MAP 561 - Automatique
Exam. Monday Mars 25th 2013

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1 Linear system

Consider the linear control system,

$$\begin{aligned}\dot{x}_1 &= 3x_2 + 7x_3 + u,\\ \dot{x}_2 &= 5x_1 + 8x_3,\\ \dot{x}_3 &= 9x_1 + u,\end{aligned}$$

where $u \in \mathbf{R}$.

A. Study its controllability.

B. Find a Brunovsky output and determine a control steering the point $(1, 1, 1)$ to the point $(2, 3, 4)$.

C. Consider the output map $y = x_2$. Is the system observable?

D. Is the system stabilizable by a feedback linear in the output? If not, build an asymptotic observer and a linear dynamic feedback stabilizing the system.

2 Stabilization of a non-linear system

Consider the dynamical system,

$$\begin{aligned}\dot{x}_1 &= -x_2x_3 + 1,\\ \dot{x}_2 &= x_1x_3 - x_2,\\ \dot{x}_3 &= x_3^2(1 - x_3).\end{aligned}$$

A. Show that the system has a unique equilibrium point P.

B. Show that P is locally asymptotically stable.

C. Is P globally asymptotically stable?

3 Harmonic Oscillator

We consider the following controlled harmonic oscillator,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + u, \end{aligned} \qquad |u| \leq 1$$

A. Is it controllable? Is it small-time locally controllable ? (i.e. is it possible to reach a neighborhood of the starting point in an arbitrarily small time?)

B. Find the time-optimal synthesis (i.e. the collection of all time-optimal trajectories) starting from the point $(2, 0)$.

4 Abnormal Extremals

Consider the minimization problem (with T fixed)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ y(y-1-x^2) \end{pmatrix}, \quad \int_0^T (u_1(t)^2 + u_2(t)^2)\, dt \to \min,$$
$$(x(0), y(0)) = (x_0, y_0), \quad (x(T), y(T)) = (x_1, y_1).$$

A. Is it controllable?

B. For every initial and final conditions (x_0, y_0), (x_1, y_1) say if there is an abnormal extremal joining them.

C. For each abnormal extremal (namely $(x(.), y(.), P(.), p^0)$, with $p^0 = 0$) found in part B, say if it is also normal, i.e., if there exists $\bar{P}(.)$ and $p^0 \neq 0$ such that $(x(.), y(.), \bar{P}(.), p^0)$ is a normal extremal.

5 Non-linear Controllability

Consider the following control system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y\, f(x) \\ -x\, f(x) \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $f(x)$ is a never vanishing smooth function and $u \in \mathbf{R}$.

A. Is it controllable? Is it small-time locally controllable?

6 Optimal Control with drift

Consider the control system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad u_1, u_2 \in \mathbf{R}$$

A. Is it controllable? Is it small-time locally controllable?

Consider the problem of minimizing the cost $\int_0^1 (u_1^2 + u_2^2)\,dt$ with fixed initial and final points. We assume that there is existence of optimal trajectories.

B. Are there abnormal extremals?

C. Find the optimal trajectory trajectory defined on $[0, 1]$ joining $(0, 0)$ with $(1, 0)$. Is it unique?