SISM2 - MAP 561
Answers for the 2012 MAP pale

Ugo Boscain and Yacine Chitour

Exercise 1

1. Remark first that $f_i(x) = 0$ implies that $x = 0$ and use the mean value theorem.
2. Consider $V(x_1, x_2, x\text{"}3) = \frac{x_2^2+x_3^2}{2} + F_1(x_1)$.
3. Lasalle theorem and standard computations.
4. One should assume that F_1 tends to infinity as $|x|$ tends to infinity. Choosing $f_1(x) = \frac{2x}{1+x^4}$ creates problems...

Exercise 2

Pure and mechanical computations. No tricks whatsoever.

Exercise 3

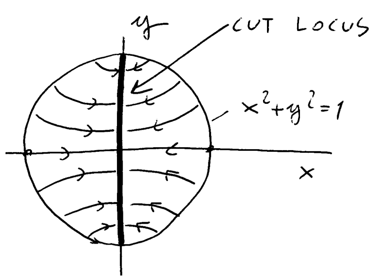
Check that (A, B) is controllable, $U > 0$ and $Q, W \geq 0$. Notice that the trajectory to follow is actually a trajectory of the control system for the input zero and starting from the initial condition $(1, 0, 3)$. The system is not observable but when you do output tracking everything reduces to a 2D system where the variable "z" disappears. From then, pure computations.

Exercise 4

1. Yes, since it is Lie-bracket generated.

2.1 There are no abnormals since with one bracket all directions are generated.

2.2, 2.3 See picture.



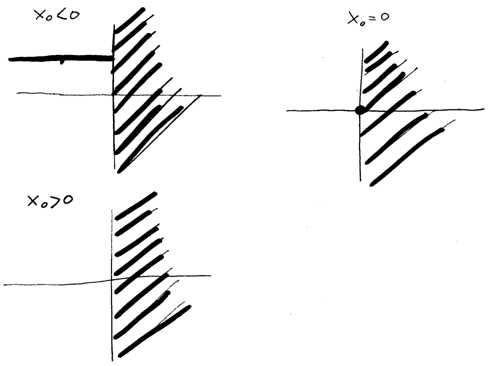
Exercise 5

We will provide answers using several remarks. Set $\mathcal{A}_{x_0}$ the reachable set from $x_0 \in$ R for the control system $\dot{x} = 1 + u_1 x$.

1. The function f verifies that, for every integer n and negative or zero real number x, then $f^{(n)}(x) = 0$.
2. $\mathcal{A}_{x_0}$ is equal to R if $x_0 < 0$, R_+ if $x_0 = 0$ and R_+^* if $x_0 > 0$.
3. The region $S = \mathrm{R}_+^* \times \mathrm{R}$ is invariant by the control system.

(To see 2., simply use appropriate constant controls.)

Then, for every $q_0 = (x_0, y_0) \in \mathrm{R}^2$, one has $\mathcal{A}_{q_0}$ is equal to S plus the half-line $\mathrm{R} \times \{y_0\}$ if $x_0 < 0$ or the point $(0, 0)$ if $x_0 = 0$. To see that first of all notice that if $q_1 \in \mathcal{A}_{q_0} \cap S$, then by 3., one never exits S. Therefore, $\mathcal{A}_{q_0}$ is equal to some subset of S plus the half-line $\mathrm{R} \times \{y_0\}$ if $x_0 < 0$ or the point $(0, 0)$ if $x_0 = 0$. It remains to show that $\mathcal{A}_{q_0}$ is equal to S for every $q_0 \in S$. It is enough to show that, starting from any $q_0 \in S$, one can reach the vertical line defined by $x = x_0$. If $\cos(x_0) \neq 0$, this is clear by taking $u_1 = 0$ and $u_2 = \pm 1$. If $\cos(x_0) = 0$, simply move horizontally to some q_1 with $\cos(x_1) \neq 0$, then to any point of the vertical line $x = x_1$ and then come back horizontally to any point of the vertical line $x = x_0$ by controls $(u_1, 0)$.



Exercise 6

The Reachable set and the optimal synthesis is shown in the following picture