MAP 561 - Automatique
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1 Linear system

Consider the linear system

$$\begin{aligned}\dot{x}_1 &= 8x_2 + 5x_3 + 2u,\\ \dot{x}_2 &= 3x_1 - 4x_3,\\ \dot{x}_3 &= 6x_1 + u.\end{aligned}$$

A. Study its controllability.

B. Find a Brunovsky output and determine a control steering the point $(1, 1, 1)$ to the point $(2, 3, 4)$.

C. Consider the output map $y = x_2$. Is the system observable?

D. Is the system stabilizable by a feedback linear in the output? If not, build an asymptotic observer and a linear dynamic feedback stabilizing the system.

2 Stability of a neural network

Consider the n-dimensional system given by

$$\dot{x}_i = h_i(x_i)[-\sum_j T_{ij}x_j + g_i(x_i) + I_i], \quad 1 \leq i \leq n, \tag{1}$$

where $T := (T_{ij})$ is a real-symmetric positive definite function, $I := (I_i)$ is a constant vector and h_i, g_i are smooth functions such that $h_i > 0$, and the g_i's are bounded over $\mathbf{R}^+$. Define the vector field G on $\mathbf{R}^n$ by $G(x) = (g_i(x_i))_{i=1\ldots n}$.

In the sequel, we will assume that this system admits equilibrium points, which are necessarily isolated. We want to prove that trajectories of the system are defined for all positive times and each of them converges to an equilibrium point.

A. Show that the vector field F defined on $\mathbf{R}^n$ by $F(x) = Tx - G(x) - I$ is a gradient vector field, i.e., there exists a smooth real-valued function V defined on $\mathbf{R}^n$ such that $\nabla V(x) = F(x)$ for every $x \in \mathbf{R}^n$.

B. Prove that V is bounded below and radially unbounded.

C. Prove the thesis. Write precisely your arguments.

3 Nonlinear controllability

Consider the following control system on the plane,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x+1 \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ x \end{pmatrix} + u_2 \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$$

Here u_1, u_2 are L^∞ functions taking values in $\mathbf{R}$ and

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

A. Find $\mathcal{A}(1,0)$, i.e. the reachable set starting from the point $(1,0)$.

B. Find the set of points that can be reached in an arbitrarily small time i.e.

$$\begin{aligned} \mathcal{A}^+(1,0) \quad := \quad & \{(\bar{x},\bar{y}) \in \mathbf{R^2} \text{ such that for every } T>0, \exists \text{ an admissible trajectory} \\ & (x(.),y(.)) \text{ defined on } [0,T] \text{ such that } (x(0),y(0)) = (1,0),\ (x(T),y(T)) = (\bar{x},\bar{y})\}. \end{aligned}$$

4 Time optimal stabilization to a target

Consider the following optimal control problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad T \to \min,$$
$$(x(0),y(0)) = (\bar{x},\bar{y}), \quad (x(T),y(T)) \in \tau := \{(x,y) \in \mathbf{R}^2 \mid x = 0, |y| \leq 1\}.$$

Here u_1 and u_2 are L^∞ functions satisfying $u_1(t)^2 + u_2(t)^2 \leq 1$ for almost every t.

A. Transform the problem in a sub-Riemannian problem with trajectories parameterised by arc-length.

B. By inverting the time, transform the problem in an optimal synthesis problem starting from the set τ.

C. What are the transversality conditions satisfied by the initial covectors?

D. Find all extremals satisfying the transversality conditions.

E Are all extremals optimal?

F. Draw a picture of the solution to the original problem, i.e. the optimal stabilising synthesis. In other words, for each $(\bar{x},\bar{y})$ draw the time optimal trajectory starting from $(\bar{x},\bar{y})$ and reaching the origin in minimum time for the original control system).

5 Sub-Riemannian problem

Consider the following sub-Riemannian problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \int_0^T (u_1(t)^2 + u_2(t)^2 + u_3(t)^2)\, dt \to \min,$$
$$(x(0), y(0), z(0)) = (0,0,0), \quad (x(T), y(T), z(T)) = (x_1, y_1, z_1).$$

Here u_1, u_2, u_3 are L^∞ functions taking values in $\mathbf{R}$. The final time T is fixed in such a way that trajectories are parameterised by arc length ($u_1(t)^2 + u_2(t)^2 + u_3(t)^2 = 1$).

A. Is it controllable? Is it small-time locally controllable ? (i.e. in an arbitrary small time is it possible to reach a neighborhood of the starting point ?)

B. Are there abnormal extremals? If yes describe them.

D. Find all extremals starting from the origin.

D. Find where extremals starting from the origin stop to be optimal (facultative).

E. Draw the optimal synthesis (i.e. the collection of all optimal trajectories) starting from the origin (facultative).

6 Minimum time with one bounded control

Consider the following optimal control problem on the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{3}y^3 \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad T \to \min,$$
$$(x(0), y(0)) = (0,0), \quad (x(T), y(T)) = (x_1, y_1).$$

Here u is a L^∞ function taking values in the interval $[-1, 1]$.

A. Is it controllable? Is it small-time locally controllable ?

B. Compute the functions Δ_A, Δ_B and f_s and divide the plane in regions where only certain commutations are admitted.

C. Find all extremals starting from the origin. Are there singular trajectories ?

E. Draw the time-optimal synthesis starting from the origin.