \theta^+$. We have to check if at the time a such that $\theta(a) = \theta(0) + 2\pi$ we have $\rho(a) < 1$. Due to the symmetries of the system, this happens iff at the time $\bar{t}$ such that $\theta(\bar{t}) = \theta^+ + \pi$ we have $\rho_{RC} := \rho(\bar{t}) < 1$. Notice that $\bar{t} = a/2$. The trajectory $\gamma^M(t)$ corresponds to the constant switching function $u = +1$ up to the time t' in which $\theta(t') = \theta^- + \pi$. This time is defined by the equations

$$
\begin{aligned}
&x(t') = \rho_0 e^{-\delta_A t'} \cos(\omega_A t' + \theta_E^+),\\
&y(t') = \rho_0 E e^{-\delta_A t'} \sin(\omega_A t' + \theta_E^+),\\
&\theta_E^+ = \arctan\left(\frac{m^+}{E}\right) \in \left\{ \begin{array}{l} [-\pi/2, \pi/2[\text{ if } \theta^+ \in [-\pi/2, \pi/2[,\\ ]\pi/2, 3\pi/4[\text{ if } \theta^+ \in]\pi/2, 3\pi/4[, \end{array}\right.\\
&\rho_0 = (\cos^2(\theta_E^+) + E^2 \sin^2(\theta_E^+))^{-1/2},\\
&y(t') = m^- x(t').
\end{aligned}
$$

It follows that $\tan(\omega_A t' + \theta_E^+) = m^-/E$. If we set $\theta_E^- = \arctan(m^-/E) \in]\theta_E^+, \theta_E^+ + \pi[$, we have $t' = (\theta_E^- - \theta_E^+)/\omega_A$.

After time t', $\gamma^M(t)$ corresponds to the constant switching function $u = 0$ up to the first time $\bar{t}$ in which $\theta(\bar{t}) = \theta^+ + \pi$. This time is defined by the equations

$$
\begin{aligned}
&x(\bar{t}) = \rho(t') e^{-\delta_B(\bar{t}-t')} \cos(\omega_B(\bar{t} - t') + \theta^- + \pi),\\
&y(\bar{t}) = \rho(t') e^{-\delta_B(\bar{t}-t')} \sin(\omega_B(\bar{t} - t') + \theta^- + \pi),\\
&\rho(t') = \rho_0 e^{-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+)} \sqrt{\cos^2(\theta_E^-) + E^2 \sin^2(\theta_E^-)},\\
&y(\bar{t}) = m^+ x(\bar{t}).
\end{aligned}
$$

It follows that $\tan(\omega_B(\bar{t} - t') + \theta^- + \pi) = \tan(\omega_B(\bar{t} - t') + \theta^-) = m^+ = \tan(\theta^+)$, and we have $\bar{t} = (\theta^+ - \theta^-)/\omega_B + t'$. Finally,

$$
\bar{\rho} = \rho(\bar{t}) = \rho(t') e^{-\frac{\delta_B}{\omega_B}(\bar{t}-t')} = e^{-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+) - \frac{\delta_B}{\omega_B}(\theta^+ - \theta^-)} \sqrt{\frac{\cos^2\theta_E^- + E^2 \sin^2(\theta_E^-)}{\cos^2\theta_E^+ + E^2 \sin^2(\theta_E^+)}}.
$$

This formula is not in a good form because it is not explicitly invariant for the exchange of δ_A, ω_A with δ_B, ω_B and because the quantity E does not appear only in the form $E + 1/E$. Recalling the definition of $\rho_A, \rho_B, \mathcal{K}$ (see Definition 2.1) and using the equality

$$
\arctan a - \arctan b = \arctan\left(\frac{ab + 1}{b - a} + \pi/2\right),
$$

which holds for $a > b$, it is possible to obtain the relations

$$-\frac{\delta_A}{\omega_A}(\theta_E^- - \theta_E^+) = -\rho_A \left(\arctan\left(\frac{-\rho_A \mathcal{K} + \rho_B}{\sqrt{\mathcal{D}}} \right) + \pi/2 \right),$$
$$-\frac{\delta_B}{\omega_B}(\theta^+ - \theta^-) = -\rho_B \left(\arctan\left(\frac{\rho_A - \rho_B \mathcal{K}}{\sqrt{\mathcal{D}}} \right) + \pi/2 \right).$$

Moreover, with elementary computation we can show that

$$\sqrt{\frac{\cos^2\theta_E^- + E^2\sin^2(\theta_E^-)}{\cos^2\theta_E^+ + E^2\sin^2(\theta_E^+)}} = \sqrt{\frac{\rho_A\rho_B + \sqrt{\mathcal{D}}}{\rho_A\rho_B - \sqrt{\mathcal{D}}}}.$$

Formula (5) is obtained.

Appendix B: Proof of formula (6). To obtain a result that explicitly does not depend on the choice of the system of coordinates, we need to write the formulas of section 3.2 in a more invariant way. Set

$$\psi = \sqrt{\frac{E\cos^2\varphi + 1/E\sin^2\varphi}{E\sin^2\varphi + 1/E\cos^2\varphi}},$$

and make the coordinates transformation

$$x \to \Psi(\psi)x, \quad \text{where } \Psi(\psi) := \begin{pmatrix} 1 & 0 \\ 0 & \psi \end{pmatrix}.$$

In this case ($E \geq 1$), the new coordinates A, B, and $\theta^\pm$ have the expressions

$$A = \Psi^{-1}(\psi)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\Psi(\psi) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$
$$B = \Psi^{-1}(\psi)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Psi(\psi) = \begin{pmatrix} -\delta - \omega\mathcal{K}/i & -\omega\sqrt{-\mathcal{K}^2+1} \\ +\omega\sqrt{-\mathcal{K}^2+1} & -\delta + \omega\mathcal{K}/i \end{pmatrix},$$
$$\theta^\pm = \arctan m^\pm, \quad m^\pm = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{2\frac{\lambda_2}{\lambda_1-\lambda_2}\sqrt{-\mathcal{K}^2+1}} = \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)\sqrt{-\mathcal{K}^2+1}}.$$

Equivalently, we can use the expressions (15), (16), (18) for $A, B, \theta^\pm$ with $E \geq 1$ and $\varphi = \pi/4$.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$
$$B = R^{-1}(\pi/4)\begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix}R(\pi/4),$$

(29) $$\theta^\pm = \arctan m^\pm, \quad m^\pm := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{\frac{\lambda_2}{\lambda_1-\lambda_2}(E+1/E)} = 2\frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i - 1)(E+1/E)}.$$

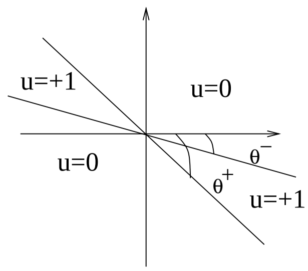
The relation between $\mathcal{K}$ and E is

$$\mathcal{K} = i\frac{1}{2}(E - 1/E), \quad E = \mathcal{K}/i + \sqrt{-\mathcal{K}^2+1}.$$

Moreover, we are considering the case $\chi < 0$ so that $\theta^+, \theta^- \in]-\pi/2, 0[$. From (29) it follows that $\theta^+ < \theta^-$.

In this case, $\gamma^M(.)$ corresponds to the feedback (see the following figure):

$$u(x) = \begin{cases} 1 \text{ if } \theta \in]\theta^+, \theta^-[\text{ or } \theta \in]\theta^+ + \pi, \theta^- + \pi[, \\ 0 \text{ if } \theta \in]\theta^-, \theta^+ + \pi[\text{ or } \theta \in]\theta^- + \pi, \theta^+ + 2\pi[. \end{cases}$$



Make the following coordinates transformation: $x \to \bar{x} = R(\pi/4)x$. We have

$$\begin{aligned} &A \to \bar{A} = R(\pi/4) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R^{-1}(\pi/4), \\ &B \to \bar{B} = \begin{pmatrix} -\delta & -\omega/E \\ \omega E & -\delta \end{pmatrix}, \\ &\theta^{\pm} \to \bar{\theta}^{\pm} = \theta^{\pm} - \pi/4 = \arctan \bar{m}^{\pm} \in [3/4\pi, \pi/4[, \quad \bar{m}^{\pm} := \frac{m^{\pm} - 1}{m^{\pm} + 1}. \end{aligned}$$

Similarly to Appendix A, we compute γ^M in polar coordinates with the initial condition $\rho(0) = 1$, $\theta(0) = \bar{\theta}^-$. Let t' be the first time such that $\theta(t') = \bar{\theta}^+ + \pi$. We have

$$\begin{aligned} &t' = (\xi^+ - \xi^-)/\omega, \\ &\rho(t') = e^{-\frac{\delta}{\omega}(\xi^+ - \xi^-)} \sqrt{\frac{\cos^2 \xi^+ + E^2 \sin^2 \xi^+}{\cos^2 \xi^- + E^2 \sin^2 \xi^-}}, \end{aligned}$$

where $\xi^{\pm} := \arctan(\bar{m}^{\pm}/E)$, $\xi^+ \in]\xi^-, \xi^- + \pi[$.

Now we come back to the old coordinates ($\bar{x} \to x = R^{-1}(\pi/4)\bar{x}$), and we integrate Bx up to the first time $\bar{t}$ such that $\theta(\bar{t}) = \theta^- + \pi$. We have

$$\begin{aligned} &x(\bar{t}) = \rho(t') \cos(\theta^+ + \pi) e^{\lambda_1(\bar{t} - t')}, \\ &y(\bar{t}) = \rho(t') \sin(\theta^+ + \pi) e^{\lambda_2(\bar{t} - t')}, \\ &y(\bar{t}) = m^- x(\bar{t}). \end{aligned}$$

It follows that

$$m^+ e^{((\lambda_2 - \lambda_1)(\bar{t} - t'))} = n^- \Longrightarrow \bar{t} - t' = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{m^-}{m^+}\right).$$

Finally,

$$\begin{aligned} \rho_{RC} &:= \rho(\bar{t}) = \rho(t') \sqrt{\cos^2 \theta^+ e^{\frac{\lambda_1}{\lambda_2 - \lambda_1} \ln(m^-/m^+)} + \sin^2 \theta^+ e^{\frac{\lambda_2}{\lambda_2 - \lambda_1} \ln(m^-/m^+)}} \\ &= e^{-\frac{\delta}{\omega}(\xi^+ - \xi^-)} \sqrt{\frac{\cos^2 \xi^+ + E^2 \sin^2 \xi^+}{\cos^2 \xi^- + E^2 \sin^2 \xi^-}} \end{aligned}$$

$$\times \sqrt{\cos^2\theta^+ \left(\frac{m^-}{m^+}\right)^{\frac{\lambda_1}{\lambda_2-\lambda_1}} + \sin^2\theta^+ \left(\frac{m^-}{m^+}\right)^{\frac{\lambda_2}{\lambda_2-\lambda_1}}}$$

$$= e^{-\rho_B(\xi^+-\xi^-)} \sqrt{\frac{\cos^2\xi^+ + E^2\sin^2\xi^+}{\cos^2\xi^- + E^2\sin^2\xi^-}}$$

$$\times \sqrt{\cos^2\theta^+ \left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i+1)} + \sin^2\theta^+ \left(\frac{m^+}{m^-}\right)^{\frac{1}{2}(-\rho_A/i-1)}},$$

which is formula (6). This formula is complicated but acceptable because there are no further symmetries.

Acknowledgment. The author is grateful to Andrei Agrachev for suggesting the problem and for helpful discussions that contributed to finding the right invariant.

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