MAP 561 - Automatique
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1 Linear control system

Some brain activities are performed by interaction between several neuronal populations. A simple modeling of interaction between Population 1 which is exciting and Population 2 which is inhibiting is given as follows :

$$\dot{x}_1 = -\frac{1}{\tau_1}x_1 + \frac{M_1}{\tau_1}s(-a_1x_2 + b_1u) \tag{1}$$

$$\dot{x}_2 = -\frac{1}{\tau_2}x_2 + \frac{M_2}{\tau_2}s(a_2x_1), \tag{2}$$

where $\tau_1, \tau_2 > 0$ are time constants associated to each population, $a_1, a_2 > 0$ are gains for synaptic interconnexion between the two populations, $M_1, M_2 > 0$ are amplitudes of the activation functions, $x_1(t)$ and $x_2(t)$ represent respective instantaneous activity of each population (number of action potentials by time unit), $u(t)$ is a control signal coming from an electrical stimulation device and $b_1 > 0$ measures the impact of such a stimulation on Population 1. The function s is the saturation function

$$s(x) = \begin{cases} 1 & \text{si} \quad x \geq 1 \\ x & \text{si} \quad x \in [-1; 1] \\ -1 & \text{si} \quad x \leq -1. \end{cases}$$

In some cases, the interaction between these populations can lead to spurious oscillations. This is the case for instance in Parkinson's disease characterized by oscillations comprised between 20 and 30Hz. These oscillations are damaging when their frequencies are not in this interval. The goal consists here in designing a feedback law which attenuates these pathological oscillations.

1. For a constant input u^*, give the equilibrium point (x_1^*, x_2^*) verifying $x_1^* \in]-M_1; M_1[$ and $x_2^* \in]-M_2; M_2[$.
2. In the sequel we assume that $u^* = 0$. Show that the linearized system at this equilibrium point is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1/\tau_1 & -M_1a_1/\tau_1 \\ M_2a_2/\tau_2 & -1/\tau_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} M_1/\tau_1 \\ 0 \end{pmatrix} u. \tag{3}$$

3. Is this system completely controllable ?
4. To counter the pathological oscillations, propose a feedback $u(t) = -Kx(t)$ so that the roots of the closed-loop system are $-\xi \pm i\omega$, with $\xi = 0.6$ and $\omega = 60\,\text{rad}.s^{-1}$.

5. Assume that one can only measure the evolution of Population 1. Is the corresponding input-output system completely observable ? If yes, build an asymptotic observer recovering the properties of the previous feedback law ?

2 Stability of a non linear system

Let N be a positive integer. We are looking for stability conditions on the constants $\tau_i > 0$, $i = 1, \cdots, 3$ and the $N \times N$ real symmetric matrix W of the following system

$$\begin{aligned} \tau_1 \frac{dx_1(t)}{dt} &= -x_1(t) + Wx_1 - S(x_3(t)) \\ \tau_2 \frac{dx_2(t)}{dt} &= -x_2(t) + x_1(t) \\ \tau_3 \frac{dx_3(t)}{dt} &= x_2(t) \end{aligned}$$

where

— each $x_i(t)$ is a N dimensional vector, describing the dynamical system. The system is then $3N$-dimensional.

— $S(y) = (s(y_1), \cdots, s(y_N))$ where $s(\cdot)$ is the saturation function given in the exercice "Linear control system" ;

1. Compute the equilibrium points and find the conditions on $\tau_i > 0$, $i = 1, \cdots, 3$ and W insuring check local asymptotic stability.
2. Do you find stronger conditions on $\tau_i > 0$, $i = 1, \cdots, 3$ and W to insure the stability of the full non linear system ? One could use a Lyapunov function.

3 Controllability

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a smooth function such that $f(x, y) > 0$ for every $(x, y) \in \mathbf{R}^2$. Consider the control system on $\mathbf{R}^2$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y\, f(x,y) \\ (-x+1)f(x,y) \end{pmatrix} + u \begin{pmatrix} 0 \\ x \end{pmatrix}$$

where u is L^∞ functions taking values in $[-1, 1]$. For every $(x_0, y_0) \in \mathbf{R}^2$, find the reachable set $\mathcal{A}_{(x_0,y_0)}$.

4 Sub-Riemannian

Consider the control system on $\mathbf{R}^2$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} x \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

where u_1, u_2 are L^∞ functions taking values in $\mathbf{R}$.

— **Q1** For every $(x_0, y_0) \in \mathbf{R}^2$ describe the reachable set $\mathcal{A}_{(x_0,y_0)}$.

Consider the cost (with fixed initial and terminal conditions)

$$\int_0^T \sqrt{u_1^2 + u_2^2}\, dt \to \min, \quad T \text{ free} \tag{5}$$

- **Q2** Are there abnormal extremals ? If yes, describe all of them.
- **Q3** For arclength parametrized trajectories, write the Hamiltonian equations given by the PMP and find their solutions as function of the initial condition and of the initial covector. (recall the the Hamiltonian is a constant of the motion to integrate them).
- **Q4** Find all arclength-parameterized solutions of the PMP which are candidates to minimize the distance (4) from the circle $\mathcal{S} = \{x^2+y^2 = 1\}$ (apply transversality conditions for trajectories going inside and outside the circle).
- **Q5** Can you say something about optimality of these trajectories ?

5 Minimum time

Consider the control system on the plane :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ x\,y \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where u is a L^∞ function taking values in $[-1, 1]$.

- **Q0** Is it controllable ? Find the reachable set starting from $(1, 1)$

Consider the problem of finding the time optimal synthesis starting from $(1, 1)$ (i.e. all optimal trajectories starting from $(1, 1)$).

- **Q1** Are there singular trajectories ?
- **Q2** Compute and draw the two trajectories corresponding to control $+1$ and -1 starting from $(1, 1)$.
- **Q3** Compute the functions Δ_A and Δ_B and study which switching are permitted in the different connected components of $\mathbf{R}^2 \setminus \{\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)\}$.
- **Q4** For every value of the initial covector $(p_1(0), p_2(0))$ find explicitly the switching function and the corresponding extremal trajectory. Draw it.
- **Q5** Are all solutions of the PMP optimal ?
- **Q6** Draw the time optimal synthesis.