An Introduction to Optimal Control

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The aim of these notes is to give an introduction to the Theory of Optimal Control for finite dimensional systems and in particular to the use of the Pontryagin Maximum Principle towards the construction of an Optimal Synthesis. In Section 1, we introduce the definition of Optimal Control problem and give a simple example. In Section 2 we recall some basics of geometric control theory as vector fields, Lie bracket and controllability. In Section 3, that is the core of these notes, we introduce Optimal Control as a generalization of Calculus of Variations and we discuss why, if we try to write the problem in Hamiltonian form, the dynamics makes the Legendre transformation not well defined in general. Then we briefly introduce the problem of existence of minimizers and state the Pontryagin Maximum Principle. As an application we consider a classical problem of Calculus of Variations and show how to derive the Euler Lagrange equations and the Weierstraß condition. Then we discuss the difficulties in finding a complete solution to an Optimal Control Problem and how to attack it with geometric methods. In Section 4 we give a brief introduction to the theory of Time Optimal Synthesis on two dimensional manifolds developed in [14]. We end with a bibliographical note and some exercises.

1 Introduction

Control Theory deals with systems that can be *controlled*, i.e. whose evolution can be influenced by some external agent. Here we consider control systems that can be defined as a system of differential equations depending on some parameters $u \in U \subseteq \mathbb{R}^m$:

$$\dot{x} = f(x, u), \tag{1}$$

where x belongs to some n–dimensional smooth manifold or, in particular, to $\mathbb{R}^n$. For each initial point x_0 there are many trajectories depending on the choice of the control parameters u.

One usually distinguishes two different ways of choosing the control:

- *open loop*. Choose u as function of time t,
- *closed loop* or *Feedback*. Choose u as function of space variable x.

The first problem one faces is the study of the set of points that can be reached, from x_0, using open loop controls. This is also known as the *controllability* problem.

If controllability to a final point x_1 is granted, one can try to reach x_1 minimizing some cost, thus defining an Optimal Control Problem:

$$\min \int_0^T L(x(t), u(t))\, dt, \qquad x(0) = x_0,\ x(T) = x_1, \tag{2}$$

where $L : \mathbb{R}^n \times U \to \mathbb{R}$ is the *Lagrangian* or *running cost*. To have a precise definition of the Optimal Control Problem one should specify further: the time T fixed or free, the set of admissible controls and admissible trajectories, etc. Moreover one can fix an initial (and/or a final) set, instead than the point x_0 (and x_1).

Fixing the initial point x_0 and letting the final condition x_1 vary in some domain of $\mathbb{R}^n$, we get a family of Optimal Control Problems. Similarly we can fix x_1 and let x_0 vary. One main issue is to introduce a concept of solution for this family of problems and we choose that of Optimal Synthesis. Roughly speaking, an Optimal Synthesis is a collection of optimal trajectories starting from x_0, one for each final condition x_1. As explained later, building an Optimal Synthesis is in general extremely difficult, but geometric techniques provide a systematic method to attack the problem.

In Section 3.1 Optimal Control is presented as a generalization of Calculus of Variations subjects to nonholonomic constraints.

Example Assume to have a point of unitary mass moving on a one dimensional line and to control an external bounded force. We get the control system:

$$\ddot{x} = u, \qquad x \in \mathbb{R},\ |u| \leq C,$$

where x is the position of the point, u is the control and C is a given positive constant. Setting $x_1 = x$, $x_2 = \dot{x}$ and, for simplicity, $C = 1$, in the phase space the system is written as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u. \end{cases}$$

One simple problem is to drive the point to the origin with zero velocity in minimum time. From an initial position $(\bar{x}_1, \bar{x}_2)$ it is quite easy to see that the optimal strategy is to accelerate towards the origin with maximum force on some interval $[0, t]$ and then to decelerate with maximum force to reach the origin at velocity zero. The set of optimal trajectories is depicted in Figure 1.A: this is the simplest example of Optimal Synthesis for two dimensional systems. Notice that this set of trajectories can be obtained using the following *feedback*, see Figure 1.B. Define the curves $\zeta^{\pm} = \{(x_1, x_2) : \mp x_2 > 0, x_1 = \pm x_2^2\}$ and let ζ be defined as the union $\zeta^{\pm} \cup \{0\}$. We define A^+ to be the region below ζ and A^- the one above. Then the feedback is given by:

$$u(x) = \begin{cases} +1 & if\ (x_1, x_2) \in A^+ \cup \zeta^+ \\ -1 & if\ (x_1, x_2) \in A^- \cup \zeta^- \\ 0 & if\ (x_1, x_2) = (0, 0). \end{cases}$$

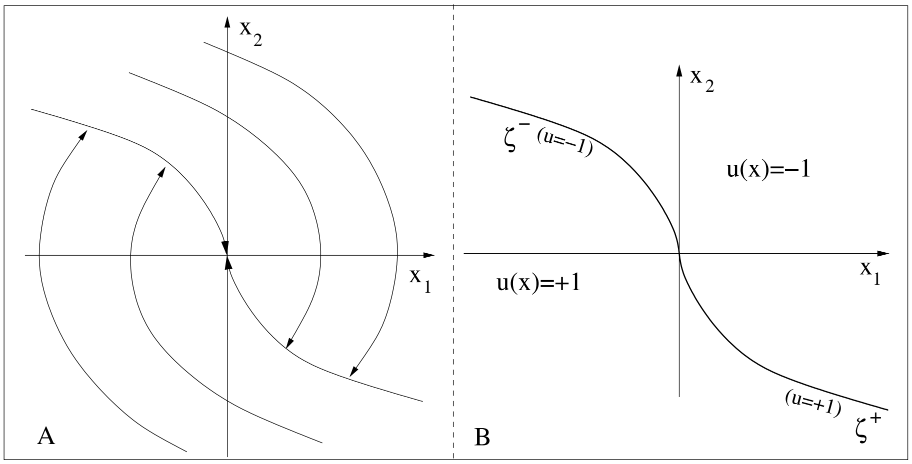

Figure 1: The simplest example of Optimal Synthesis and corresponding feedback.

Notice that the feedback u is discontinuous.

2 Basic Facts on Geometric control

This Section provides some basic facts about Geometric Control Theory. This is a brief introduction that is far from being complete: we illustrate some of the main available results of the theory, with few sketches of proofs. For a more detailed treatment of the subject, we refer the reader to the monographs [3, 29].

Consider a control system of type (1), where x takes values on some manifold M and $u \in U$. Along these notes, to have simplified statements and proofs, we assume more regularity on M and U:

(H0) M is a closed n-dimensional submanifold of $\mathbb{R}^N$ for some $N \geq n$. The set U is a measurable subset of $\mathbb{R}^m$ and f is continuous, smooth with respect to x with Jacobian, with respect to x, continuous in both variables on every chart of M.

A point of view, very useful in geometric control, is to think a control system as a family of assigned vector fields on a manifold:

$$\mathcal{F} = \{F_u(\cdot) = f(\cdot, u)\}_{u \in U}.$$

We always consider smooth vector fields, on a smooth manifold M, i.e. smooth mappings $F : x \in M \mapsto F(x) \in T_xM$, where T_xM is the tangent space to M at x. A vector field can be seen as an operator from the set of smooth functions on M to $\mathbb{R}$. If $x = (x_1, ..., x_n)$ is a local system of coordinates, we have:

$$F(x) = \sum_{i=1}^{n} F^i \frac{\partial}{\partial x^i}.$$

The first definition we need is of the concept of *control* and of *trajectory* of a control system.

Definition 1 *A control is a bounded measurable function $u(\cdot) : [a, b] \to U$. A trajectory of (1) corresponding to $u(\cdot)$ is a map $\gamma(\cdot) : [a, b] \to M$, Lipschitz continuous on every chart, such that (1) is satisfied for almost every $t \in [a, b]$. We write $Dom(\gamma)$, $Supp(\gamma)$ to indicate respectively the domain and the support of $\gamma(\cdot)$. The initial point of γ is denoted by $In(\gamma) = \gamma(a)$, while its terminal point $Term(\gamma) = \gamma(b)$*

Then we need the notion of reachable set from a point $x_0 \in M$.

Definition 2 *We call* reachable set within time $T > 0$ *the following set:*

$$\begin{aligned} \mathcal{R}_{x_0}(T) \quad := \quad & \{x \in M : \textit{ there exists } t \in [0, T] \textit{ and a trajectory} \\ & \gamma : [0, t] \to M \textit{ of (1) such that } \gamma(0) = x_0,\ \gamma(t) = x\}. \end{aligned} \tag{3}$$

Computing the reachable set of a control system of the type (1) is one of the main issues of control theory. In particular the problem of proving that $\mathcal{R}_{x_0}(\infty)$ coincide with the whole space is the so called *controllability problem*. The corresponding local property is formulated as:

Definition 3 (Local Controllability) *A control system is said to be locally controllable at x_0 if for every $T > 0$ the set $\mathcal{R}_{x_0}(T)$ is a neighborhood of x_0.*

Various results were proved about controllability and local controllability. We only recall some definitions and theorems used in the sequel.

Most of the information about controllability is contained in the structure of the Lie algebra generated by the family of vector fields. We start giving the definition of Lie bracket of two vector fields.

Definition 4 (Lie Bracket) *Given two smooth vector fields X, Y on a smooth manifold M, the Lie bracket is the vector field given by:*

$$[X, Y](f) := X(Y(f)) - Y(X(f)).$$

In local coordinates:

$$[X, Y]_j = \sum_i \left(\frac{\partial Y_j}{\partial x_i} X_i - \frac{\partial X_j}{\partial x_i} Y_i \right).$$

In matrix notation, defining $\nabla Y := \Big(\partial Y_j / \partial x_i\Big)_{(j,i)}$ (j row, i column) and thinking to a vector field as a column vector we have $[X, Y] = \nabla Y \cdot X - \nabla X \cdot Y$.

Definition 5 (Lie Algebra of $\mathcal{F}$) *Let $\mathcal{F}$ be a family of smooth vector fields on a smooth manifold M and denote by $\chi(M)$ the set of all $\mathcal{C}^\infty$ vector fields on M. The Lie algebra $Lie(\mathcal{F})$ generated by $\mathcal{F}$ is the smallest Lie subalgebra of $\chi(M)$ containing $\mathcal{F}$. Moreover for every $x \in M$ we define:*

$$Lie_x(\mathcal{F}) := \{X(x) : \ X \in Lie(\mathcal{F})\}. \tag{4}$$

Remark **1** In general $Lie(\mathcal{F})$ is a infinite-dimensional subspace of $\chi(M)$. On the other side since all $X(x) \in T_xM$ (in formula (4)) we have that $Lie_x(\mathcal{F}) \subseteq T_xM$ and hence $Lie_x(\mathcal{F})$ is finite dimensional.

Remark **2** $Lie(\mathcal{F})$ is built in the following way. Define: $D_1 = Span\{\mathcal{F}\}$, $D_2 = Span\{D_1 + [D_1, D_1]\}, \cdots D_k = Span\{D_{k-1} + [D_{k-1}, D_{k-1}]\}$. D_1 is the so called *distribution generated by* $\mathcal{F}$ and we have $Lie(\mathcal{F}) = \cup_{k \geq 1} D_k$. Notice that $D_{k-1} \subseteq D_k$. Moreover if $[D_n, D_n] \subseteq D_n$ for some n, then $D_k = D_n$ for every $k \geq n$.

A very important class of families of vector fields are the so called *Lie bracket generating* (or completely *nonholonomic*) systems for which:

$$Lie_x\mathcal{F} = T_xM, \ \ \forall\, x \in M. \tag{5}$$

For instance *analytic systems* (i.e. with M and $\mathcal{F}$ analytic) are always Lie bracket generating on a suitable immersed analytic submanifold of M (the so called *orbit* of $\mathcal{F}$). This is the well know Hermann-Nagano Theorem (see for instance [29], pp. 48).

If the system is *symmetric*, that is $\mathcal{F} = -\mathcal{F}$ (i.e. $f \in \mathcal{F} \Rightarrow -f \in \mathcal{F}$), then the controllability problem is more simple. For instance condition (5) with M connected implies complete controllability i.e. for each $x_0 \in M$, $\mathcal{R}_{x_0}(\infty) = M$ (this is a corollary of the well know Chow Theorem, see for instance [3]).

On the other side, if the system is not symmetric (as for the problem treated in Section 4), the controllability problem is more complicated and controllability is not guaranteed in general (by (5) or other simple conditions), neither locally. Anyway, important properties of the reachable set for Lie bracket generating systems are given by the following theorem (see [37] and [3]):

Theorem 1 (Krener) *Let $\mathcal{F}$ be a family of smooth vector fields on a smooth manifold M. If $\mathcal{F}$ is Lie bracket generating, then, for every $T \in]0, +\infty]$, $\mathcal{R}_{x_0}(T) \subseteq Clos(Int(\mathcal{R}_{x_0}(T))$. Here $Clos(\cdot)$ and $Int(\cdot)$ are taken with respect to the topology of M.*

Krener Theorem implies that the reachable set for Lie bracket generating systems has the following properties:

- It has nonempty interior: $Int(\mathcal{R}_{x_0}(T)) \neq \emptyset, \ \ \forall\, T \in]0, +\infty]$.
- Typically it is a manifold with or without boundary of full dimension. The boundary may be not smooth, e.g. have corners or cuspidal points.

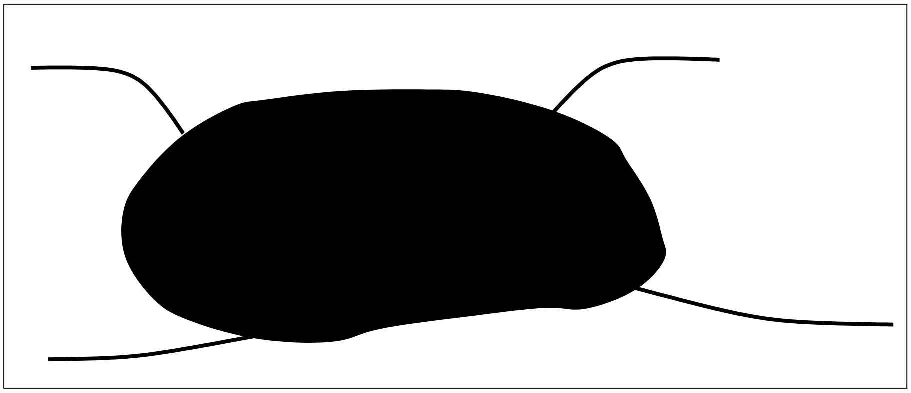

Figure 2: A prohibited reachable set for a Lie bracket generating systems.

In particular it is prohibited that reachable sets are collections of sets of different dimensions as in Figure 2. These phenomena happen for non Lie bracket generating systems, and it is not know if reachable sets may fail to be stratified sets (for generic smooth systems) see [28, 29, 54].

Local controllability can be detached by linearization as shown by the following important result (see [40], p. 366):

Theorem 2 *Consider the control system $\dot{x} = f(x, u)$ where x belongs to a smooth manifold M of dimension n and let $u \in U$ where U is a subset of $\mathbb{R}^m$ for some m, containing an open neighborhood of $u_0 \in \mathbb{R}^m$. Assume f of class $\mathcal{C}^1$ with respect to x and u. If the following holds:*

$$\begin{aligned} & f(x_0, u_0) = 0, \\ & rank[B, AB, A^2B, ..., A^{n-1}B] = n, \\ & \textit{where } A = \big(\partial f/\partial x\big)(x_0, u_0) \textit{ and } B = \big(\partial f/\partial u\big)(x_0, u_0), \end{aligned} \tag{6}$$

then the system is locally controllable at x_0.

Remark **3** Condition (6) is the well know *Kalman condition* that is a necessary and sufficient condition for (global) controllability of *linear systems*:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ A \in \mathbb{R}^{n\times n}, \ B \in \mathbb{R}^{n\times m}, \ u \in \mathbb{R}^m.$$

In the local controllable case we get this further property of reachable sets:

Lemma 1 *Consider the control system $\dot{x} = f(x, u)$ where x belongs to a smooth manifold M of dimension n and let $u \in U$ where U is a subset of R^m for some m.*

Assume f of class $\mathcal{C}^1$ with respect to x and continuous with respect to u. If the control system is locally controllable at x_0 then for every $T, \varepsilon > 0$ one has:

$$\mathcal{R}_{x_0}(T) \subseteq Int(\mathcal{R}_{x_0}(T+\varepsilon)). \tag{7}$$

Proof. Consider $x \in \mathcal{R}_{x_0}(T)$ and let $u_x : [0,T] \to U$ be such that the corresponding trajectory starting from x_0 reaches x at time T. Moreover, let Φ_t be the flux associated to the time varying vector field $f(\cdot, u(t))$ and notice that Φ_t is a diffeomorphism. By local controllability at x_0, $\mathcal{R}_{x_0}(\varepsilon)$ is a neighborhood of x_0. Thus $\Phi_T(\mathcal{R}_{x_0}(\varepsilon))$ is a neighborhood of x and, using $\Phi_T(\mathcal{R}_{x_0}(\varepsilon)) \subset \mathcal{R}_{x_0}(T+\varepsilon)$, we conclude.□

3 Optimal Control

In this section we give an introduction to the theory of Optimal Control. Optimal Control can be seen as a generalization of the Classical Calculus of Variations ($min \int_0^T L(x(t), \dot{x}(t))$) to systems with *nonholonomic* constrains of the kind $\dot{x} = f(x,u), \quad u \in U$.

Remark **4** We recall that a constraint on the velocity is said to be nonholonomic if it cannot be obtained as consequence of a (holonomic) constraint on the position of the kind:

$$\psi_i(x) = 0, \;\; i = 1, ..., n', \;\; n' < n.$$

where the real functions $\psi_i(x)$ are sufficiently regular to define a submanifold $M' \subset M$. Clearly since holonomic constraints can be eliminated simply by restricting the problem to M', the interesting case is the nonholonomic one. In the following when we speak about nonholonomic constraints we always refer to nonholonomic constraints of the kind $\dot{x} = f(x,u), \;\; u \in U$.

The most important and powerful tool to look for an explicit solution to an Optimal Control Problem is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [3, 29, 49]) that give a first order necessary condition for optimality. PMP is very powerful for the following reasons:

- it generalizes the Euler Lagrange equations and the Weierstraß condition of Calculus of Variations to variational problems with nonholonomic constrains;
- it provides a pseudo-Hamiltonian formulation of the variational problem in the case in which the standard Legendre transformation is not well defined (as in the case of Optimal Control, see below).

Roughly speaking PMP says the following. If a trajectory of a control system is a minimizer, then it has a lift to the cotangent bundle, formed by vector-covector pairs, such that:

- it is a solution of an pseudo-Hamiltonian system,
- the pseudo Hamiltonian satisfies a suitable maximization condition.

Here we speak of a pseudo-Hamiltonian system since the Hamiltonian depends on the control (see below). In the regular cases the control is computed as function of the state and costate using the maximization condition.

It is worth to mention that giving a complete solution to an optimization problem (that for us means to give an *optimal synthesis*, see Definition 7) in general is extremely difficult for several reasons:

A the maximization condition not always provide a unique control. Moreover PMP gives a two point boundary value problem with some boundary conditions given at initial time (state) and some given at final time (state and covector);

B one is faced with the problem of integrating a pseudo–Hamiltonian system (that generically is not integrable except for very special dynamics and costs);

C a key role is played by some special classes of extremals called *abnormal* (extremals independent from the cost) and *singular* (extremals that are singularity of the End-Point Mapping). See Section 3.2.3;

D even if one is able to find all the solutions of the PMP it remains the problem of *selecting* among them the *optimal trajectories*.

Usually **A**, **B** and **C** are very complicated problems and **D** may be even more difficult. For this reason (out from the so called *linear quadratic problem*, see for instance [3, 29]) one can hope to find a complete solution of an Optimal Control Problem only in low dimensions, unless the system presents a lot of symmetries. For instance most of the problems in dimension 3 are still open also for initial and final conditions close one to the other.

In Section 3.1 we give a brief introduction to the PMP in Optimal Control as a generalization of the classical first order conditions in Calculus of Variations.

In Section 3.2 we briefly discuss the problem of existence of minimizers, state the PMP, define abnormal and singular trajectories. In Section 3.3 we show how to get, in the case of the Calculus of Variations, the Euler Lagrange equations and the Weierstraß condition. In Section 3.4, as an application we show in some detail how to compute geodesics for a famous singular Riemannian problem using the PMP. In Section 3.5 the problem of constructing an Optimal Synthesis with geometric methods is treated.

3.1 Introduction

In this Section we first recall the Euler Lagrange equations, how to transform them in an Hamiltonian form and discuss in which case this transformation is applicable (the Legendre transformation must be invertible).

In Section 3.1.2 it is shown that for nonholonomic constraints (i.e. for an Optimal Control problem with nontrivial dynamics) the Euler Lagrange equation cannot be written in standard form and the Legendre transformation is never well defined. Finally we explicate some connection with the Lagrangian formulation of mechanical systems.

Here, for simplicity, the state space is assumed to be $\mathbb{R}^n$ and not an arbitrary manifold.

3.1.1 The Legendre Transformation

Consider a standard problem in Calculus of Variations:

$$\begin{cases} \text{minimize} \quad \int_0^T L(x(t), \dot{x}(t))dt, \\ x(0) = x_0, \\ x(T) = x_T, \end{cases} \tag{8}$$

where $x = (x^1, ..., x^n) \in \mathbb{R}^n$ and $L : \mathbb{R}^{2n} \to \mathbb{R}$ is a $\mathcal{C}^2$ function. It is a standard fact that if a $\mathcal{C}^2$ minimizer $x(.)$ exists, then it must satisfy the Euler Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}, \tag{9}$$

that for our purpose is better to write more precisely as:

$$\frac{d}{dt}\left(\left.\frac{\partial L(x,u)}{\partial u^i}\right|_{(x(t),\dot{x}(t))} \right) = \left.\frac{\partial L(x,u)}{\partial x^i}\right|_{(x(t),\dot{x}(t))}. \tag{10}$$

$$\tag{11}$$

Euler Lagrange equations are second order ODEs very difficult to solve in general, also numerically. In fact, the theory of differential equation is much more developed for first order than for second order differential equations. For this reason it is often convenient to transform equations (10) into a system of ODEs of a special form (Hamiltonian equations) via the so called Legendre transformation.

The problem of finding solutions to a system of ODEs is simplified if the system admits constants of the motion (in involution). Roughly speaking this permits to decouple the problem to the corresponding level sets. The most important advantage of passing from the Lagrangian to the Hamiltonian formulation is that, in Hamiltonian form, it is easier to recognize constants of the motion.

The Legendre transformation consists in the following.

- We first reduce the system (10) of n second order ODEs to a system of $2n$ first order ODEs introducing the variable $u := \dot{x}$:

$$\begin{cases} \frac{d}{dt}\left.\frac{\partial L(x,u)}{\partial u^i}\right|_{(x(t),u(t))} = \left.\frac{\partial L(x,u)}{\partial x^i}\right|_{(x(t),u(t))} \\ \dot{x}(t) = u(t). \end{cases} \tag{12}$$

- Then we make the change of coordinates in $\mathbb{R}^n$:

$$(x,u) \to (x,p), \text{ where } p^i = \Phi^i(x,u) := \frac{\partial L(x,u)}{\partial u^i}.$$

This change of coordinates is well defined if it realizes a $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$, i.e. we must have:

$$det \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline \frac{\partial \Phi(x,u)}{\partial x} & \frac{\partial \Phi(x,u)}{\partial u}. \end{array} \right) = det \left(\frac{\partial^2 L(x,u)}{\partial u^i \partial u^j} \right) \neq 0, \qquad (13)$$

for every $(x,u) \in \mathbb{R}^{2n}$. If the condition (13) is satisfied, the Legendre transformation is said *invertible*. In this case the inverse transformation is $u = \Phi^{-1}(x,p)$ and the Lagrangian is called *regular*.

- Define the function (called Hamiltonian):

$$H(x,p) := p\Phi^{-1}(x,p) - L(x, \Phi^{-1}(x,p)). \qquad (14)$$

In the (x,u) coordinates the Hamiltonian takes the form $\frac{\partial L}{\partial u} u - L(x,u)$ and usually one remembers it in the "mixed coordinates" form $pu - L$.

After the Legendre transformation (if it is invertible), the Euler Lagrange equations are written as the Hamiltonian equations:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \qquad (15)$$

In fact using carefully the chain rule we have for the first:

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial p}\left(p\Phi^{-1}(x,p) - L(x,\Phi^{-1}(x,p))\right) \\ &= \Phi^{-1}(x,p) + p\frac{\partial \Phi^{-1}(x,p)}{\partial p} - \left.\frac{\partial L(x,u)}{\partial u}\right|_{(x,\Phi^{-1}(x,p))} \frac{\partial \Phi^{-1}(x,p)}{\partial p}, \end{aligned}$$

and using the fact that $u = \Phi^{-1}(x,p)$ and $p = \frac{\partial L}{\partial u}$ we get the second of (12). Similarly for the second of (15) we have:

$$\begin{aligned} \dot{p} &= -\frac{\partial}{\partial x}\left(p\Phi^{-1}(x,p) - L(x,\Phi^{-1}(x,p))\right) \\ &= -\left(p\frac{\partial \Phi^{-1}(x,p)}{\partial x} - \frac{\partial L}{\partial x} - \left.\frac{\partial L(x,u)}{\partial u}\right|_{(x,\Phi^{-1}(x,p))} \frac{\partial \Phi^{-1}(x,p)}{\partial x}\right). \end{aligned}$$

Again using the fact that $p = \frac{\partial L}{\partial u}$ we get the first of (12) and hence (10).

3.1.2 Optimal Control as a Generalization of Calculus of Variations

An Optimal Control Problem can be thought as a generalization of a problem of Calculus of Variations (8) in the case in which:

- a nonholonomic constraint is added (i.e. a dynamic $\dot{x} = f(x, u)$, $u \in U \subset \mathbb{R}^m$).
- the Lagrangian L is a function of (x, u) instead than function of $(x, \dot{x})$. Clearly this is a generalization, since from (x, u) we can always find $(x, \dot{x})$ using the dynamics $\dot{x} = f(x, u)$.

Usually one considers also more general costs and targets (see below), but this is out of the purpose of this introduction. The consequences of these generalizations are the following:

- In the case of nonholonomic constraints the first order necessary condition can not be written in the form (10). One can still write a Lagrange formulation of the problem using Lagrange multipliers, but this procedure works only under suitable regularity conditions (see the recent book [7] and references therein).
- One could try to write the problem in Hamiltonian form, but the dynamics $\dot{x} = f(x, u),\ \ u \in U$ (unless it is equivalent to the trivial one $\dot{x} = u$, $u \in \mathbb{R}^n$) renders the Legendre transformation not well defined. This is simply consequence of the fact that the $\dot{x}^i$'s may be not independent (or equivalently $f(x, U)$ is a proper subset of $\mathbb{R}^n$) and so operating with the Legendre transformation we may get not enough p^i's. In other words the Legendre transformation would be:

$$(x, u) \to (x, p), \ \text{where } p^i = \Phi^i(x, v) := \frac{\partial L(x, u)}{\partial \dot{u}^i}, \tag{16}$$

  but $\frac{\partial^2 L(x,u)}{\partial u^i \partial u^j}$ is not a $n \times n$ matrix (if $U \subset \mathbb{R}^m$ it is $m \times m$) and so the Legendre transformation can not be inverted.

There is a way of getting a well defined Legendre transformation also in the case of Optimal Control, starting from the Lagrange equations written with Lagrange multipliers. But again this procedure works only under suitable regularity conditions, see [7].

The Pontryagin Maximum Principle (that will be stated precisely in the next section) permits to write the minimization problem directly in a pseudo-Hamiltonian form (pseudo because the Hamiltonian still depends on the controls that must be determined with a suitable maximization condition, see Definition 8, p. 33).

Remark **5 (nonholonomic mechanics)** At this point it is worth to mention the fact that Optimal Control Theory can not be used as a variational formulation of nonholonomic mechanics. More precisely, any conservative mechanical problem, subject only

	Lagrange Formulation	Legendre Transf. $\longrightarrow$ invertible	Hamilt. Formulation	connections with mechanics
Calulus of Variation $\downarrow$ geneneralization	In standard form	if $\frac{\partial^2 L(x,v)}{\partial u \partial u} \neq 0$	in standard form, but possible only if the Legendre transf is invertible	systems with holonomic constraints
Optimal Control	NOT in standard form	the Legendre transf. is NOT well defined	in a pseudo Hamilt. form. but ALWAYS possible	NO

Figure 3:

to holonomic constraints, can be formulated as a variational problem in the following sense. Writing $L = T - V$, where T is the kinetic energy and V the potential energy, the equations the motion are the Euler Lagrange equations corresponding to L. On the other side, if we have a mechanical system subjects to nonholonomic constraints (e.g. a body rolling without slipping) then *the equations of motion are not given by the solution of the* corresponding *Optimal Control Problem*. To get a variational formulation of a nonholonomic mechanical problem, one should impose the constraints after the variation, see [7].

In the Figure 3 we collect the ideas presented in this section.

3.2 The Theory of Optimal Control

An Optimal Control Problem in Bolza form for the system (1), p. 19, is a problem of the following type:

$$\begin{cases} \text{minimize} \quad \int_0^T L(x(t), u(t))dt + \psi(x(T)), \\ x(0) = x_0, \\ x(T) \in \mathcal{T}, \end{cases} \tag{17}$$

where $L : M \times U \to \mathbb{R}$ is the *Lagrangian* or *running cost*, $\psi : M \to \mathbb{R}$ is the *final cost*, $x_0 \in M$ is the *initial condition* and $\mathcal{T} \subset M$ the *target*. The minimization is taken on the set of all *admissible* trajectories of the control system (1) (the admissibility conditions to be specified), that start at the initial condition and end at the target in finite time T that can be fixed or free, depending on the problem. Notice that this Optimal Control Problem is autonomous, hence we can always assume trajectories to be defined on some interval of the form $[0, T]$.

Of great importance for applications are the so called *control affine* systems:

$$\dot{x} = F_0 + \sum_{i=1}^{m} u_i F_i, \quad u_i \in \mathbb{R}, \tag{18}$$

with *quadratic cost*:

$$\text{minimize} \int_0^T \sum_{i=1}^{m} u_i^2 dt, \tag{19}$$

or with cost equal to 1 (that is the *minimum time* problem) and *bounded controls* $|u_i| \leq 1$. The term F_0 in (18) is called *drift* term and, for the so called *distributional systems* (or *driftless*), is equal to zero. *Subriemannian* problems are distributional systems with quadratic cost. See also Exercise 3, p. 60. *Single–input systems* are control affine systems with only one control, i.e. $m = 1$.

Definition 6 *We call $\mathcal{P}_{x_1}$, $x_1 \in M$, the Optimal Control Problem given by the dynamics (1) and the minimization (17) with $\mathcal{T} = x_1$.*

Beside (H0), (see p. 21) we make the following basic assumptions on (1) and (17).

(H1) L is continuous, smooth with respect to x with Jacobian, with respect to x, continuous in both variables on every chart of M;

(H2) ψ is a $\mathcal{C}^1$ function.

We are interested in solving the family of control problems $\{\mathcal{P}_{x_1}\}_{x_1 \in M}$ and for us a solution is given by an Optimal Synthesis that is

Definition 7 (Optimal Synthesis) *Given $\Omega \subset M$, an* optimal synthesis *on Ω for the family of Optimal Control Problems $\{\mathcal{P}_{x_1}\}_{x_1 \in \Omega}$ is a collection $\{(\gamma_{x_1}, u_{x_1}) : x_1 \in \Omega\}$ of trajectory–control pairs such that (γ_{x_1}, u_{x_1}) provides a solution to $\mathcal{P}_{x_1}$.*

Remark **6** One can invert the role of x_0 and x_1 letting x_1 be fixed and x_0 vary. Moreover, we can consider a generalization fixing an initial manifold $\mathcal{S}$, called *source*. In this case the initial condition reads $x_0 \in \mathcal{S}$.

Remark **7** In many cases it happens that an Optimal Synthesis, defined on $\mathcal{R}(T)$, is generated by a piecewise smooth feedback $u(x)$, that is a map from $\mathcal{R}(T)$ to U.

3.2.1 Existence

In order to prove existence for $\mathcal{P}_{x_1}$, we first give the next:

Theorem 3 *Assume f bounded, U compact and let the space of admissible controls be the set of all measurable maps $u(\cdot) : [a,b] \to U$. If the set of velocities $V(x) = \{f(x,u)\}_{u\in U}$ is convex, then the reachable set $\mathcal{R}(T)$ is compact.*

Remark **8** The fact that $\mathcal{R}(T)$ is relatively compact is an immediate consequence of the boundedness and continuity of f. One can also replace the boundedness with a linear grows condition. The convexity of $V(x)$ is the key property to guarantee that $\mathcal{R}(T)$ is closed. For a proof see [23].

From this theorem, we get immediately the following:

Theorem 4 *Assume f bounded, U compact, and let the space of admissible controls be the set of all measurable maps $u(\cdot) : [a,b] \to U$. Moreover assume that the set $\{f(x,u), L(x,u)\}_{u\in U}$ is convex. If $L \geq C > 0$ and $x_1 \in \mathcal{R}_{x_0}(\infty)$, then the problem $\mathcal{P}_{x_1}$ has a solution.*

Sketch of the Proof. There exists $T > 0$ such that every trajectory ending at x_1 after T is not optimal. Consider the augmented system obtained adding the extra variable y such that:

$$\dot{y} = L(x,u).$$

Since f is bounded, all trajectories defined on $[0,T]$ have bounded costs. Thus applying the previous Theorem to the augmented system, we get that $\mathcal{R}_{(x_0,0)}(T)$, the reachable set for the augmented system of (x,y), is compact. Therefore there exists a minimum for the function $(x,y) \to y + \psi(x)$ on the set $\mathcal{R}_{(x_0,0)}(T) \cap \{(x,y) : \ x = x_1\}$. The trajectory, reaching such a minimum, is optimal. $\square$

3.2.2 Pontryagin Maximum Principle

The standard tool to determine optimal trajectories is the well known Pontryagin Maximum Principle, see [3, 29, 49], that gives a first order condition for optimality.

Pontryagin Maximum Principle can be stated in several forms depending on the following:

i) the final time is fixed or free (for fixed final time see Exercise 4, p. 61),

ii) dimension and regularity of the source and of the target,

iii) the cost contains only the running part, only the final part, or both,

iv) the source and/or the target depend on time.

Here we state a version in which: **i)** the final time is free, **ii)** the source is zero dimensional and the target $\mathcal{T}$ is a smooth submanifold of M of any dimension, **iii)** there are both running and final cost, **iv)** the source and the target do not depend on time. Let us introduce some notation.

For every $(x, p, \lambda_0, u) \in T^*M \times \mathbb{R} \times U$ we define:

$$\mathcal{H}(x, p, \lambda_0, u) = \langle p, f(x, u) \rangle + \lambda_0 L(x, u), \tag{20}$$

and

$$H(x, p, \lambda_0) = \max\{\mathcal{H}(x, p, \lambda_0, u) : u \in U\}.$$

Definition 8 (Extremal Trajectories) *Consider the Optimal Control Problem (1), (17) and assume (H0), (H1) and (H2). Let* $u : [0, T] \to U$ *be a control and* γ *a corresponding trajectory. We say that* γ *is an* extremal trajectory *if there exist a Lipschitz continuous map called* covector $\lambda : t \in [0, T] \mapsto \lambda(t) \in T^*_{\gamma(t)}M$ *and a constant* $\lambda_0 \le 0$*, with* $(\lambda(t), \lambda_0) \neq (0, 0)$ *(for all* $t \in [0, T]$*), that satisfy:*

(PMP1) for a.e. $t \in [0, T]$*, in a local system of coordinates, we have*

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x}(\gamma(t), \lambda(t), \lambda_0, u(t));$$

(PMP2) for a.e. $t \in [0, T]$*, we have* $H(\gamma(t), \lambda(t), \lambda_0) = \mathcal{H}(\gamma(t), \lambda(t), \lambda_0, u(t)) = 0$*;*

(PMP3) for every $v \in T_{\gamma(T)}\mathcal{T}$*, we have* $\langle \lambda(T), v \rangle = \lambda_0 \langle \nabla \psi(\gamma(T)), v \rangle$ *(*transversality condition*).*

In this case we say that (γ, λ) *is an* extremal pair.

Pontryagin Maximum Principle (briefly PMP) states the following:

Theorem 5 (Pontryagin Maximum Principle) *Consider the Optimal Control Problem (1), (17) and assume (H0), (H1) and (H2). If* $u(\cdot)$ *is a control and* γ *a corresponding trajectory that is optimal, then* γ *is extremal.*

Remark **9** Notice that the dynamic $\dot{x} = f(x, u)$ and equation *(PMP1)* can be written in the pseudo-Hamiltonian form:

$$\begin{aligned} \dot{x}(t) &= \frac{\partial \mathcal{H}}{\partial p}(x(t), p(t), \lambda_0, u(t)), \\ \dot{p}(t) &= -\frac{\partial \mathcal{H}}{\partial x}(x(t), p(t), \lambda_0, u(t)). \end{aligned}$$

Remark **10** Notice that the couple (λ, λ_0) is defined up to a positive multiplicative factor, in the sense that if the triple $(\gamma, \lambda, \lambda_0)$ represents an extremal, than the same happens for the triple $(\gamma, \alpha\lambda, \alpha\lambda_0)$, $\alpha > 0$. If $\lambda_0 \neq 0$, usually one normalizes $(\lambda(\cdot), \lambda_0)$ by $\lambda_0 = -1/2$ or $\lambda_0 = -1$.

Remark **11** If the target $\mathcal{T}$ is a point then the transversality condition (PMP3) is empty.

Remark **12** In the case in which we have also a source: $\gamma(0) \in \mathcal{S}$, we have also a transversality condition at the source:

(PMP3') For every $v \in T_{\gamma(0)}\mathcal{S}$, $\langle \lambda(0), v\rangle = 0$.

Remark **13** The maximized Hamiltonian $H(x,p)$ can be written as

$$H(x,p) = \mathcal{H}(x,p,u(x,p)),$$

where $u(x,p)$ is one of the values of the control realizing the maximum. Now if the set of controls U is an open subset of $\mathbb{R}^m$ and $\mathcal{H}$ is smooth with respect to u then $u(x,p)$ is one of the solutions of:

$$\frac{\partial \mathcal{H}(x,p,u)}{\partial u} = 0. \tag{21}$$

A useful fact is the following. If moreover (21) permits to find in a unique way u as a smooth function of (x,p), then we can write the Hamiltonian equations with H instead of $\mathcal{H}$ getting a true Hamiltonian system with no dependence on the control. In fact in this case:

$$\begin{aligned}
\frac{\partial H(x,p)}{\partial x} &= \frac{\partial \mathcal{H}(x,p,u(x,p))}{\partial x} = \left.\frac{\partial \mathcal{H}(x,p,u)}{\partial x}\right|_{(x,p,u(x,p))} \\
&+ \left.\frac{\partial H(x,p,u)}{\partial u}\right|_{(x,p,u(x,p))} \frac{\partial u(x,p)}{\partial x} = \left.\frac{\partial \mathcal{H}(x,p,u)}{\partial x}\right|_{(x,p,u(x,p))} \\
\frac{\partial H(x,p)}{\partial p} &= \frac{\partial \mathcal{H}(x,p,u(x,p))}{\partial p} = \left.\frac{\partial \mathcal{H}(x,p,u)}{\partial p}\right|_{(x,p,u(x,p))} \\
&+ \left.\frac{\partial H(x,p,u)}{\partial u}\right|_{(x,p,u(x,p))} \frac{\partial u(x,p)}{\partial p} = \left.\frac{\partial \mathcal{H}(x,p,u)}{\partial p}\right|_{(x,p,u(x,p))}
\end{aligned}$$

where the last equality follows from (21). Then one gets the controls plugging the solution of this Hamiltonian system in the expression $u(x,p)$. This nice situation happens always for normal extremals for distributional systems ($\dot{x} = \sum_i u_i F_i(x)$) with quadratic cost ($\min \int_0^T \sum_i u_i^2\, dt$). See Section 3.4, p. 38 for an example.

3.2.3 Abnormal Extremals and Endpoint Singular Extremals

Assume now that f is differentiable also with respect to u. The following sets of trajectories have very special features:

Definition 9 (Endpoint Singular Trajectories and Endpoint Singular Extremals) *We call* endpoint singular trajectories*, solutions to the following equations:*

$$\dot{x}(t) = \frac{\partial \bar{\mathcal{H}}}{\partial p}(x(t), p(t), u(t)),$$
$$\dot{p}(t) = -\frac{\partial \bar{\mathcal{H}}}{\partial x}(x(t), p(t), u(t)),$$
$$\frac{\partial \bar{\mathcal{H}}}{\partial u}(x(t), p(t), u(t)) = 0,$$

where $\bar{\mathcal{H}}(x, p, u) := \langle p, f(x, u)\rangle$*,* $p(t) \neq 0$ *and the constraint* $u \in U \in \mathbb{R}^m$ *is changed in* $u \in Int(U) \neq \emptyset$*. Endpoint singular trajectories that are also extremals are called* endpoint singular extremals*.*

Remark **14** Notice that, although endpoint singular trajectories do not depend on the cost and on the constraints on the control set, endpoint singular extremals (that for our minimization problem are the interesting ones) do depend.

The name endpoint singular trajectories comes from the fact that they are singularities of the *endpoint mapping* that, fixed an initial point and a time T, associates to a control function $u(\cdot)$, defined on $[0, T]$, the end point of the corresponding trajectory γ:

$$\mathcal{E}^{x_0,T} : u(\cdot) \mapsto \gamma(T).$$

By singularity of the end point mapping, we mean a control at which the Fréchet derivative of $\mathcal{E}^{x_0,T}$ is not surjective. For more details see [11]. Roughly speaking, this means that the reachable set, locally, around the trajectory, does not contain a neighborhood of the end point $\gamma(T)$.

In the case of a minimum time problem for a control affine system (18) with $|u_i| \leq 1$, endpoint singular trajectories satisfy $< p(t), F_i(\gamma(t)) >= 0$. Endpoint singular extremals, are just endpoint singular trajectories for which there exists $\lambda_0 \leq 0$ satisfying $< p(t), F_0 > +\lambda_0 = 0$ and corresponding to admissible controls.

In Section 4, we consider control systems of the form $\dot{x} = F(x) + uG(x)$ where $|u| \leq 1$. In this case, under generic conditions, endpoint singular extremals are arcs of extremal trajectories corresponding to controls not constantly equal to $+1$ or -1.

Definition 10 (Abnormal Extremals) *We call* abnormal extremals *extremal trajectories for which* $\lambda_0 = 0$*.*

Remark **15** Abnormal extremals do not depend on the cost, but depend on the constraint $u \in U$.

Remark **16** In some problems, like in subriemannian geometry or in distributional problems for the minimum time with bounded controls, endpoint singular extremals and abnormal extremals coincide, see Exercise 7, p. 61.

In other problems (e.g. minimum time for control affine systems) the two definitions are different, but coincide for some very special class of trajectories. These trajectories are usually called *singular exceptional* (see [13]).

Remark **17** (High Order Conditions) PMP is used in Synthesis Theory to attempt a *finite dimensional reduction* of the minimization problem, as explained in Step 2 of Section 3.5, p. 41. Of course high order conditions can be very useful for further restriction of candidate optimal trajectories.

There are several possible higher order variations. For instance the *high order principle* of Krener [38] and the *envelope theory* developed by Sussmann [52, 53]. Other high order conditions are based on different techniques: symplectic geometric methods, conjugate points, Generalized Index Theory. For a list of references see [14].

High order conditions are out of the purpose of this notes. See for instance [14] and references therein.

3.3 Calculus of Variations

The classical case of Calculus of Variations can be formulated as an Optimal Control Problem for the dynamics $\dot{x} = u$, $u \in T_xM$, fixed final time T, $\psi = 0$ and $\mathcal{T} = x_1 \in M$:

$$\begin{cases} \dot{x} = u \\ \min \int_0^T L(x(t), u(t))\, dt \\ x(0) = x_0, \\ x(T) = x_T. \end{cases} \qquad (22)$$

Here, together with (H0) and (H1), we assume also $L(x, u) \in \mathcal{C}^1$ and we are looking for Lipschitz minimizers. For this problem there are not abnormal extremals and end-point singular trajectories. Moreover one can deduce from PMP the Euler Lagrange equations and the Weierstraß condition.

Indeed from (22) the Hamiltonian reads:

$$\mathcal{H}(x, p, \lambda_0, u) = \langle p, u \rangle + \lambda_0 L(x, u). \qquad (23)$$

In this case $\bar{\mathcal{H}}(x, p, u) = \langle p, u \rangle$ thus $\partial\bar{\mathcal{H}}/\partial u$ implies $p \equiv 0$. Hence there are not endpoint singular trajectories (cfr. Definition 9, p. 35). Moreover (PMP2) with $\lambda_0 = 0$ implies (cfr. Remark 10) $p = 0$ contradicting the fact that $(p, \lambda_0) \neq 0$. Therefore there are not abnormal extremals and it is convenient to normalize $\lambda_0 = -1$ (in the following we drop the dependence from λ_0):

$$\mathcal{H}(x, p, u) = \langle p, u \rangle - L(x, u). \qquad (24)$$

Since there are is no constraint on the control set ($u \in \mathbb{R}^n$) and L is differentiable with respect to u, the maximization condition (PMP2) implies a.e.:

$$\frac{\partial \mathcal{H}}{\partial u}(x(t), p(t), u(t)) = 0 \;\; \Rightarrow \;\; p(t) = \frac{\partial L}{\partial u}(x(t), u(t)), \qquad (25)$$

while the Hamiltonian equations are:

$$\begin{cases} \dot{x}(t) = \frac{\partial \mathcal{H}}{\partial p}(x(t), p(t), u(t)) = u(t), \quad a.e., \\ \dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), p(t), u(t)) = +\frac{\partial L}{\partial x}(x(t), u(t)), \quad a.e. \end{cases} \tag{26}$$

The first equation is the dynamics, while the second together with (25) give the Euler Lagrange equations in the case of a Lipschitz minimizer. (We recall that in general $(\partial L/\partial u)(x(t), u(t))$ is not absolutely continuous, but it coincides a.e. with an absolutely continuous function, whose derivative is a.e. equal to $(\partial L/\partial x)(x(t), u(t))$). If $x(.) \in \mathcal{C}^2$ (that implies $u(.) \in \mathcal{C}^1$) and $\partial L/\partial u$ is $\mathcal{C}^1$, then $p(t)$ is $\mathcal{C}^1$ and one gets the Euler Lagrange equations in standard form:

$$\frac{d}{dt}\frac{\partial L}{\partial u}(x(t), u(t)) = \frac{\partial L}{\partial x}(x(t), u(t)),$$

but it is well known that they hold everywhere also for $x(.) \in \mathcal{C}^1$ and for a less regular L (cfr. for instance [22, 26]).

Remark **18** Notice that, in the case of Calculus of Variations, the condition to get a true Hamiltonian system in the sense of Remark 13 (i.e. $\partial \mathcal{H}(x, p, u)/\partial u$ solvable with respect to u) is the same condition of invertibility of the Legendre transformation:

$$det\left(\frac{\partial^2 L(x, u)}{\partial u^i \partial u^j}\right) \neq 0.$$

From PMP one gets also the stronger condition known as Weierstrass condition. From the maximization condition:

$$\mathcal{H}(x(t), p(t), u(t)) = \max_{v \in \mathbb{R}^n} \mathcal{H}(x(t), p(t), v),$$

we get for every $v \in \mathbb{R}^n$:

$$\langle p(t), u(t)\rangle - L(x(t), u(t)) \geq \langle p(t), v\rangle - L(x(t), v),$$

and using the second of (26):

$$L(x(t), v) - L(x(t), u(t)) \geq \langle \frac{\partial L}{\partial u}(x(t), u(t)), (v - u(t))\rangle. \tag{27}$$

Condition (27) is known as Weierstraß condition. Weierstraß condition is stronger (in a sense it is "more global") than Euler Lagrange equations, because it is a necessary conditions for a minimizer to be a strong local minimum (while Euler Lagrange equations are necessary conditions for a weak local minimum). We recall that given the minimization problem (22), $x(.) \in \mathcal{C}^1$ is a strong local minimizer (respectively a weak local minimizer) if there exists $\delta > 0$ such that for every $\bar{x}(.) \in \mathcal{C}^1$ satisfying the boundary conditions and $\|x - \bar{x}\|_{L^\infty} \leq \delta$ (respectively $\|x - \bar{x}\|_{L^\infty} + \|\dot{x} - \dot{\bar{x}}\|_{L^\infty} \leq \delta$)

we have $\int_0^T L(x(t), \dot{x}(t))\, dt \le \int_0^T L(\bar{x}(t), \dot{\bar{x}}(t))\, dt$. A classical example of a weak local minimizer that is not a strong local minimizer is $x(.) \equiv 0$ for the minimization problem:

$$\begin{cases} \min_{x(.)\in\mathcal{C}^1[0,1]} \int_0^1 (\dot{x}^2 - \dot{x}^4)\, dt \\ x(0) = 0, \;\; x(1) = 0. \end{cases}$$

For more details on the Weierstraß condition see any book of Calculus of Variations, for instance [22, 26].

3.4 An Detailed Application: the Grusin's Metric

A nice and simple application of the PMP is the computation of geodesics for the so called *Grusin's metric*. The Grusin's metric is a Riemannian metric on the plane (x_1, x_2) having singularities along the x_2-axis.

The problem of computing geodesics for a Riemannian metric on a manifold M of dimension n can be presented as an optimal control problem with n controls belonging to $\mathbb{R}$ (in fact it is a problem of calculus of variations):

$$\dot{x} = \sum_{j=1}^{n} u_j F_j(x) \tag{28}$$

$$\min_{u(.)} \int_0^T \sqrt{\sum_{j=1}^{n} u_j(t)^2}\; dt, \tag{29}$$

where $\{F_j(x)\}$ is an orthonormal basis for the given metric $g_x : T_xM \times T_xM \to \mathbb{R}$, that is $g_x(F_j(x), F_k(x)) = \delta_{j,k}$ (we recall that giving a metric on a manifold is equivalent to give an orthonormal base).

One immediately verify that the cost (29) is invariant by a smooth reparameterization of the controls. Moreover every minimizer parametrized by constant speed (i.e. $\sum_{j=1}^{n} u_j(t)^2 = const$) is also a minimizer of the so called "energy cost":

$$\min_{u(.)} \int_0^T \sum_{j=1}^{n} u_j(t)^2\; dt. \tag{30}$$

This is a simple consequence (exercise) of the Cauchy-Schwartz inequality (that holds for two integrable functions $f, g : [a, b] \to \mathbb{R}$):

$$\begin{aligned} \left(\int_a^b f(t)g(t)\, dt\right)^2 &\le \int_a^b f^2(t)dt \int_a^b g^2(t)dt \\ \left(\int_a^b f(t)g(t)\, dt\right)^2 &= \int_a^b f^2(t)dt \int_a^b g^2(t)dt \;\; \Leftrightarrow f(t) = c\, g(t) \end{aligned} \tag{31}$$

a.e. in $[a,b]$, for some $c \in \mathbb{R}$. The cost (30) is "nicer" than the cost (29) (is smooth and quadratic) and usually, in Riemannian geometry and in control theory, one deals with (30). The only thing to keep in mind is that, in (30), the final time must be fixed otherwise there is no minimum (cfr. Exercise 3). Thus if we want to find minimizers via the PMP (in this case one can use also Euler-Lagrange equations), then the Hamiltonian must be equal to a (strictly) positive constant (cfr. Exercises 4).

If the vector fields $\{F_j(x)\}$ are not always linearly independent, then the corresponding metric g_x presents singularities at points x where $span\{F_j(x)\}$ is a proper subset of T_xM. In this case the problem can not be treated as a problem of calculus of variations, since there are nonholonomic constraints at the singular points.

An example is given by Grusin's metric on the plane (x_1, x_2), that is the metric corresponding to the frame $F_1 = (1,0)^T$ and $F_2 = (0, x_1)^T$. In the language of control theory, this translates into the following problem:

$$\begin{aligned}
&\dot{x} = u_1 F_1(x) + u_2 F_2(x),\\
&x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \;\; F_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; F_2(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}.\\
&\min_{u(.)} \int_0^T \left(u_1(t)^2 + u_2(t)^2 \right) \, dt.
\end{aligned}$$

The vector field F_2 is zero on the x_2 axis. In other words, on the x_2 axis, every trajectory of the system has velocity parallel to the x_1 axis. Therefore the Grusin's metric explodes when one is approaching the x_2 axis (see Figure 4).

Let us compute the geodesics starting from the origin using PMP. In this case the Hamiltonian is (cfr. equation 20, p. 33):

$$\mathcal{H}(x,p,\lambda_0,u) = \langle p, f(x,u) \rangle + \lambda_0 L(x,u) = \langle p, u_1 F_1 + u_2 F_2 \rangle + \lambda_0 (u_1^2 + u_2^2). \quad (32)$$

There are not non trivial abnormal extremals. In fact let $x(.)$ be an extremal trajectory and $p(.)$ the corresponding covector. If $\lambda_0 = 0$, then it is possible to maximize (32) with respect to the control only if

$$\langle p(t), F_1(x(t) \rangle = \langle p(t), F_2(x(t)) \rangle = 0, \quad (33)$$

(otherwise the supremum is infinity). Being $\lambda_0 = 0$, $p(t)$ can not vanish. It follows that (33) can hold only in the region where the two vector fields are linearly dependent (i.e. on the axis $x_1 = 0$). But the only admissible trajectories, whose support belongs to this set, are trivial (i.e. $\gamma(t) = (0, const)$). We then normalize $\lambda_0 = -1/2$.

In this case, since the controls belong to $\mathbb{R}$, the maximization condition:

$$\mathcal{H}(x(t), p(t), \lambda_0, u(t)) = \max_v \mathcal{H}(x(t), p(t), \lambda_0, v)$$

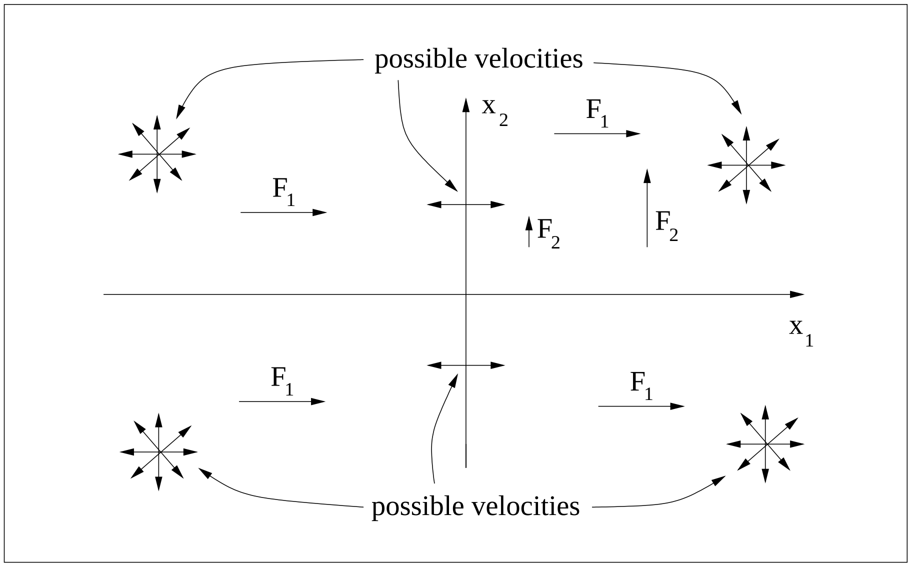

Figure 4: The Grusin's orthonormal frame

can be rewritten as (cfr. Remark 13, p. 34):

$$\frac{\partial \mathcal{H}}{\partial u} = 0. \tag{34}$$

From this condition we get controls as functions of (x, p):

$$\begin{aligned} u_1 &= p_1, \\ u_2 &= p_2 x_1, \end{aligned}$$

where $p = (p_1, p_2)$. In this way the maximized Hamiltonian becomes:

$$H = \frac{1}{2}(p_1^2 + p_2^2 x_1^2).$$

Since the maximization condition define u uniquely as a smooth function of x and p, we can write the Hamiltonian equations for H instead than $\mathcal{H}$ (cfr. again Remark (13)). We have:

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = p_1, \tag{35}$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = p_2 x_1^2, \tag{36}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -p_2^2 x_1, \tag{37}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 0. \tag{38}$$

From (38), we have $p_2 = a \in \mathbb{R}$. Equations (35),(37) are the equation for an harmonic oscillator whose solution is:

$$\begin{aligned} x_1 &= A \sin(at + \phi_0), \\ p_1 &= aA \cos(at + \phi_0). \end{aligned}$$

Since we are looking for geodesics starting from the origin we have $\phi_0 = 0$ and integrating (36), with $x_2(0) = 0$, we have:

$$x_2(t) = \frac{aA^2}{2} t - \frac{A^2}{4} \sin(2at). \tag{39}$$

In the expressions of $x_1(t), x_2(t), p_2(t)$ there are still the two real constants A and a. Requiring that the geodesics are parameterized by arclength ($u_1^2(t) + u_2^2(t) = 1$), that corresponds to fix the level $1/2$ of the Hamiltonian, we have:

$$\frac{1}{2} = H = \frac{1}{2}(p_1^2(t) + p_2^2(t) x_1^2(t)) = \frac{1}{2} a^2 A^2.$$

This means that $A = \pm 1/a$. Finally the formulas for extremal trajectories and controls are given by the following two families (parametrized by $a \in \mathbb{R}$):

$$\left\{ \begin{array}{lcl} x_1(t) &=& \pm\frac{1}{a}\sin(at) \\ x_2(t) &=& \frac{1}{2a}t - \frac{1}{4a^2}\sin(2at) \\ u_1(t) &=& p_1(t) = \pm\cos(at) \\ u_2(t) &=& a x_1(t) = \pm\sin(at) \end{array} \right. \tag{40}$$

For $a = 0$, the definition is obtained by taking limits.

Due to the symmetries of the problem, and checking when extremal trajectories self intersects, one can verify that each trajectory of (40) is optimal exactly up to time $t = \pi/|a|$. In Figure 5A geodesics for some values of a are portrayed. While Figure 5B, illustrates some geodesics and the set of points reached in time $t = 1$. Notice that, opposed to the Riemannian case, this set is not smooth.

3.5 Geometric Control Approach to Synthesis

Geometric control provides a standard method toward the construction of an Optimal Synthesis for the family of problems $\{\mathcal{P}_{x_1}\}_{x_1 \in M}$, of Definition 6, p. 31.

The approach is illustrated for systems with a compact control set as for the problem treated in Section 4. The following scheme, consisting of four steps, elucidates the procedure for building an Optimal Synthesis.

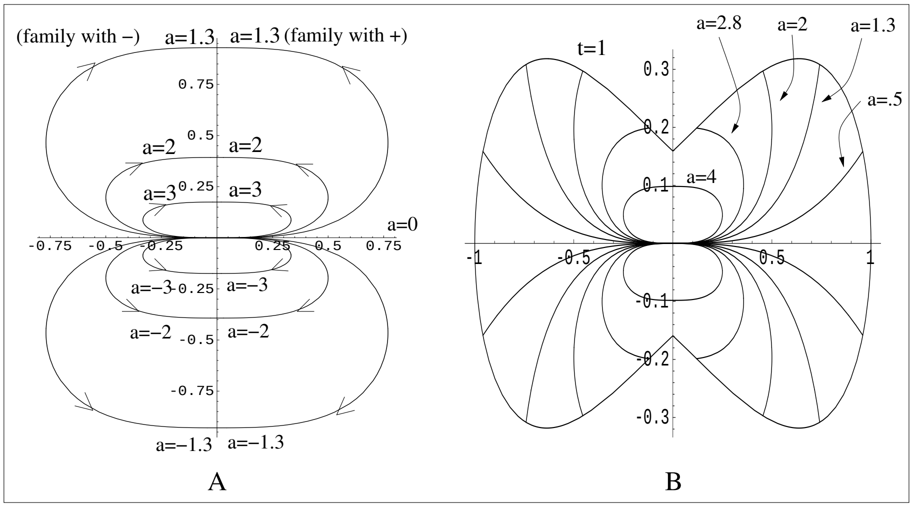

Figure 5: Geodesics and minimum time front (for $t = 1$) for the Grusin's metric

Step 1. Use PMP and high order conditions to study the properties of optimal trajectories.

Step 2. Use Step 1. to obtain a finite dimensional family of candidate optimal trajectories.

Step 3. Construct a synthesis, formed of extremal trajectories, with some regularity properties.

Step 4. Prove that the regular extremal synthesis is indeed optimal.

Let us describe in more detail each Step.

Step 1.

We stated the PMP that in some cases gives many information on optimal trajectories. Beside high order maximum principle and envelopes, there are various higher order conditions that can be used to discard some extremal trajectories that are not locally optimal. These conditions come from: symplectic geometric methods, conjugate points, degree theory, etc.

Step 2. (finite dimensional reduction)

The family of trajectories and controls, on which the minimization is taken, is clearly an infinite dimensional space. Thanks to the analysis of Step 1, in some cases it is possible to narrow the class of candidate optimal trajectories to a finite dimensional family. This clearly reduces drastically the difficulty of the problem and allows a deeper analysis. More precisely, one individuates a finite number of smooth controls $u_i(x)$, such that every optimal trajectory is a finite concatenation of trajectories corresponding to the vector fields $f(x, u_i(x))$.

Step 3.

Once a finite dimensional reduction is obtained, one may construct a synthesis in the following way. Assume that on the compact set $\mathcal{R}(\tau)$, there is a bound on the number of arcs (corresponding to controls u_i) that may form, by concatenation, an optimal trajectory. Then one can construct, by induction on n, trajectories that are concatenations of n arcs and cut the not optimal ones. The latter operation produces some special sets, usually called *cut loci* or *overlaps*, reached optimally by more than one trajectory.

The above procedure is done on the base space M, however extremals admit lifts to T^*M. Thus another possibility is to construct the set of extremals in the cotangent bundle and project it on the base space. In this case, projection singularities are responsible for singularities in the synthesis.

Step 4.

Even if a finite dimensional reduction is not obtained, one can still fix a finite dimensional family of extremal trajectories and construct a synthesis on some part of the state space. If the synthesis is regular enough then there are sufficiency theorems ensuring the optimality of the synthesis.

These sufficiency theorems fit well also inside the framework of viscosity solution to the corresponding Hamilton–Jacobi–Bellman equation.

The above approach is quite powerful, but in many cases it is not known how to reach the end, namely to produce an Optimal Synthesis. For the problem treated in Section 4, we are able not only to construct an Optimal Synthesis under generic assumptions, but also to give a topological classification of singularities of the syntheses and of the syntheses themselves, see [14].

Remark **19** For general problems (not necessarily with bound on the control, e.g. with quadratic cost), Steps 1 and 4 are still used, while Step 3 is not based on a finite dimensional reduction.

3.6 Fuller Phenomenon and Dubins' Car With Angular Acceleration.

A major problem in using the Geometric Control Approach for the construction of Optimal Syntheses is that the finite dimensional reduction, of **Step 2.**, may fail to exits. The most famous example is that given by Fuller in [25]. The author considered the infinite horizon Optimal Control problem:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \leq 1,$$

$$\min \int_0^{+\infty} x_1(t)^2 \, dt.$$

Optimal trajectories reach the origin in finite time after an infinite number of switchings between the control $+1$ and the control -1. Thus they are optimal also for the finite horizon case.

The genericity of Fuller phenomenon, that is the presence of optimal trajectories with an infinite number of switchings, was extensively analyzed in [39] and [61].

For the problem considered by Fuller, we have anyway the existence of an Optimal Synthesis that is quite regular, see [48]. We give now an example of an optimal control problems, coming from car–like robot models, for which Fuller phenomenon is present and it is not known how to construct an Optimal Synthesis.

Dubins' car with angular acceleration. One of the simplest model for a car–like robot is the one known as Dubins' car. In this model, the system state is represented by a pair $((x, y), \theta)$ where $(x, y) \in \mathbb{R}^2$ is the position of the center of the car and $\theta \in S^1$ is the angle between its axis and the positive x-axis. The car is assumed to travel with constant (in norm) speed, the control is on the angular velocity of the steering and is assumed to be bounded, thus we obtain the system:

$$\begin{cases} \dot{x} &= \cos(\theta) \\ \dot{y} &= \sin(\theta) \\ \dot{\theta} &= u \end{cases}$$

where $|u| \leq C$ (usually for simplicity one assumes $C = 1$).

A modified version of the Dubins' car, in which the control is on the angular acceleration of the steering, is given by:

$$\begin{cases} \dot{x} &= \cos(\theta) \\ \dot{y} &= \sin(\theta) \\ \dot{\theta} &= \omega \\ \dot{\omega} &= u. \end{cases} \tag{41}$$

with $|u| \leq 1$, $(x, y) \in \mathbb{R}^2$, $\theta \in S^1$ and $\omega \in \mathbb{R}$. We use the notation $\mathbf{x} = ((x, y), \theta, \omega)$ and $M = \mathbb{R}^2 \times S^1 \times \mathbb{R}$.

Pontryagin Maximum Principle conditions are now written as:

(PMP1) $\dot{p}_1 = 0,\ \dot{p}_2 = 0,\ \dot{p}_3 = p_1 \sin(\theta) - p_2 \cos(\theta),\ \dot{p}_4 = -p_3,$

(PMP2) $H(\mathbf{x}(t), \mathbf{p}(t), \lambda_0) = \mathcal{H}(\mathbf{x}(t), \mathbf{p}(t), \lambda_0, u(t)) = 0$, that is $p_4(t)u(t) = |p_4(t)|$, where $\mathbf{p}$= $(p_1, ..., p_4)$.

The function $p_4(.)$ is the so called switching function (cfr. Section 4.1, p. 46). In fact from (PMP2) it follows that:

- if $p_4(t) > 0$ (resp < 0) for every $t \in [a,b]$, then $u \equiv 1$ (resp. $u \equiv -1$) on $[a,b]$. In this case the corresponding trajectory $\mathbf{x}(.)|_{[a,b]}$ is an arc of clothoid in the (x,y) space.

- if $p_4(t) \equiv 0$ for every $t \in [a,b]$, then $u \equiv 0$ in $[a,b]$. In this case the trajectory $\mathbf{x}(.)|_{[a,b]}$ is a straight line in the (x,y) space.

The main feature of this highly nongeneric problem is that *an optimal trajectory cannot contain points where the control jumps from* ± 1 *to* 0 *or from* 0 *to* ± 1. In other words a singular arc cannot be preceded or followed by a bang arc.

In [58] it is proved the presence of Fuller phenomenon. More precisely, there exist extremal trajectories $\mathbf{x}(.)$ defined on some interval $[a,b]$, which are singular on some interval $[c,b]$, where $a < c < b$, and such that $p_4(.)$ does not vanish identically on any subinterval of $[a,c]$. Moreover the set of zeros of $p_4(.)$ consists of c together with an increasing sequence $\{t_j\}_{j=1}^{\infty}$ of points converging to c. At each t_j, $\mathbf{x}(.)$ is switches the control from $+1$ to -1 or viceversa.

An optimal path can thus have at most a finite number of switchings only if it is a finite concatenation of clothoids, with no singular arc. Existence of optimal trajectories (not only extremal) presenting chattering was proved by Kostov and Kostova in [36]. More precisely if the distance between the initial and final point is big enough, then the shortest path can not be a finite concatenation of clothoids.

4 The Minimum Time Problem for Planar Control Affi ne Systems

In this section we give an introduction the theory of "optimal synthesis for planar control affine systems" developed by Sussmann, Bressan, and the authors in [15, 16, 17, 18, 19, 46, 47, 55, 57], and recently collected in [14]. The main ingredients of that theory are described next.

Consider a control system on $\mathbb{R}^2$:

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad |u| \leq 1, \tag{42}$$

and the problem of reaching every point of the plane in minimum time starting from the origin.

To guarantee completeness of the vector fields and to be able to compute derivatives of arbitrary order, we assume the following:

(H) F, G are $\mathcal{C}^{\infty}$ vector fields, uniformly bounded with all their partial derivatives on $\mathbb{R}^2$.

The PMP takes a particularly simple form for (42) and one can easily see that (under generic conditions) controls are always either bang-bang, that is corresponding to constant control $u = +1$ or $u = -1$, or singular (i.e. corresponding to trajectories whose switching functions vanishes, see Definition 12 below and Remark 22). The latter can happen only on the set where the vector field G and the Lie bracket $[F, G]$ are parallel (this generically is a one dimensional submanifold of the plane).

Moreover one can also predict (see Section 4.2) which kind of "switchings" (See Definition 12) can happen on each region of the plane. We first need to state the PMP for the system (42) and introduce some notations.

4.1 Pontryagin Maximum Principle and Switching Functions

The Pontryagin Maximum Principle (see Section 3.2.2, p. 32) in this special case states the following.

Remark **20 (Notation in the following version of the PMP)**

- $T_x^*\mathbb{R}^2 = (\mathbb{R}^2)_*$ denotes the set of row vectors.
- the duality product $< \ . \, , \ . \ >$ is now simply the matrix product and it is indicated by " $\cdot$ ".
- the Hamiltonian does not include the cost factor $\lambda_0 L$, (that in this case is just λ_0). The condition $\mathcal{H} = 0$ become then $\mathcal{H} + \lambda_0 = 0$. With this new definition of Hamiltonian, abnormal extremals are the zero levels of the $\mathcal{H}$.

Theorem 6 (PMP for the minimum time problem for planar control affine systems)
Define for every $(x, p, u) \in \mathbb{R}^2 \times (\mathbb{R}^2)_* \times [-1, 1]$*:*

$$\mathcal{H}(x, p, u) = p \cdot F(x) + u\, p \cdot G(x)$$

and:

$$H(x, p) = \max\{p \cdot F(x) + u\, p \cdot G(x) : u \in [-1, 1]\}. \tag{43}$$

If $\gamma : [0, a] \to \mathbb{R}^2$ *is a (time) optimal trajectory corresponding to a control* $u : [0, a] \to [-1, 1]$*, then there exist* a nontrivial field of covectors along γ*, that is a Lipschitz function* $\lambda : [0, a] \to (\mathbb{R}^2)_*$ *never vanishing, and a constant* $\lambda_0 \leq 0$ *such that for a.e.* $t \in Dom(\gamma)$*:*

i) $\dot{\lambda}(t) = -\lambda(t) \cdot (\nabla F + u(t)\nabla G)(\gamma(t))$*,*

ii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) + \lambda_0 = 0$*,*

iii) $\mathcal{H}(\gamma(t), \lambda(t), u(t)) = H(\gamma(t), \lambda(t))$*.*

Remark **21** Notice that, since the Lagrangian cost is constantly equal to 1, the condition $(\lambda(t), \lambda_0) \neq (0, 0)$, for all $t \in [0, T]$, given in Definition 8, p. 33, now becomes $\lambda(t) \neq 0$ for all t. In fact $\lambda(t) \equiv 0$ with the condition *(PMP2)* of Definition 8 implies $\lambda_0 = 0$. To see what happens for vanishing Lagrangians cfr. Exercise 8, p. 61).

Definition 11 (Switching Function) *Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2 \times (\mathbb{R}^2)_*$ be an extremal pair. The corresponding switching function is defined as $\phi(t) := \lambda(t) \cdot G(\gamma(t))$. Notice that $\phi(\cdot)$ is absolutely continuous.*

From the PMP one immediately get that the switching function describes when the control switches from +1 to -1 and viceversa. In fact we have:

Lemma 2 *Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2 \times (\mathbb{R}^2)_*$ be an extremal pair and $\phi(\cdot)$ the corresponding switching function. If $\phi(t) \neq 0$ for some $t \in]0, \tau[$, then there exists $\varepsilon > 0$ such that γ corresponds to a constant control $u = sgn(\phi)$ on $]t - \varepsilon, t + \varepsilon[$.*

Proof. There exists $\varepsilon > 0$ such that ϕ does not vanish on $]t - \varepsilon, t + \varepsilon[$. Then from condition **iii)** of PMP we get $u = \mathrm{sgn}(\phi)$. □

Reasoning as in Lemma 2 one immediately has:

Lemma 3 *Assume that ϕ has a zero at t, $\dot{\phi}(t)$ is strictly greater than zero (resp. smaller than zero) then there exists $\varepsilon > 0$ such that γ corresponds to constant control $u = -1$ on $]t - \varepsilon, t[$ and to constant control $u = +1$ on $]t, t + \varepsilon[$ (resp. to constant control $u = +1$ on $]t - \varepsilon, t[$ and to constant control $u = -1$ on $]t, t + \varepsilon[$).*

We are then interested in differentiating ϕ:

Lemma 4 *Let $(\gamma, \lambda) : [0, \tau] \to \mathbb{R}^2 \times (\mathbb{R}^2)_*$ be an extremal pair and ϕ the corresponding switching function. Then $\phi(\cdot)$ is continuously differentiable and it holds:*

$$\dot{\phi}(t) = \lambda(t) \cdot [F, G](\gamma(t)).$$

Proof. Using the PMP we have for a.e. t:

$$\begin{aligned}
\dot{\phi}(t) &= \frac{d}{dt}(\lambda(t) \cdot G(\gamma(t))) = \dot{\lambda}(t) \cdot G(\gamma(t)) + \lambda \cdot \dot{G}(\gamma(t)) \\
&= -\lambda(t)(\nabla F + u(t)\nabla G)(\gamma(t)) \cdot G(\gamma(t)) + \lambda \cdot \nabla G(\gamma(t))(F + u(t)G)(\gamma(t)) \\
&= \lambda(t) \cdot [F, G](\gamma(t)).
\end{aligned}$$

Since $\phi(\cdot)$ is absolutely continuous and $\lambda(t) \cdot [F, G](\gamma(t))$ is continuous, we deduce that ϕ is $\mathcal{C}^1$. □

Notice that if $\phi(\cdot)$ has no zeros then u is a.e. constantly equal to $+1$ or -1. Next we are interested in determining when the control may change sign or may assume values in $] - 1, +1[$. For this purpose we give first the following definition:

Definition 12 *Let $u(.) : [a,b] \to [-1,1]$ be a control for the control system (42).*

- $u(.)$ *is said to be a* bang control *if for almost every $t \in [a,b]$, $u(t)$ is constant and belongs to a point of the set $\{+1,-1\}$.*

- *A* switching *time of $u(.)$ is a time $t \in [a,b]$ such that for every $\varepsilon > 0$, $u(.)$ is not bang on $(t-\varepsilon, t+\varepsilon) \cap [a,b]$.*

- *If $u_A : [a_1,a_2] \to [-1,1]$ and $u_B : [a_2,a_3] \to [-1,1]$ are controls, their* concatenation $u_B * u_A$ *is the control:*

$$(u_B * u_A)(t) := \begin{cases} u_A(t) \text{ for } t \in [a_1,a_2] \\ u_B(t) \text{ for } t \in]a_2,a_3]. \end{cases}$$

  The control $u(.)$ is called bang-bang *if it is a finite concatenation of bang arcs.*

- *A trajectory of (42) is a* bang trajectory, *(resp.* bang-bang trajectory*), if it corresponds to a bang control, (resp. bang-bang control).*

- *An extremal trajectory γ defined on $[c,d]$ is said to be* singular *if the switching function ϕ vanishes on $[c,d]$.*

Remark **22** A singular extremal in the sense above is also an endpoint singular extremal in the sense of Section 3.2.3, p. 34. In fact for these trajectories the Hamiltonian is independent from the control. In the following we use the term *singular trajectories* with the same meaning of *endpoint singular extremal*.

Remark **23** On any interval where ϕ has no zeroes (respectively finitely many zeroes) the corresponding control is bang (respectively bang-bang).

Singular trajectories are studied in details below. In Figure 6 we give an example to illustrate the relationship between an extremal trajectory and the corresponding switching function. The control φ, corresponding to the singular arc, is computed below, see Lemma 6, p. 51.

4.2 Singular Trajectories and Predicting Switchings

In this section we give more details on singular curves and explain how to predict which kind of switching can happen, using properties of the vector fields F and G.

A key role is played by the following functions defined on $\mathbb{R}^2$:

$$\Delta_A(x) := Det(F(x), G(x)) = F_1(x)G_2(x) - F_2(x)G_1(x), \tag{44}$$
$$\Delta_B(x) := Det(G(x), [F,G](x)) = G_1(x)[F,G]_2(x) - G_2(x)[F,G]_1(x) \tag{45}$$

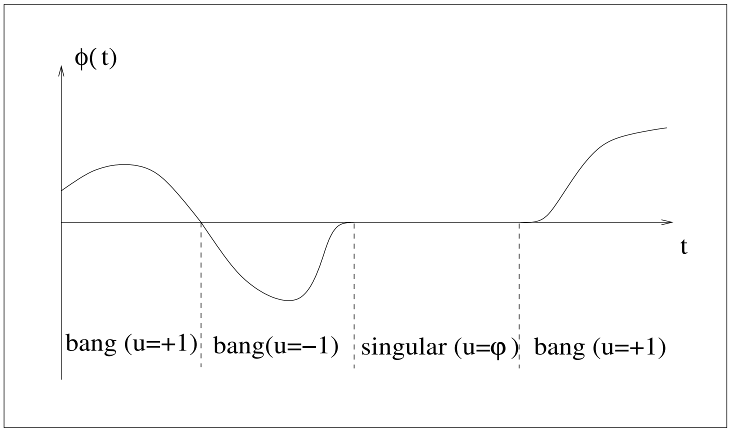

Figure 6: The switching function

The set of zeros $\Delta_A^{-1}(0), \Delta_B^{-1}(0)$, of these two functions, are respectively the set of points where F and G are parallel and the set of points where G is parallel to $[F, G]$. These loci are fundamental in the construction of the optimal synthesis. In fact, assuming that they are smooth one dimensional submanifolds of $\mathbb{R}^2$, we have the following:

- on each connected component of $M \setminus (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0))$, every extremal trajectory is bang-bang with at most one switching. Moreover, at a switching time, the value of the control switches from -1 to $+1$ if $f_S := -\Delta_B^{-1}(0) \setminus \Delta_A^{-1}(0) > 0$ and from $+1$ to -1 if $f_S < 0$;
- the support of singular trajectories is always contained in the set $\Delta_B^{-1}(0)$.

The synthesis is built by induction on the number of switchings of extremal trajectories, following the classical idea of canceling, at each step, the non optimal trajectories. We refer to Chapter 2 of [14], for more details about this construction.

First we study points out of the regions $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$.

Definition 13 *A point $x \in \mathbb{R}^2$ is called an* ordinary point *if $x \notin \Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$. If x is an ordinary point, then $F(x), G(x)$ form a basis of $\mathbb{R}^2$ and we define the scalar functions f, g to be the coefficients of the linear combination: $[F, G](x) = f(x)F(x) + g(x)G(x)$.*

The function f is crucial in studying which kind of switchings can happen near ordinary points. But first we need a relation between f, Δ_A and Δ_B:

Lemma 5 *Let x an ordinary point then:*

$$f(x) = -\frac{\Delta_B(x)}{\Delta_A(x)}. \tag{46}$$

Proof. We have:

$$\begin{aligned}\Delta_B(x) &= Det\big(G(x), [F,G](x)\big) = Det(G(x), f(x)F(x) + g(x)G(x))\\ &= f(x)Det\big(G(x), F(x)\big) = -f(x)\Delta_A(x).\end{aligned}$$

□

On a set of ordinary points the structure of optimal trajectories is particularly simple:

Theorem 7 *Let $\Omega \in \mathbb{R}^2$ be an open set such that every $x \in \Omega$ is an ordinary point. Then, in Ω, all extremal trajectories γ are bang-bang with at most one switching. Moreover if $f > 0$ (resp. $f < 0$) throughout Ω then γ corresponds to control $+1, -1$ or has a $-1 \to +1$ switching (resp. has a $+1 \to -1$ switching).*

Proof. Assume $f > 0$ in Ω, the opposite case being similar. Let (γ, λ) be an extremal pair such that γ is contained in Ω. Let $\bar{t}$ be such that $\lambda(\bar{t}) \cdot G(\gamma(\bar{t})) = 0$, then:

$$\dot{\phi}(\bar{t}) = \lambda(\bar{t}) \cdot [F,G](\gamma(\bar{t})) = \lambda(\bar{t}) \cdot (fF + gG)(\gamma(\bar{t})) = f(\gamma(\bar{t}))\,\lambda(\bar{t}) \cdot F(\gamma(\bar{t})).$$

Now from PMP, we have $H(\gamma(\bar{t}), \lambda(\bar{t})) = \lambda(\bar{t}) \cdot F(\gamma(\bar{t})) \geq 0$. Hence $\dot{\phi} > 0$, since $F(\gamma(\bar{t}))$, and $G(\gamma(\bar{t}))$ are independent. This proves that ϕ has at most one zero with positive derivative at the switching time and gives the desired conclusion. □

We are now interested in understanding what happens at points that are not ordinary.

Definition 14 *A point $x \in \mathbb{R}^2$ is called an* non ordinary point *if $x \in \Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$.*

In the following we study some properties of singular trajectories in relation to non ordinary points on which $\Delta_B = 0$.

Definition 15 • *An* non ordinary arc *is a $\mathcal{C}^2$ one-dimensional connected embedded submanifold S of $\mathbb{R}^2$ with the property that every $x \in S$ is a non ordinary point.*

• *A non ordinary arc is said* isolated *if there exists a set Ω satisfying the following conditions:*

(C1) *Ω is an open connected subset of $\mathbb{R}^2$.*

(C2) *S is a relatively closed subset of Ω.*

(C3) *If $x \in \Omega \setminus S$ then x is an ordinary point.*

(C4) *The set $\Omega \setminus S$ has exactly two connected components.*

• *A* turnpike *(resp.* anti-turnpike*) is an isolated non ordinary arc that satisfies the following conditions:*

(S1) *For every $x \in S$ the vectors $X(x)$ and $Y(x)$ are not tangent to S and point to opposite sides of S.*

(S2) *For every $x \in S$ one has $\Delta_B(x) = 0$ and $\Delta_A(x) \neq 0$.*

(S3) *Let Ω be an open set which satisfies* **(C1)–(C4)** *above and $\Delta_A \neq 0$ on Ω. If Ω_- and Ω_+ are the connected components of $\Omega \setminus S$ labeled in such a way $F(x) - G(x)$ points into Ω_- and $F(x) + G(x)$ points into Ω_+, then the function f satisfies*

$$f(x) > 0 \textit{ (resp. } f(x) < 0) \textit{ on } \Omega_+$$
$$f(x) < 0 \textit{ (resp. } f(x) > 0) \textit{ on } \Omega_-$$

The following Lemmas describes the relation between turnpikes, anti-turnpikes and singular trajectories. In Lemma 6 we compute the control corresponding to a trajectory whose support is a turnpike or an anti-turnpike. In Lemma 7 we prove that if this control is admissible (that is the turnpike or the anti-turnpike is regular, see Definition 16) then the corresponding trajectory is extremal and singular. In Lemma 8 we show that anti-turnpike are locally not optimal.

Lemma 6 *Let S be a turnpike or an anti-turnpike and $\gamma : [c, d] \to \mathbb{R}^2$ a trajectory of (42) such that $\gamma(c) = x_0 \in S$. Then $\gamma(t) \in S$ for every $t \in [c, d]$ iff γ corresponds to the feedback control (called* singular*):*

$$\varphi(x) = -\frac{\nabla \Delta_B(x) \cdot F(x)}{\nabla \Delta_B(x) \cdot G(x)}, \tag{47}$$

Proof. Assume that $\gamma([c, d])) \subset S$ and let u be the corresponding control, that is $\dot{\gamma}(t) = F(\gamma(t)) + u(t)G(\gamma(t))$, for almost every t. From $\Delta_B(\gamma(t)) = 0$, for a.e. t we have:

$$0 = \frac{d}{dt}\Delta_B(\gamma(t)) = \nabla \Delta_B \cdot (F(\gamma(t)) + u(t)G(\gamma(t))).$$

This means that at the point $x = \gamma(t)$ we have to use control $\varphi(x)$ given by (47). $\square$

Definition 16 (regular turnpike or anti-turnpike) *We say that a turnpike or anti-turnpike S is* regular *if $|\varphi(x)| < 1$ for every $x \in S$.*

Lemma 7 *Let $(\gamma, \lambda) : [0, \bar{t}] \to \mathbb{R}^2$ be an extremal pair that verifies $\gamma(\bar{t}) = x$, $x \in S$ where S is a turnpike or an anti-turnpike, and $\lambda(\bar{t}) \cdot G(\gamma(\bar{t})) = 0$. Moreover let $\gamma' : [0, t'] \to \mathbb{R}^2$ $(t' > \bar{t})$ be a trajectory such that:*

- $\gamma'|_{[0,\bar{t}]} = \gamma$,
- $\gamma'([\bar{t}, t']) \subset S$.

Then γ' is extremal. Moreover if ϕ' is the switching function corresponding to γ' then $\phi'|_{[\bar{t},t']} \equiv 0$.

For the proof see [14].

Lemma 8 *Let S be an anti-turnpike and $\gamma : [c, d] \to \mathbb{R}^2$ be an extremal trajectory such that $\gamma([c, d]) \subset S$. Then γ is not optimal.*

Proof. Choose an open set Ω containing $\gamma([c, d])$ such that $\Delta_A \neq 0$ on Ω and define the differential form ω on Ω by $\omega(F) = 1$, $\omega(G) = 0$. Let $\gamma_1 : [c, d_1] \to \Omega$ be any trajectory such that $\gamma_1(c) = \gamma(c)$, $\gamma_1(d_1) = \gamma(d)$, $\gamma_1(t) \notin S$ for every $t \in]c, d_1[$. Notice that $d - c = \int_\gamma \omega$, $d_1 - c = \int_{\gamma_1} \omega$. Hence $d - d_1 = \int_{\gamma_1^{-1} * \gamma} \omega$ where γ_1^{-1} is γ_1 run backward. Assume that $\gamma_1^{-1} * \gamma$ is oriented counterclockwise, being similar the opposite case. Then by Stokes' Theorem, $d - d_1 = \int_A d\omega$ where A is the region enclosed by $\gamma_1^{-1} * \gamma$. Now $d\omega(F, G) = F \cdot \nabla\omega(G) - G \cdot \nabla\omega(F) - \omega([F, G]) = -f$, by definition of ω. Since $d\omega(F, G) = \Delta_A d\omega(\partial_x, \partial_y)$, we get

$$d - d_1 = \int_A \left(-\frac{f}{\Delta_A} \right) dx\, dy.$$

If Δ_A is positive (resp. negative) then $F + G$ (resp. $F - G$) points to the side of S where γ_1 is contained and, by definition of anti-turnpike, f is negative (resp. positive) on A. We conclude that $d > d_1$ and γ is non optimal. □

One finally gets:

Theorem 8 *Let $\gamma : [0, \bar{t}] \to \mathbb{R}^2$ be an optimal trajectory that it is singular on some interval $[c, d] \subset [0, \bar{t}]$. Then, under generic conditions, $Supp(\gamma|_{[c,d]})$ is contained in a regular turnpike S.*

Proof. From $\phi \equiv 0$ on $[c, d]$ it follows $\dot{\phi} \equiv 0$ on $[c, d]$. By Lemma 4, $Supp(\gamma|_{[c,d]}) \subset \Delta_B^{-1}(0)$. Under generic conditions, $\Delta_B^{-1}(0) \cap \mathcal{R}(\bar{t})$ is formed by a finite number of turnpikes, anti-turnpikes and isolated points (at intersections with $\Delta_A^{-1}(0)$). By Lemma 8 we conclude. □

Figure 7 illustrates the possible switchings in the connected regions of $\mathbb{R}^2 \setminus (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0))$ in relation with the sign of f, and an example of extremal trajectory containing a singular arc.

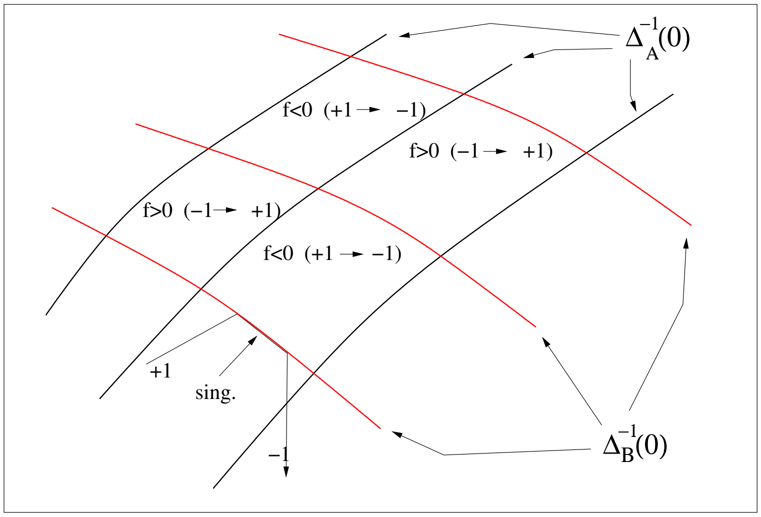

Figure 7: Possible switchings in the connected regions of $\mathbb{R}^2 \setminus (\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0))$ in relation with the sign of f, and an example of extremal trajectory containing a singular arc.

4.3 The Optimal Synthesis: A Brief Summary of Results

Consider the control system (42) assume **(H)**, and focus on the problem of reaching every point of the plane in minimum time starting from the origin $x_0 = (0, 0)$, under the additional hypothesis that $F(x_0) = 0$.

Remark **24** The hypothesis $F(x_0) = 0$, i.e. x_0 is a stable point for F, is very natural. In fact, under generic assumptions, it guarantees local controllability. Moreover if we reverse the time, we obtain the problem of stabilizing in minimum time all the points of M to x_0. For this time-reversed problem, the stability of x_0 guarantees that once reached the origin, it is possible to stay.

In [46, 47] (see also [14]) it was proved that this problem with F and G generic, admits an optimal synthesis in finite time T. More precisely, if we define the reachable set in time T:

$$\begin{aligned}\mathcal{R}(T) \quad := \quad & \{x \in \mathbb{R}^2 : \exists\, b_x \in [0, T] \text{ and a trajectory} \\ & \gamma_x : [0, b_x] \to M \text{ of (42) such that } \gamma_x(0) = x_0,\ \gamma_x(b_x) = x\}, \quad (48)\end{aligned}$$

an *optimal synthesis in time* T, for the control system (42) under the additional hypothesis $F(x_0) = 0$, is a family of time optimal trajectories $\Gamma = \{\gamma_x : [0, b_x] \mapsto \mathbb{R}^2,$

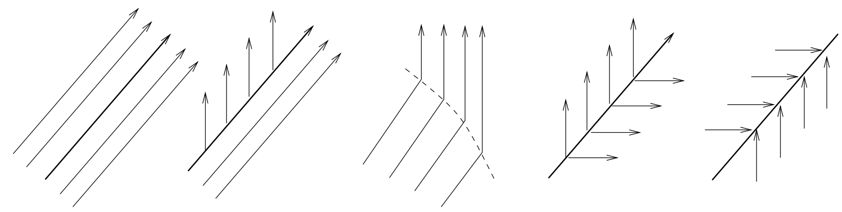

Figure 8:

$x \in \mathcal{R}(T) : \;\; \gamma(0) = x_0, \; \gamma(b_x) = x\}$. Moreover it was also proved that there exists a stratification of $\mathcal{R}(T)$ (roughly speaking a partition of $\mathcal{R}(T)$ in manifolds of different dimensions, see [14] for more details) such that the optimal synthesis can be obtained from a feedback $u(x)$ satisfying:

- on strata of dimension 2, $u(x) = \pm 1$,
- on strata of dimension 1, called *frame curves* (FC for short), $u(x) = \pm 1$ or $u(x) = \varphi(x)$, where $\varphi(x)$ is defined by (47).

The strata of dimension 0 are called *frame points* (FP). Every FP is an intersection of two FCs. A FP x, which is the intersection of two frame curves F_1 and F_2 is called an (F_1, F_2) Frame Point. In [47] (see also [14]), it is provided a complete classification of all types of FPs and FCs, under generic conditions. All the possible FCs are:

- FCs of kind Y (resp. X), corresponding to subsets of the trajectories γ^+ (resp. γ^-) defined as the trajectory exiting x_0 with constant control $+1$ (resp. constant control -1);
- FCs of kind C, called *switching curves*, i.e. curves made of switching points;
- FCs of kind S, i.e. *singular trajectories*;
- FCs of kind K, called *overlaps* and reached optimally by two trajectories coming from different directions;
- FCs which are arcs of optimal trajectories starting at FPs. These trajectories "transport" special information.

The FCs of kind Y, C, S, K are depicted in Fig. 8.
There are eighteen topological equivalence classes of FPs. They are showed in Figure 9. A detailed description can be found in [14, 47].

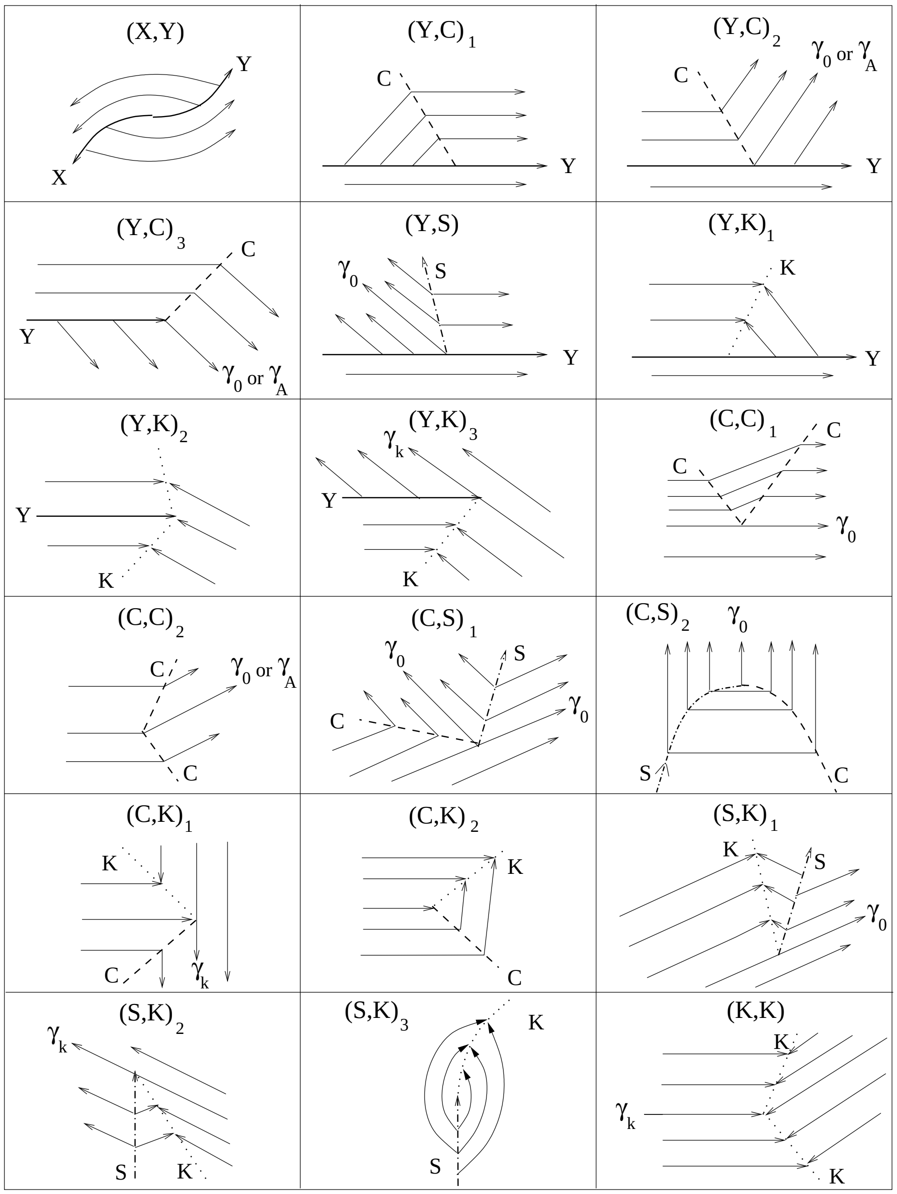

Figure 9: The FPs of the Optimal Synthesis.

4.4 Some Examples

As first example, let us compute the time optimal synthesis for the problem given in the introduction.

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = u. \end{array} \right. , \quad |u| \leq 1$$

With respect to the introduction, here we reverse the time and we consider the problem of reaching every point of the plane starting from $x_0 = (0,0)$. Writing the system as $\dot{x} = F + uG$ we have:

$$F = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [F,G] = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Hence:

$$\Delta_A(x) = det(F,G) = x_2, \quad \Delta_B(x) = det(G,[F,G]) = 1,$$

$$f(x) = -\Delta_B/\Delta_A = -1/x_2. \tag{49}$$

Since $\Delta_B^{-1}(0) = \emptyset$ there are not singular trajectories (cfr. Theorem 8, p. 52). Moreover, in the region $x_2 > 0$ (resp. $x_2 < 0$), the only switchings admitted are from control $+1$ to control -1 (resp. from control -1 to control $+1$, cfr. Theorem 7). The trajectories corresponding to control $+1$ (resp. -1) and starting from the point (x_{10}, x_{20}) are the two parabolas:

$$\left\{ \begin{array}{l} x_1 = x_{10} + x_{20}t \pm \frac{1}{2}t^2 \\ x_2 = x_{20} \pm t. \end{array} \right. \tag{50}$$

To compute the switching times, we have to solve the equation for the covector $\dot{p} = -p(\nabla F + u\nabla G)$:

$$\left\{ \begin{array}{l} p_1 = p_{10} \\ p_2 = p_{20} - p_{10}t \end{array} \right. \tag{51}$$

The switching function is then $\phi(t) = p(t) \cdot G(x(t)) = p_{20} - p_{10}t$. It is clear that every time optimal trajectory has at most one switching because, for every initial conditions p_{10} and p_{20}, the function ϕ has at most one zero. After normalizing $p_{10}^2 + p_{20}^2 = 1$, the switchings are parametrized by an angle belonging to the interval $[0, 2\pi]$. The time optimal synthesis is showed in Figure 10 A. In Figure 10 B it is presented the time optimal synthesis for the same control system, but starting from a ball of radius $1/2$. Computing the explicit expression of the switching curves is an exercise for the reader (see Exercise 9, p. 61).

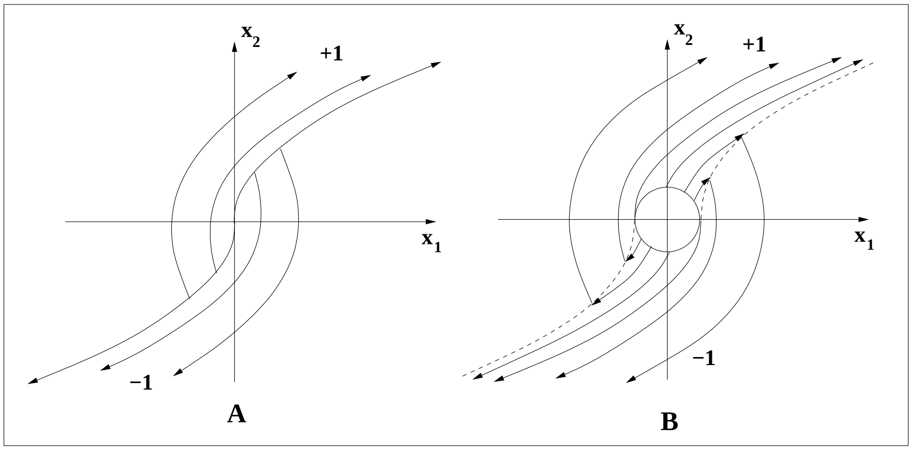

Figure 10:

A more rich example presenting the FPs of kind (X, Y), $(Y, K)_1$, (Y, S) is the following:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + \frac{1}{2}x_1^2 \end{cases} .$$

Let us build the time optimal synthesis for a fixed time $T > 2$. The $X-$ and $Y-$ trajectories can be described giving x_2 as a function of x_1 and are, respectively, cubic polynomials of the following type:

$$\begin{aligned} x_2 &= -\frac{x_1^3}{6} - \frac{x_1^2}{2} + \alpha \qquad \alpha \in \mathbb{R} \\ x_2 &= \frac{x_1^3}{6} + \frac{x_1^2}{2} + \alpha \qquad \alpha \in \mathbb{R}. \end{aligned} \tag{52}$$

With a straightforward computation we obtain:

$$[F, G] = \begin{pmatrix} 0 \\ -1 - x_1 \end{pmatrix}$$

then the system is locally controllable and:

$$\Delta_B(x) = \det \begin{pmatrix} 1 & 0 \\ 0 & -1 - x_1 \end{pmatrix} = -1 - x_1. \tag{53}$$

From equation (53) it follows that every turnpike is subset of $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = -1\}$. Indeed, the synthesis contains the turnpike:

$$S = \left\{ (x_1, x_2) : x_1 = -1, x_2 \leq -\frac{1}{3} \right\}.$$

Given b, consider the trajectories $\gamma_1 : [0,b] \mapsto \mathbb{R}^2$ for which there exists $t_0 \in [0,b]$ such that $\gamma_1|[0,t_0]$ is a Y–trajectory and $\gamma_1|[t_0,b]$ is an X–trajectory, and the trajectories $\gamma_2 : [0,b] \mapsto \mathbb{R}^2$, $b > 2$, for which there exists $t_1 \in [2,b]$ such that $\gamma_2|[0,t_1]$ is an X–trajectory and $\gamma_2|[t_1,b]$ is a Y–trajectory. For every $b > 2$, these trajectories cross each other in the region of the plane above the cubic (52) with $\alpha = 0$ and determine an overlap curve K that originates from the point $(-2,-\frac{2}{3})$. We use the symbols $x^{+-}(b,t_0)$ and $x^{-+}(b,t_1)$ to indicate, respectively, the terminal points of γ_1 and γ_2 above. Explicitly we have:

$$x_1^{+-} = 2t_0 - b \qquad x_2^{+-} = -\frac{(2t_0-b)^3}{6} - \frac{(2t_0-b)^2}{2} + t_0^2 + \frac{t_0^3}{3} \tag{54}$$

$$x_1^{-+} = b - 2t_1 \qquad x_2^{-+} = \frac{(b-2t_1)^3}{6} + \frac{(b-2t_1)^2}{2} - t_1^2 + \frac{t_1^3}{3}. \tag{55}$$

Now the equation:

$$x^{+-}(b,t_0) = x^{-+}(b,t_1), \tag{56}$$

as b varies in $[2,+\infty[$, describes the set K. From (54), (55) and (56) it follows:

$$t_0 = b - t_1 \qquad t_1\Big(-2t_1^2 + (2+3b)t_1 + (-b^2-2b)\Big) = 0.$$

Solving for t_1 we obtain three solutions:

$$t_1 = 0, \qquad t_1 = b, \qquad t_1 = 1 + \frac{b}{2}. \tag{57}$$

The first two of (57) are trivial, while the third determines a point of K, so that:

$$K = \left\{(x_1,x_2) : x_1 = -2, x_2 \geq -\frac{2}{3}\right\}.$$

The set $\mathcal{R}(T)$ is portrayed in Fig. 11.

Bibliographical Note

For the problem of controllability, classical results are found in the papers by Krener [37], and Lobry [41]. For the problem of controllability on Lie groups see the papers by Jurdjevic-Kupka [30, 31], Jurdjevic-Sussmann [32], Gauthier-Bornard [27], and Sachkov [60].

The issue of existence for Optimal Control as well as for Calculus of Variations is a long standing research field. For a review of classical available results we refer to [22, 23, 26].

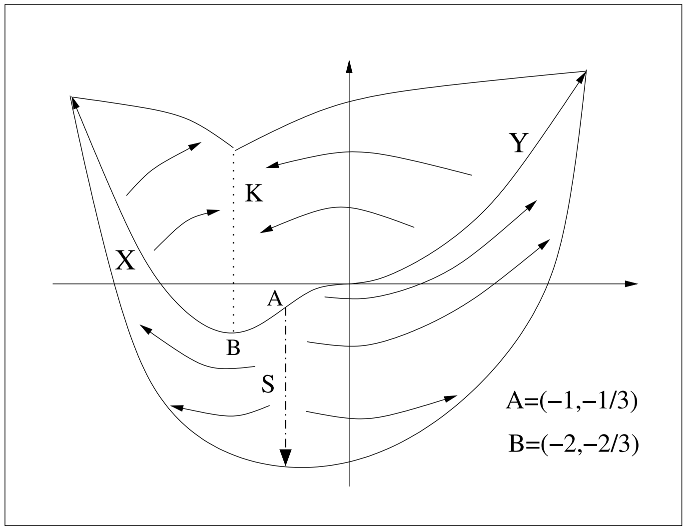

Figure 11: Example 1, (X, Y), $(Y, K)_1$, (Y, S) Frame Points

The terms Abnormal Extremals and Singular Extremals (in our case called Endpoint Singular Extremals) are used with several different meanings. For some results see [1, 2, 11, 12, 13, 44, 45].

The first sufficiency theorem, for extremal synthesis, was given by Boltyanskii in [10]. Then various generalization appeared, [20, 21, 48]. In particular the result of [48], differently from the previous ones, can be applied to systems presenting Fuller phenomena.

In **Step 3** of Section 3.5, an alternative method for the construction of the Optimal Synthesis is indicated. Namely, to construct all extremals in the cotangent bundle. This second method is more involved, but induces a more clear understanding of the fact that projection singularities are responsible for singularities in the Optimal Synthesis. This approach was used in [15, 16, 33, 34].

For sufficiency theorems, in the framework of viscosity solutions to Hamilton–Jacobi–Bellman equation (mentioned in **Step 4** of Section 3.5), see [6].

Dubins' car problem was originally introduced by Markov in [43] and studied by Dubins in [24]. In particular Dubins proved that every minimum time trajectory is concatenation of at most three arcs, each of which is either an arc of circle or a straight line. If we consider the possibility of non constant speed and admit also backward motion, then we obtain the model proposed by Reed and Shepp [50]. A family of time optimal trajectories, that are sufficiently reach to join optimally any two points, was given in [59]. Now the situation is more complicated since there are 46 possible

combination of straight lines and arcs of circles. Then a time optimal synthesis was built by Soueres and Laumond in [51].
Time optimal trajectories for the system (41) were studied mainly by Boissonnat, Cerezo, Kostov, Kostova, Leblond and Sussmann, see [8, 9, 35, 36, 58].

The concept of synthesis was introduced in the pioneering work of Boltianski [10]. Properties of extremal trajectories for single input affine control systems on the plane were first studied by Baitman in [4, 5] and later by Sussmann in [55, 56].

The existence of an optimal synthesis for analytic systems and under generic conditions for $\mathcal{C}^\infty$ systems was proved by Sussmann in [57] and by Piccoli in [46] respectively.

Finally, a complete classification of generic singularities and generic optimal synthesis is given by Bressan and Piccoli in [19, 47].

Exercises

Exercise 1 Consider the control system on the plane:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \sin(x) \\ 0 \end{pmatrix} + u \begin{pmatrix} G_1(x,y) \\ G_2(x,y) \end{pmatrix}, \quad u \in \mathbb{R}, \tag{58}$$

where $G_1, G_2 \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and $G_1(x,y), G_2(x,y) > 0$ for every $(x,y) \in \mathbb{R}^2$. Find the set of points where Theorem 2, p. 24 permits to conclude that there is local controllability. Is it true that in all other points the system is not locally controllable?

Exercise 2 Consider the control system $\dot{x} = F(x) + uG(x)$, $u \in \mathbb{R}$ where:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x_1^2 + x_2^2 + x_3^2 = 1,$$

$$F(x) = \begin{pmatrix} x_2 \cos(\alpha) \\ -x_1 \cos(\alpha) \\ 0 \end{pmatrix}, \quad G(x) = \begin{pmatrix} 0 \\ x_3 \sin(\alpha) \\ -x_2 \sin(\alpha) \end{pmatrix}, \quad \alpha \in]0, \pi/2[.$$

Find where the system is locally controllable and prove that it is (globally) controllable.
[Hint: to prove global controllability, find a trajectory connecting each couple of points.]

Exercise 3 Prove that, if the controls take values in the whole $\mathbb{R}^m$, then there exists no minimum for the minimum time problem of system (18). Moreover, show that, for a distributional problem with quadratic cost (the so called *subriemannian problem*), if we do not fix the final time, then there exists no minimum.

Exercise 4 Prove that if we fix the final time in the PMP, then condition (PMP2) becomes

(PMP2bis) For a.e. $t \in [0,T]$,

$$H(\gamma(t),\lambda(t),\lambda_0) = \mathcal{H}(\gamma(t),\lambda(t),\lambda_0,u(t)) = const \geq 0;$$

and find the value of the constant for a given γ.

Exercise 5 Consider the distributional system $\dot{x} = u_1F_1(x) + u_2F_2(x)$, $x \in M$, $u_1, u_2 \in \mathbb{R}$, with quadratic cost (19) and fixed final time T. Prove that (λ, λ_0) can be chosen up to a multiplicative positive constant and if $\mathcal{H}(\gamma(t),\lambda(t),\lambda_0,u(t)) = \frac{1}{2}$ with $\lambda_0 = -\frac{1}{2}$ (see (PMP2bis) of the previous Exercise), then the extremal is parametrized by arclength (i.e. $u_1^2(t) + u_2^2(t) = 1$).
[Hint: Use the condition $\frac{\partial \mathcal{H}}{\partial u} = 0$ implied by the condition (PMP2bis).]

Exercise 6 Consider the control system $\dot{x} = f(x,u)$, $x \in M$ and $u \in U = \mathbb{R}^m$, and the problem (17) with $\mathcal{T} = \{x_1\}$, $x_1 \in M$. Prove that if γ is an endpoint singular trajectory and, for every $t \in Dom(\gamma)$, the function $u \to \langle p(t), f(\gamma(t),u)\rangle$ is strictly convex, then γ is an abnormal extremal.

Exercise 7 Consider a distributional system $\dot{x} = \sum_i u_iF_i(x)$. Prove that for the problem (17), with $\psi \equiv 0$, $\mathcal{T} = \{x_1\}$ and the quadratic cost $L = \sum_i u_i^2$, every endpoint singular extremal is an abnormal extremal and viceversa (what happens if $x_0 = x_1$?). Prove the same for minimum time with bounded control.

Exercise 8 Consider the system $\dot{x} = F(x) + uG(x)$, $u \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$ and the optimal control problem with Lagrangian $L(x) = x_2^2$, $\psi = 0$, initial point $(0,0)$ and terminal point $(c,0)$. Assume that, for every x, we have $F_1(x) > 0$, $G_2(x) > 0$, $\Delta_A(x) := F_1(x)G_2(x) - F_2(x)G_1(x) > 0$. Prove that there exists a monotone function $\phi(t)$, such that $\gamma(t) = (\phi(t), 0)$ is the unique optimal trajectory and show that it is extremal for every covector $\lambda \equiv 0$ and $\lambda_0 < 0$.
[Hint: find a control such that the corresponding trajectory has zero cost.]

Exercise 9 Compute the time optimal synthesis for the control system on the plane $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $|u| \leq 1$ with source the unit closed ball.

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