MAP 561 - Automatique
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1 Stability

Consider the differential equation on $\mathbb{R}^n$ given by

$$(S): \quad \dot{x} = -Q\phi(x),$$

where Q is a positive definite symmetric matrix and ϕ is a function of class C^1 such that, for every $x = (x_1, \cdots, x_n)^T$ in $\mathbb{R}^n$,

$$\phi(x) = (\phi_1(x_1), \cdots, \phi_n(x_n)).$$

Suppose that $\phi(0) = 0$ and there exists an open neighborhood D of $0 \in \mathbb{R}^n$ such that, for every $1 \leq i \leq n$ and $y \in D$ non zero, we have $y\phi_i(y) > 0$.

Q1 Show that 0 is locally asymptotically stable.

Q2 Under which conditions one has that 0 is globally asymptotically stable?

Q3 Application for $n = 2$. Take

$$\phi_1(x_1) = x_1 - x_1^2, \quad \phi_2(x_2) = x_2 + x_2^3, \quad Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Is the origin locally asymptotically stable? Is the origin globally asymptotically stable?

2 Linear System

Consider the linear control system given by

$$\begin{aligned} \dot{x}_1 &= 2x_2 + 3x_3 + u, \\ \dot{x}_2 &= 4x_1 + 5x_3, \\ \dot{x}_3 &= 6x_1 + u. \end{aligned}$$

Q1 Study its controllability.

Q2 Find a Brunovsky output (if possible).

Q3 Determine a control law steering $(0, 0, 0)$ to $(2, 0, 1)$ in time 1.

Q4 Assume that the output map is $y = x_2$. Is the system observable?

Q5 Recall what a static feedback law is for the above control system together with the output map y. Show that such a control system cannot be stabilized with a static feedback law. Build an asymptotic observer and a dynamic feedback law which stabilizes the system.

3 Controllability

Consider the control system on $\mathbf{R}^3$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix} + u_1(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + u_2(t) \begin{pmatrix} 0 \\ z \\ x^2 z \end{pmatrix} \tag{1}$$

where $u_1, u_2 \in L^\infty(\mathbf{R}, \mathbf{R})$.

Q1 For every initial condition $(x_0, y_0, z_0) \in \mathbf{R}^3$, find the reachable set $A_{(x_0,y_0,z_0)}$.

4 Minimum time for a linear system

Consider the control system on the plane:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} + u(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where u is a L^∞ function taking values in $[-1, 1]$.

Q1 Compute and draw the minimum-time optimal synthesis starting from the set $\{y = 0\}$, i.e., all time optimal trajectories starting from $\{y = 0\}$ and satisfying the transversality conditions.

5 Sub-Riemannian

Consider the sub-Riemannian problem on $(0, \infty) \times (0, \infty)$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} \sqrt{x} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ \sqrt{y} \end{pmatrix}, \quad \int_0^T (u_1(t)^2 + u_2(t)^2) dt \to \min \tag{2}$$

where $u_1, u_2 \in L^\infty(\mathbf{R}, \mathbf{R})$. The initial and final point are fixed and T is fixed in such a way that the Hamiltonian given by the Pontryagin Maximum Principle is equal to $1/2$ (i.e. in such a way that $u_1(t)^2 + u_2(t)^2 = 1$).

Q1 Are there abnormal extremals?

Q2 Find the optimal synthesis (i.e. the collection of all optimal trajectoiries) starting from the point $(1,1)$.

Q3 Let now consider the system (2) on $\mathbf{R}^2$. In this case the vector fields are not smooth and to one control and initial condition it may correspond more than one trajectory (no uniqueness of the Cauchy problem). Can we say that the system is controllable in the sense that for every pair of points there exists continuous controls and a corresponding C^1 trajectory connecting the two points?

6 Minimum time for a non-linear system

Consider the control system on the plane:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 + \frac{1}{2}x^2 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where u is a L^∞ function taking values in $[-1,1]$.

Q0 Is it controllable?

Consider the problem of finding the time optimal synthesis starting from the origin.

Q1 Are there singular trajectories?

Q2 Compute and draw the two trajectories corresponding to control $+1$ and -1 starting from the origin.

Q3 Compute the functions Δ_A and Δ_B and study which switching are permitted in the different connected components of $\mathbf{R}^2 \setminus \{\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)\}$.

Q4 Compute the reachable set from the origin.

Q5 Compute and draw the time optimal synthesis. (facultative)