MAP 561 - Automatique
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Documents are allowed. Pictures are welcome.

1 Controllability

Consider the control system in $\mathbf{R}^3$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + u(t) \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$$

where $u(\cdot)$ is a L^∞ function taking values in $[-1, 1]$.

Q1 Compute the reachable set $\mathcal{R}$ starting from the point $(0, 0, 1)$.

2 Stabilisation via backstepping

In this exercise we would like to stabilize globally the the system (S)

$$\dot{x} = z^3,$$
$$\dot{z} = u,$$

via a technique called "backstepping". Here x, z, u are reals.

Q1 We consider first the system $\dot{x} = u^3$. Show that this system is globally stabilizable in zero with the feedback $u = -x$.

Q2 Let $V(x) = x^2$ and $W(x, z) = V(x) + (z + x)^2/2$. Show that one can choose $u = k(x, z)$ polynomial such that W is a strict Lyapunov function for the system (S) with the feedback $u = k(x, z)$.

3 Quadratic cost

Consider the control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix} + u(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $u(.)$ is a L^∞ function taking values in $\mathbf{R}$.

Q1 Is it controllable?

Consider the problem of minimizing $\int_0^T u(t)^2$ with $(x(0), y(0)) = (0, 0)$, $(x(T), y(T)) = (\bar{x}, \bar{y})$ and T fixed. We assume that this problem admits existence of minimizers.

Q2 Are there abnormal extremals?

Q3 Find the expression of all solutions of the Pontryagin Maximum Principle starting from the origin and parameterized by the initial covector.

Q4 Find all the solutions of the Pontryagin Maximum principle with $T = 1$ satisfying $(x(0), y(0)) = (0, 0)$, $(x(1), y(1)) = (0, 1)$.

Q5 (more diffucult) Which of the solutions of the Pontryagin Maximum Principle found in **Q4** are optimal?

4 Minimum time

Consider the control system on the plane:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} + u_1(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $u_1(.), u_2(.)$ are L^∞ function taking values in $[-1, 1]$.

Q1 Study its controllability.

Consider the problem of minimizing the time (with initial condition fixed at $(0, 0)$ and fixed terminal condition).

Q2 Write the maximization condition given by the Pontryagin Maximum Principle and find the controls as function of the covector $p(t) = (p_1(t), p_2(t))$. Solve explicitly the equation for the covector in term of the initial condition $(p_1(0), p_2(0))$.

Q3 For each initial convector $(p_1(0), p_2(0))$, compute the corresponding solution of the PMP starting from the point $(0, 0)$ and draw it.

Q4 Are all solutions of the Pontryagin Maximum Principle optimal? Draw the time optimal synthesis.

5 Règlage du pilotage d'un missile

See attached pages