UNIVERSITÉ DE BOURGOGNE

Laboratoire Le2i.

Mémoire

présenté pour obtenir le diplôme d'

Habilitation à diriger des recherches

Spécialité : MATHEMATIQUE ET AUTOMATIQUE

par

Ugo Boscain

Sujet: Motion Planning and Optimal Control for Quantum Mechanical Systems

Soutenue le 14 Février devant le jury composé de :

M. Jacques Descusse	Examinateur
M. Jean-Paul Gauthier	Examinateur
M. Luc Dugard	Rapporteur
M. Michele Paindavoine	Examinateur
M. Pierre Rouchon	Rapporteur
M. Mohammed M'Saad	Rapporteur

December 13, 2006

Acknowledgments

I would like to thank Prof. Jean-Paul Gauthier, for proposing me to join the LE2i and for his offer of scientific collaboration. I am deeply grateful to him for all what he taught me in modern control theory and beyond.

I would like to thank Prof. Benedetto Piccoli, my Ph.D supervisor, who initiated me to control theory and always proposed me very challenging problems and projects, like that of writing a book. I also thank him for being a constant source of suggestions and help during my carrier.

I am deeply grateful to Prof. Andrei A. Agrachev for his scientific generosity. He was always very close to me, proposing nice research projects and giving illuminating ideas.

I am also greatly indebted to Prof. Luc Dugard, Prof. Pierre Rouchon and Prof. Mohammed M' Saad that accepted to be reviewers of this mémoire and took the time to read it.

I am grateful to Prof. Jacques Descusse and to Prof. Michel Paindavoine who accepted to be in the jury of this HDR.

A special thank goes to Prof. Yacine Chitour, one of my closest collaborators. With him I solved the most complicated problems.

I am also grateful to my collegue Claudio Altafini for his careful reading of the manuscript and for many suggestions.

I finally thank my closest collaborators and friends Riccardo Adami, Grégoire Charlot, Mario Sigalotti and my students Paolo Mason and Francesco Rossi, for the time spent together in front of a blackboard.

A Mario Sigalotti aggiungo un ringraziamento speciale per essere stato come un fratello nell'ultimo anno.

A Fobo

Contents

1 Introduction

In this mémoire I present my scientific activity in the years 2000-2006. It is in the framework of geometric control theory and concerns mainly problems of optimization, stability and motion planning of control systems on spaces of small dimension or with some special structure (Lie Groups).

From a theoretical point of view, I focused mainly on optimal control problems for non linear systems. In particular on optimal syntheses for 2-D systems and on optimal control problems (including sub-Riemannian geometry) on semisimple Lie groups. I worked also on problems of stability of switching systems for arbitrary switchings.

About applications I worked mainly in the field of control of quantum mehcanical systems (quantum control). This research has applications in *physical chemistry*, where the aim is to selectively break/form bonds in molecules, and in *quantum information processing* for the realization of *quantum gates* (namley the quantum counterpart of the elementary operations AND, OR, NOT).

The study of quantum control involved also motion planning questions and controllability problems in infinite dimensional spaces.

In the following I present a list of research topics divided in theoretical and applications, with the list of main collaborators.

Theoretical Topics:

- Stability of Switched Systems for Arbitrary Switchings. In collaboration with:
  - Yacine Chitour, Paris XI University (Paris-sud, Orsay), Laboratoire Signaux et Systemes
  - Paolo Mason, SISSA student, Trieste
  - Gregoire Charlot, Grenoble University, Institut Fourier.
  - Mario Sigalotti, INRIA, Nancy,
- Time Optimal Syntheses on 2-D Manifolds. In collaboration with:
  - Benedetto Piccoli, IAC, CNR Rome
  - Igor Nikolaev, Montreal University,
- Time Optimal Control and Sub-Riemannian Geometry on Lie Groups. In collaboration with:
  - Andrei Agrachev, SISSA, Trieste, Functional Analysis Sector.
  - Jean-Paul Gauthier, Bourgogne University, “aile de l’ ingenieur”, laboratoire Le2i.
  - Yacine Chitour, Paris XI University (Paris-sud, Orsay), Laboratoire Signaux et Systemes
  - Francesco Rossi, SISSA student, Trieste.

Applications:

- Motion Planning and Optimal Control of Finite Dimensional Quantum Systems. In collaboration with:
  - Jean-Paul Gauthier, Bourgogne University, “aile de l’ ingenieur”, laboratoire Le2i.
  - Thomas Chambrion, Nancy University.
  - Gregoire Charlot, Grenoble University, Institut Fourier.
  - Stephane Guerin, Bourgogne University, Laboratoire de Physique.
  - Hans–Rudolf Jauslin, Bourgogne University, Laboratoire de Physique.
- Controllability of the Schrödinger equation . In collaboration with:
  - Riccardo Adami, University of Milano Bicocca. Department of Mathematics.

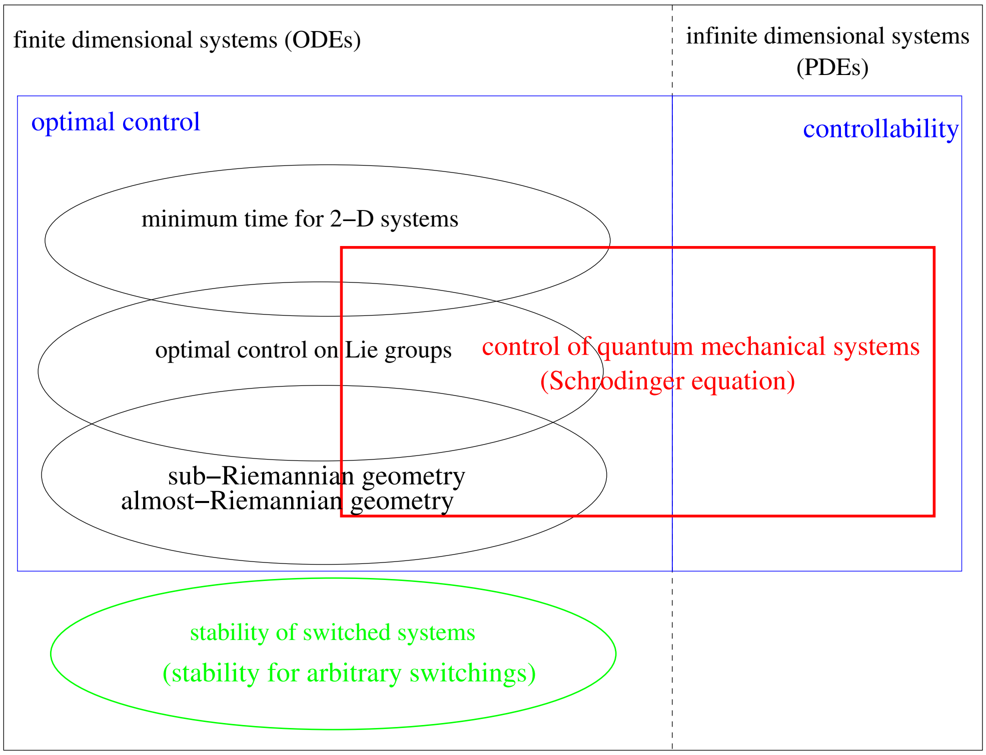

Figure 1:

In Figure 1 the interconnections among the subjects are showed.
After presenting a Curriculum Vitae (Section 2), I have decided to give a detailed description of my research about quantum control (see Section 3), including the most relevant papers:

[1] U. Boscain, G. Charlot, J.-P. Gauthier, Stéphane Guérin and Hans–Rudolf Jauslin, *Optimal Control in laser-induced population transfer for two- and three-level quantum systems*, Journal of Mathematical Physics, 43, pp. 2107-2132, 2002.

[2] U. Boscain, T. Chambrion and J.-P. Gauthier, *On the K+P problem for a three-level quantum system: Optimality implies resonance*, Journal of Dynamical and Control Systems, 8, pp. 547-572, 2002.

[3] U. Boscain, G. Charlot, *Resonance of Minimizers for n-level Quantum Systems with an Arbitrary Cost*, ESAIM COCV, pp. 593-614, 2004.

[4] U. Boscain, T. Chambrion, G. Charlot, *Nonisotropic 3-level Quantum Systems: Complete Solutions for Minimum Time and Minimal Energy*, Discrete and Continuous Dynamical Systems-B, 5 pp. 957-990, 2005.

[5] U. Boscain, P. Mason, *Time Minimal Trajectories for a Spin 1/2 Particle in a Magnetic field*, J. Math. Phys. 47, 062101 (29 pages) 2006.

[6] R. Adami, U. Boscain, *Controllability of the Schrödinger Equation via Intersection of Eigenvalues*, Proceedings of the 44rd IEEE Conference on Decision and Control December 12-15, Seville, (Spain). Also on "Control Systems: Theory, Numerics and Applications, Roma, Italia 30 Mar - 1 Apr 2005, POS, Proceeding of science.

[7] A. Agrachev, U. Boscain, M. Sigalotti, *A Gauss-Bonnet-like Formula on Two-Dimensional Almost-Riemannian Manifolds*, PREPRINT SISSA 55/2006/M, submitted to Discrete and Continuous Dynamical Systems-A.

The papers [**1**]–[**5**] focus on optimal control problems for finite dimensional quantum systems, while the paper [**6**] treats a problem of controllability of the Schrödinger Equation as a PDE. Finally the paper [**7**] is on a problem of almost-Riemannian geometry that naturally arises in the field of quantum control. All these papers have been written after the defence of my ph.D Thesis in 2000.

In Section 4, I give a brief description of my other research topics.

2 Curriculum Vitae

Professional Address
LE2i, CNRS UMR5158,
Universite de Bourgogne,
9, avenue Alain Savary- BP 47870
21078 DIJON - CEDEX FRANCE
Tel. +33 380 39 58 38

Present Position

From 1st November 2006, Chargé de recherche of first class (CR1) at the Laboratoire Le2i of Dijon University, in the CNRS section 07.

Personal Data

Birthday	:	October 24th 1968
Birthplace	:	Ivrea (Turin), Italy
Nationality	:	Italian
Military Service	:	from 13/3/1993 to 22/3/1994

2.1 Academic Positions, Fellowships

2.1.1 Qualification

- February 2006: "Qualification" as professor of class 26 (Applied Mathematics) in France.

2.1.2 Academic Positions

- From 1st November 2006, Chargé de recherche of first class (CR1) at the Laboratoire Le2i of the Dijon University in the CNRS section 07. (Second in the national list, four positions available.)
- April 2002–October 2006: Full time Permanent Researcher at the International School for Advanced Studies (SISSA-ISAS), Trieste. Sector: Functional Analysis and Applications. Activity: Applied Mathematics.

2.1.3 Post Doc Positions

- 19th December 2001 to 18th December 2003: Individual Marie Curie Post doc. fellowship, at Bourgogne University.
- 1st December 2000 – 30th November 2001: TMR (Training and Mobility of Researchers) Post Doc fellowship of the Non Linear Control Network (NCN), Bourgogne University.

2.1.4 Visiting Positions

- From 1th November 2000 to 30th November 2000: Research Contract, University of Salerno.
- From 2001, about one month invitation (every year), to Paris XI (Orsay).

2.2 Education

2.2.1 PH.D

November 1996, October 2000, Ph.D in Mathematics SISSA – *Scuola Internazionale Superiore di Studi Avanzati*, Trieste, Italy, with fellowship.

- Dissertation: October 20th 2000.
- Supervisor of the Thesis: Prof. Benedetto Piccoli, IAC, CNR Rome
- Referee of the Thesis: Prof. Gianna Stefani, Università degli Studi di Firenze, Italy
- Title of the Thesis: *Extremal Synthesis and Morse Property for Minimum Time.*
- Member of Thesis Committee:

  Prof. Andrei Agrachev, SISSA, Trieste, Italy, and Steklov Institute of Mathematics, Moscow, Russia
  Prof. Antonio Ambrosetti, SISSA, Trieste, Italy
  Prof. Andrea Braides, Università "Tor Vergata", Roma, Italy
  Prof. Alberto Bressan, SISSA, Trieste, Italy

2.2.2 Degrees

1. July 4th, 1996: Laurea (Degree) in Theoretical Physics, University of Torino, with full marks Cum Laude and the special mention for the Thesis Dignita' di Stampa.
   - Supervisor of the Thesis: Prof. Leonardo Castellani, "Universitá del Piemonte Orientale", Italy.
   - Referee of the Thesis: Prof. Vittorio de Alfaro, "Universitá di Torino".
   - Title of the Thesis: "Integrazione su Gruppi Quantici e sul q-spazio di Minkowski"
2. September 1991: Diploma (Degree) in Music (Piano) at "Conservatorio di Alessandria", Italy.

2.2.3 Participation to Schools

- Participation to the school on "Campi vettoriali di Hörmander, equazioni differenziali ipoellittiche e applicazioni" 12-16 July 2004, Politecnico di Milano, Dipartimento di Matematica.
- Participation to the "Summer School on Mathematical Control Theory", International Center of Theoretical Physics "Abdus Salam" (ICTP), Trieste, Italy, September 2001.
- Participation to the "Second School on the Mathematics of Economics", International Center of Theoretical Physics "Abdus Salam" (ICTP), Trieste, Italy August 2000.
- Participation to the "First Non Linear Control Network Pedagogical School" (Athens September 1999).
- Participation to the "Quarter on Control Theory" (Paris, Henri Poincaré Institute, Spring 1998).

2.3 Awards

- Special Mention for the Degree Thesis: "Dignita' di Stampa", University of Torino, July 1996.

2.4 Research Activity

2.4.1 Research Areas

Geometric Control Theory: Optimal Control, Control on Lie groups, Quantum Control, Switched (Hybrid) Systems, Control of the Schrödinger equation, Applications to Robotics, Quantum information Science, Vision Problems.

2.4.2 Referee Activity

- International Journal of Control,
- SIAM J. on Control and Optimization,
- ESAIM, Control Optimization and Calculus of Variations,
- Automatica,
- Physica A,
- Mathematics of Control Signal and Systems,
- Journal of Dynamical and Control System,
- IEEE Trans. on Automatic Control,
- IEEE Transactions on Circuits and Systems,
- Preprint sever: ”Control Theory and Partial Differential Equations”
- System and Control Letters
- ACC 2002-2004, CDC 2002-2004-2005, ECC 2001, MTNS 2004.

2.4.3 Editorial Activity and Participation to Scientific Committees

- Member of the editorial board of the Proceedings of the conference *Control Systems: Theory, Numerics and Applications*, Rome 30 March - 1 April 2005.
- Member of the scientific committee of the “Third (2004) and Fourth (2005) Junior meeting on control theory and stabilization”.
- Member of the Ph.D thesis committee at SISSA in 2002, 2003, 2004,2005,2006.
- Member of the committee for Ph.D entrance examination at SISSA in 2002, 2003, 2004, 2005, 2006.

2.5 Administrative Activity

2.5.1 European Networks and Other Projects

- TMR European Network “Nonlinear control Network”, 1998-2002.
- Multi-partner Marie Curie Training Site, 2002-2006.
- Co-writing of the Italian projects: Cofin 2002, Progetto intergruppo INDAM 2004, Cofin 2004, Cofin 2006.

2.5.2 Organization and co-organization of conferences and sessions:

1. 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005 organization of a session on "Control of PDEs and Applications"
2. 22nd IFIP. Conference on System Modeling and Optimization Turin, Italy, July 18-22, 2005, Session on "Geometric methods in optimal control".
3. 42nd IEEE Conference on Decision and Control Maui, Hawaii, Usa, December 9-12, 2003, Session on "New Trends in Geometric and Optimal Control".
4. Second Junior European Meeting on: Control Theory and Stabilization, Torino, Italy, Italy, 3-5 December 2003.
5. Trimester on "Dynamical and Control Systems" SISSA-ICTP, Italy, Sept. 8 - Dec. 7, 2003.
6. International Conference PhysCon 2003 August 20-22, 2003, Saint Petersburg, Russia, Session on "Geometric and Optimal Control Methods for Quantum Dynamical Systems".
7. Workshop on "Feedback control and optimal control", Siena, Italy, July 28-31, 2003.
8. First Junior European Meeting on: Control Theory and Stabilization, Dijon, France, October 2-4 2002.
9. "European Control Conference" 4-7 September, 2001 Seminario de Vilar, Porto, Portugal (ECC2001) Session on "Optimal Control".
10. 39th IEEE Conference on Decision and Control, Sidney, December 2000, Session on "Optimal Control And Applications".
11. Workshop on "Mathematical Control Theory and Robotics" SISSA, Trieste, Italy, 25–27 June 2000.

2.6 Invited Conferences and Talks

1. November 23th 2006, Grenoble University, Institut Fourier "Almost Riemannian geometry from a control theory point of view".
2. November 07th 2006, Université Montpellier II, Département de mathématique, "Stability of Nonlinear Switched Systems".
3. Meeting on Subelliptic PDE's and Applications to Geometry and Finance. June 12-17th 2006, CORTONA (Italy). "Singular Riemannian Geometry from a Control Theory Point of View".
4. Conference on: "Geometric Control and Nonsmooth Analysis" on the occasion of the 73rd birthday of H. Hermes and of the 71st birthday of R.T. Rockafellar. INDAM, ROME, Italy, June 5 - 9, 2006. "Stability of Nonlinear Switched Systems"
5. March 7th 2006, University of Padova, Dep. of Mathematics, "Singular Riemannian geometry from a control theory point of view".
6. February 28th, 2006, University of Rome, "La Sapienza", Colloquia di Teoria del Controllo ed EDP, INDAM, "Singular Riemannian geometry from a control theory point of view".
7. Mathematisches Forschungsinstitut Oberwolfach (MFO), Research in Pairs with Mario Sigalotti on "A Gauss-Bonnet-like Theorem on 2-D Singular Riemannian Manifolds". February 12-26th, 2006.
8. 44th IEEE Conference on Decision and Control and European Control Conference ECC 2005, Seville, (Spain). 12-15 December 2005. Communications: "Stability of Nonlinear Switched Systems on the Plane".
9. 22nd IFIP, Conference on System Modeling and Optimization Turin, Italy, July 18-22, 2005, Communication"Time Minimal Trajectories for two-level Quantum Systems".
10. Universita' di Roma "La Sapienza", May 18 2005. Communication: " Stability of Dynamical Systems under Random Switchings"
11. Workshop of the INDAM intergroup project "Controllo e Numerica" Control Systems: Theory, Numerics and Applications. Rome 30-31 March, 1 April 2005. Communication: Controllability of the Schrödinger Equation via Intersection of Eigenvalues.
12. Invitation to the Mathematisches Forschungsinstitut Oberwolfach (MFO) for the workshop "Entanglement and Decoherence: Mathematics and Physics of Quantum Information and Computation", organized by Sergio Albeverio, Gianfausto Dell'Antonio, Francesco De Martini, January 23rd - January 29th, 2005.

13. Miniworkshop on Subriemannian Geometry and pde's, November 26th, 2004, Dipartimento di Matematica, Universita' di bologna. Communication: "Sub-Riemannian and Singular-Riemannian Problems on Lie Groups and Applications to Quantum Mechanics."
14. Third Junior European Meeting on "Control, Optimization and Computation", September 6-8, 2004 Department of Mathematics University of Aveiro Portugal Communication:" Stability of Switched Systems for Random Switching Functions: A Survey"
15. Politecnico di Torino, Italy, July 15th, 2004. Communication: "On the existence of a common polynomial Lyapunov function for linear switched systems"
16. INRIA, Sophia-Antipolis, France, March 18th, 2004. Communication: Stability of planar switched systems for arbitrary switchings."
17. I.A.C. CNR, Rome, February 12th, 2004. Communication: "Controllo ottimo di sistemi quanto-meccanici in dimensione finita",
18. Convegno Nazionale "Problemi Matematici in Meccanica Quantistica" Modena, 18-20, December 2003. Communication: "Controllo ottimo per sistemi quantistici finito dimensionali."
19. 42nd IEEE Conference on Decision and Control, Maui Hawaii, USA, December 9-12, 2003, Communications: "Resonance of Minimizers for N-Level Quantum Systems".
20. 42nd IEEE Conference on Decision and Control, Maui Hawaii, USA, December 9-12, 2003, Communications: "Stability of Planar Nonlinear Switched Systems".
21. 42nd IEEE Conference on Decision and Control, Maui Hawaii, USA, December 9-12, 2003, Communications: "Time Optimal Synthesis for a So(3)-Left-Invariant Control System on a Sphere."
22. Second Junior meeting on Control Theory and Stabilization, Politecnico di Torino, Italy, December 3-5, 2003,, Communication: "Resonance of Minimizers for n-level Quantum Systems with an Arbitrary Cost."
23. Workshop on: Feedback Control and Optimal Control, Siena, Italy, July 28-31, 2003, Communication: "Bound on the number of switchings for a minimum time problem on the rigid body"
24. CIMPA School and Workshop on Geometric Non-linear Control, State University of Campinas, Brazil, July 14-26, 2003. Communication: "Optimal control on a n-level quantum system"
25. Bimestre su "Probabilita' e Meccanica Statistica nella Scienza dell'Informazione", CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI, Pisa, Italy, June-July 2003. Communication: "Optimal Control of a N-level Quantum System" (9 June)
26. University of Genova, June 3, 2003, Communication: "Stability of Switched Systems"
27. IFAC 2nd Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNLC'03, Seville, SPAIN, April 3-5, 2003. Communication: "Optimal Control on a n-level Quantum System"
28. 41nd IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, December 10-13, 2002. Communications: "On the K+P Problem for a Three Level Quantum"
29. 41nd IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, December 10-13, 2002. Communications: "Stability of Planar Switched Systems: The Linear Single Input Case"
30. International Conference on Differential and Functional Differential Equations Moscow, Russia, August 11-17, 2002, (satellite conference of the International Congress of Mathematicians, 2002, Beijing, China). Communication: "On the K+P problem for a three level quantum system"
31. Fourth International Conference on Dynamical Systems and Differential Equations, Wilmington, NC, USA, May 24-27, 2002, Special session on "Stabilization and optimal control of dynamical systems". Communication "Lower and upper bounds for the number of switchings for time optimal trajectories of the Dubin's problem on SO(3)."
32. Mini-workshop on Switched Systems. Politecnico di Torino. February 26, 2002. Communication: "Stabilita' di sistemi switched bidimensionali"
33. European Control Conference, 4-7 September, 2001 Seminario de Vilar, Porto, Portugal (ECC2001). Communication: "Stability of Switched Systems: The Single Input Case."
34. 4th Nonlinear Control Network Workshop: Nonlinear and Adaptive Control, Sheffield, UK, June 25-28, 2001. Communication: "Optimal control of the Schrodinger equation with two or three levels."
35. University of Paris Orsay, France, April 19th, 2001. Communication: "Control on a Three Level Schrodinger Equation".
36. International Conference on Differential Equations and Dynamical Systems, Suzdal, August 2000. Communication: "Projection Singularities of Extremals for Planar Control Systems".
37. Workshop on "Mathematical Control Theory and Robotics", SISSA Trieste Italy, 25–27 June 2000. Communication: "Stability of Planar Switched Systems". Moreover presentation of the Poster: "Projection Singularities for planar Systems".
38. 38th IEEE Conference on Decision and Control", Phoenix–Arizona, USA, December 7-10, 1999. Communication: "Projection Singularities of Extremals for Planar systems."

2.7 Didactic Activities (Courses and short Courses)

- November 2006, Bourgogne University, Research Master, "Analyse d'image" (10 hours).
- September 2006, Padova University, Minicourse on "Introduzione ai sistemi ibridi e switching in teoria del controllo" (Introduction to Hybrid and Switching systems in Control Theory) (3 hours).
- September 2006, ICTP Diploma Program in Mathematics, Ordinary Differential Equations (7.5 hours)
- May 2006, Laurea Specialistica in Matematica, SISSA-Universita' di Trieste. Title of the course "Esercitazioni di Analisi Funzionale" (6 hours)
- March 2006, UCAD, Dakar, Senegal, Introduction to Control Theory (10 hour).
- October-November 2005, Laurea Specialistica in Matematica, SISSA-Universita' di Trieste and SISSA Ph.D program. Title of the course "Esercitazioni di equazioni differenziali ordinarie" (26 hours)
- October 2005, University of Milano Bicocca, Ph.D Program, "Introduzione alla teoria del controllo geometrico" (12 hours).
- September 2005, ICTP Diploma Program in Mathematics, Calculus in $\mathbb{R}^n$ (7.5 hours)
- September 2005, ICTP Diploma Program in Mathematics, Ordinary Differential Equations (7.5 hours)
- 28-29 June 2005, Corso di Dottorato, Universita' di Bologna. Introduction to Optimal Control Theory (4 hours).
- October-December 2004, Laurea Specialistica in Matematica, SISSA-Universita' di Trieste. Title of the course "Esercitazioni di equazioni differenziali ordinarie" (16 hours)
- April-May 2004, ICTP / SISSA joint master's degree program in Modeling and Simulation of Complex Realities, Title of the course "Control Theory". (16 hours)
- February 3,4,5 2004, INFN, University of Naples, "Introduzione alla Teoria del Controllo Quantistico" (6 hours)
- January-April 2004, SISSA Ph.D program. Title of the course: "Introduction to Control Theory" (20 hours).
- September 8 - December 7, 2003. Trimester on Dynamical and control systems, ICTP, SISSA, Italy. Title of the course "Control of Quantum Systems, " (4 hours).
- April 26 - May 8 2003, CIMPA school on *Contrôle non lineaire et application.*, Tlemcen (Algeria). Title of the course: "Optimal control and applications" (14 hours).
- May 2003, SISSA Ph.D program. Title of the course: "Optimal Synthesis and Applications to Quantum Mechanics" (18 hours)

2.8 Supervision of Students and Post Doc

Supervision of Ph.D Thesis

- Paolo Mason, SISSA Ph.D program. Co-direction with A Agrachev defense: 26th October 2006. Title: "Stability, optimization and motion planning for control affine systems". http://www.iecn.u-nancy.fr/∼mason/
- Rebecca Salmoni, Naples University and Paris XI (Orsay), Co-direction with Y. Chitour. Ph.D defense in Italy February 2th 2006. Title "Su una sintesi ottima per un sistema quantistico a due livelli". Expected Ph.D defence in France: spring 2007.

Supervision of Master Thesis

- Viktoriya Victorovna Semeshenko, ICTP / SISSA joint master's degree program in Modeling and Simulation of Complex Realities, (from August 2002 to September 2003). Title of the Master Thesis "Geometric Control Techniques for Three and Four Levels Quantum Systems".

Supervision of mémoire de la "Laurea Specialistica in Matematica"

- Francesco Rossi, SISSA-Universita' di Trieste, Laurea Magistralis in Matematica. (defence: July 19th, 2006). Title: "Problemi di Geometria sub-Riemmaniana su gruppi di Lie compatti".

2.9 Publications

Type of Pubblication	#
Books	1
Chapters of Books	1
Preprints	3
International Journals	16
Refereed Conference Proceedings:	20
Thesis	2

Almost all my papers are available on my web page:

`http://www.sissa.it/∼boscain/publications.html`

Books

[1] U. Boscain, B, Piccoli,*Optimal Synthesis for Control Systems on 2-D Manifolds*, Springer, SMAI, Vol.43, 2004.

Chapters of Books

[1] U. Boscain, B. Piccoli *A Short Introduction to Optimal Control*," in "Controle non lineaire et applications", edited by Tewfik Sari, Travaux en cours, Hermann, Paris. LES COURS DU CIMPA. pp. 19-66, 2005.

Preprints

[3] A. Agrachev, U. Boscain, M. Sigalotti, *A Gauss-Bonnet-like Formula on Two-Dimensional Almost-Riemannian Manifolds*, PREPRINT SISSA 55/2006/M, submitted to Discrete and Continuous Dynamical Systems-A.

[2] M. Balde, U. Boscain, *Stability of Planar Switched Systems: the Nondiagonalizable Case* PREPRINT SISSA, 44/2006/M, submitted to Communication on Pure and Applied Analysis.

[1] Paolo Mason, Rebecca Salmoni, Ugo Boscain, Yacine Chitour, *Limit Time Optimal Syntheses for a control-affine system on* S^2, PREPRINT SISSA 48/2006/M, submitted to SIAM SICON, Journal on Control and Optimization.

International Journals

[16] U. Boscain, P. Mason, *Time Minimal Trajectories for a Spin 1/2 Particle in a Magnetic field*, J. Math. Phys. 47, 062101 (29 pages), 2006.

[15] P. Mason, U. Boscain, Y.Chitour, *Common Polynomial Lyapunov Functions for Linear Switched Systems*, SIAM SICON, Journal on Control and Optimization, Vol. 45, n. 1, pp. 226-245, 2006

[14] U. Boscain, G. Charlot, M. Sigalotti, *Stability of Planar Nonlinear Switched Systems,* Discrete and Continuous Dynamical Systems-A, Vol 15, n.2, pp. 415-432, 2006.

[13] U. Boscain, T. Chambrion, G. Charlot, *Nonisotropic 3-level Quantum Systems: Complete Solutions for Minimum Time and Minimal Energy*, Discrete and Continuous Dynamical Systems-B, 5, pp. 957-990, 2005.

[12] U. Boscain, Y. Chitour, *Time Optimal Synthesis for Left–Invariant Control Systems on SO(3)*, SIAM SICON, Journal on Control and Optimization, 44, pp. 111-139, 2005.

[11] U. Boscain, I. Nikolaev, B. Piccoli, *Classification of Stable Time–Optimal Controls on 2-manifolds*, Journal of Mathematical Sciences. 135, pp. 3109-3124. (2006). Published in Russian on Sovrem. Mat. Prilozh. No. 21, Geom. Zadachi Teor. Upr. pp. 19–35, 2004.

[10] U. Boscain, G. Charlot, *Resonance of Minimizers for n-level Quantum Systems with an Arbitrary Cost,* ESAIM: Control, Optimization and Calculus of Variations (COCV), pp. 593-614, 2004.

[9] U. Boscain, T. Chambrion, J-P Gauthier, *On the K+P problem for a three-level quantum system: optimality implies resonance,* Journal of Dynamical and Control Systems n. 8, pp. 547–572, 2002.

[8] U. Boscain, G. Charlot, Jean-Paul Gauthier, Stephane Guerin, Hans–Rudolf Jauslin, *Optimal Control in laser-induced population transfer for two- and three-level quantum systems,* Journal of Mathematical Physics, n. 43, pp. 2107–2132, 2002.

[7] U. Boscain, Y.Chitour, *On the minimum time problem for driftless left-invariant control systems on SO(3),* Communication on Pure and Applied Analysis, n. 1 pp. 285–312, 2002.

[6] U. Boscain, *Stability of Planar Switched Systems: the linear Single Input Case,* SIAM SICON, Journal on Control and Optimization 41 pp. 89–112, 2002.

[5] U. Boscain, B. Piccoli, *On Automaton Recognizability of Abnormal Extremals,* SIAM SICON, Journal on Control and Optimization, n. 40, pp. 1333–1357, 2002.

[4] U. Boscain, *Projection singularities of extremals and Morse property for minimum time,* Proceedings of the Steklov Mathematical Institute, 236, pp.399-407, 2002.

[3] U. Boscain, B. Piccoli, *Morse Properties for the Minimum Time Function on 2–D Manifolds,* Journal of Dynamical and Control Systems, 7, pp. 385–423, 2001.

[2] U. Boscain, B. Piccoli, *Extremal Synthesis for Generic Planar Systems,* Journal of Dynamical and Control Systems, 7, pp. 209–258, 2001.

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3 Quantum control

In this section I present my research in Quantum Control and in particular the attached papers [1, 2, 3, 4, 5, 6, 7]. The section starts with a brief and by no means exhaustive recall of quantum mechanics and quantum control, for the reader not used to this language. For more details we refer the reader to any book of quantum mechanics (see for instance [54, 91]), and to the following papers of quantum control [33, 37, 54, 56, 57, 73, 80, 81, 92]. See also [78, 84, 90].

3.1 Isolated quantum mechanical systems

Let $\mathcal{H}$ be a complex separable Hilbert space of finite or infinite dimension. Given $\varphi_1, \varphi_2 \in \mathcal{H}$ let us indicate their scalar product in $\mathcal{H}$ as $< \varphi_1, \varphi_2 >$. The induced norm of a vector $\varphi \in \mathcal{H}$ is by definition $\|\varphi\| = \sqrt{< \varphi, \varphi >}$. Let us denote by $\mathbf{S}$ the unit sphere in $\mathcal{H}$.

The time evolution of quantum mechanical system (e.g. an atom, a molecule, a system of particles with spin) is described by a map $\psi : \mathbb{R} \to \mathbf{S}$, called *wave function*. Given a time t, $\psi(t)$ is usually called the *state* of the system, while $\mathbf{S}$ is called the *state space*.

The equation of evolution of the state is the so-called Schrödinger equation that, if the system is isolated, has the form (in a system of units such that the Planck constant $\hbar = 1$):

$$i\frac{d}{dt}\psi(t) = H_0\psi(t), \tag{1}$$

where H_0 is a self-adjoint operator acting on $\mathcal{H}$ called *free Hamiltonian*. The fact that the system is isolated is contained in the fact that H_0 does not depend on the time.

We recall that being H_0 self-adjoint the evolution of ψ is unitary. This means that equation (1) defines an evolution in $\mathbf{S}$. If $\mathcal{H}$ is finite dimensional, then equation (1) is an ODE, otherwise it is a PDE.

Assume for simplicity of notation that the spectrum of H_0 (that is real since H_0 is self-adjoint) is discrete and nondegenerate, with eigenvalues $E_1, E_2, \ldots$ (called *energy-levels*) and normalized eigevectors $\phi_1, \phi_2 \ldots \in \mathbf{S}$ (called *eigenstates*). Since H_0 is self-adjoint, $\phi_1, \phi_2 \ldots$ is an orthonormal base of $\mathcal{H}$. We recall that if $\phi_1, \phi_2 \ldots$ is an orthonormal base of $\mathcal{H}$ made of eigenstates of H_0, then for every $\alpha_1, \alpha_2 \ldots \in \mathbb{R}$ also $e^{i\alpha_1}\phi_1, e^{i\alpha_2}\phi_2 \ldots$ is.

When the system is isolated, once that eigenvalues and eigenvectors are known, it is very simple to compute the time evolution of a state: if $\psi(0) = \sum_j c_j^0 \phi_j$ then $\psi(t) = \sum_j c_j(t)\phi_j$, where $c_j(t) = c_j^0 e^{-iE_j t}$. On the other hand one has that $c_j(t) =< \phi_j, \psi(t) >$ and the physical interpretation of these functions is the following: $|c_j(t)|^2$ is the probability that, if we make a measure of energy on the system at time t, then we get energy E_j.

Notice, that according with this probabilistic interpretation, $\sum_j |c_j(t)|^2 = \|\psi(t)\|^2 = 1$. Moreover $|c_j(t)|^2 = |c_j^0|^2$, hence the probability of measuring a certain value of energy is constant in the time.

3.2 Controlled quantum mechanical systems

Assume now to act on the system with some external fields (e.g an electromagnetic field) whose amplitude is represented by some functions $u_1(.), \ldots u_m(.) \in L^\infty(\mathbb{R}, \mathbb{R})$. In this case the Schrödinger equation becomes

$$i\frac{d}{dt}\psi(t) = H(t)\psi(t), \tag{2}$$

where $H(t) = H_0 + \sum_{j=1}^m u_j(t)H_j$ and H_j $(j = 1, 2, \ldots, m)$ are self-adjoint operators representing the coupling between the system and the external fields. The functions $u_1(.), \ldots, u_m(.)$ are the controls. The time dependent operators $H(t)$ and $\sum_{j=1}^m u_j(t)H_j$ are called respectively the *Hamiltonian* and the *control-Hamiltonian*. In the language of control theory, the free Hamiltonian H_0 is called the drift.

The typical problem one meets in quantum control is the so called *population transfer problem*:

Population transfer problem. *assume that at time zero the system is in an eigenstate ϕ_j of the free Hamiltonian H_0. With this one means that $| < \phi_j, \psi(0) > |^2 = 1$. Design controls $u_1(t), \ldots, u_m(t)$ such that at time T (fixed or free, possibly $+\infty$, depending on the specific problem) the system is in another prescribed* *eigenstate* ϕ_l *of* H_0 *i.e.* $| < \phi_l, \psi(T) > |^2 = 1$.

Of course for some applications also initial and final states that are not eigenstates are interesting. However in my works I am usually interested to the classical problem of steering the system form an eigenstate to another.

More in general in Quantum control one is faced with the classical problems of control theory e.g. the problems of controllability, motion planning, stability and optimal control.

A big difference with respect to the classical control problems is that in quantum control usually one does not look for solutions in *feedback* form. The reason is that as a consequence of the *postulate of wave function collapse*, if one make a measure on the system then, immediately after, the system collapses in the eigenstate corresponding to the eigenvalue that has been measured. This means that the measure process modify abruptly the state of the system and one cannot compute the control by measuring the state. However in certain very special cases non-destructive measures are possible and feedbacks can be implemented. See for instance [67, 59].

Nowadays quantum control has many applications in chemical physics, in nuclear magnetic resonance (also in medicine) and it is central in the implementation of the so-called quantum gates (the basic blocks of a quantum computer). See for instance [84, 98, 78, 90]. In all these problems the transfer is induced by means of a sequence of laser pulses, or with the action of a magnetic fields and should be as efficient as possible in order to minimize the effects of relaxation or decoherence that are always present.

Typically in these problems, the number of controls acting on the system is small, independently of the dimension of $\mathcal{H}$.

3.3 Finite dimensional quantum control problems

As one could expect the difficulties that one meets in attacking the problems of controllability, motion planning, stability and optimal control are completely different if the state space is finite or infinite dimensional.

Problems that are genuinely finite dimensional are problems of controlling spin systems. Moreover in molecular dynamics, there are many infinite dimensional systems in interaction with external fields that are extremely well described by finite dimensional models.

Another point of interest for the study of finite dimensional quantum systems (in particular with few controls and with $\dim(\mathcal{H})$ large), is that one can get information which could be very useful for proving controllability results for infinite dimensional quantum systems. A research project in this direction is presented in Section 3.5.

For a finite dimensional quantum mechanical system, if n is the number of energy levels (counted with their multiplicity, if there are degeneracies) we have $\mathcal{H} = \mathbb{C}^n$ and the state space $\mathbf{S}$ is the unit sphere $S^{(2n-1)} \subset \mathbb{C}^n$. These problems are right-invariant control problems on the Lie group $SU(n)$. Indeed problems of quantum mechanics (being multilinear) can be formulated, at the level of the wave function $\psi(t)$, but also at the level of the time evolution operator (the resolvent), that is a $n \times n$ matrix, denoted here by $g(t)$, satisfying $\psi(t) = g(t)\psi(0)$, $g(0) = id$, and

$$\frac{d}{dt}g(t) = -iH(t)g(t). \tag{3}$$

The equation of evolution for $g(t)$ is still called Schrödinger equation. Since $-iH(t)$ is a skew Hermitian matrix (i.e. belonging to the Lie algebra $u(n)$) then $g(t) \in U(n)$, i.e., $g(t)$ is a unitary matrix.

However, since the evolution of the trace part is decoupled from the rest, without loss of generality we can always assume that $H(t)$ has zero trace and $g(t) \in SU(n)$. This will be done hereafter. In the following we call the quantum control problem for $\psi(t)$ and for $g(t)$ respectively the *problem downstairs* (i.e., on the homogeneous space) and the *problem upstairs* (i.e., on the corresponding group of transformations). In this kind of problems one can take advantages of working both upstairs and downstairs depending on the specific task. This approach happened to be successful in some other problems of optimal control on Lie groups, see for instance the paper [14], in collaboration with Yacine Chitour.

Therefore finite dimensional quantum systems are in fact right-invariant control systems on $SU(n)$, or on the corresponding Homogeneous space $S^{2n-1} \subset \mathbb{C}^n$. For these kinds of systems very powerful techniques were developed both for what concerns controllability [55, 61, 70, 94] and optimal control [17, 32, 69].

For finite dimensional problems the controllability problem (i.e. proving that for every couple of points in $SU(n)$ one can find controls steering the system from one point to the other) is nowadays well understood. Indeed the system is controllable if and only if

$$\mathrm{Lie}\{iH_0, iH_1, ...iH_m\} = su(n), \tag{4}$$

(see for instance [94]). However verifying condition (4) is not always simple and the problem of finding alternative necessary and sufficient conditions for controllability has been deeply studied in the literature (see for instance [33, 37, 94]). Here we just recall that the condition (4) is generic in the space of Hermitian matrices.

The situation is much more difficult in the infinite dimensional case. In the attached paper [6], in collaboration with Riccardo Adami, (see also Section 3.4) we have attacked the problem of controllability of the Schrödinger equation as a PDE, using slowing varying controls (adiabatic theory), and the intersection of the eigenvalues in the space of controls. An idea on how to use finite-dimensional techniques to prove controllability results in the infinite dimensional case (Galerkin approximation) can be found in Section 3.5.

3.3.1 Optimal control on finite-dimensional quantum systems

Once that controllability is proved one would like to steer the system, between two fixed points in the state space, in the most efficient way. Typical costs that are interesting to minimize in the applications are:

C1 Energy transfered by the controls to the system. [1] $\displaystyle\int_0^T \sum_{j=1}^m u_j^2(t)\ dt,$

C2 Time of transfer. [2] In this case one can attack two problems that are very different from a mathematical point of view: one with *bounded* and one with *unbounded* controls.

The problem of minimizing time with unbounded controls has been deeply investigated in [29, 73] and is now well understood. On the other hand the problems of minimizing time with bounded controls or energy are very difficult in general and one can hope to find a complete solution, for quantum systems with few levels only.

Indeed the most important and powerful tool for the study of optimal trajectories is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [32, 69, 89]) and Section 4.2. It is a first order necessary condition for optimality and generalizes the Weierstraß conditions of Calculus of Variations to problems with non-holonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle that is a solution to a suitable pseudo–Hamiltonian system. Giving a complete solution to an optimization problem (that for us means to give an *optimal synthesis*, see for instance [17, 46, 51, 52, 88]) is extremely difficult for the following reasons.

First, one is faced with the problem of integrating a Hamiltonian system (that generically is not integrable except for very special costs). Second, one should manage with some special solutions of the PMP, the so called *abnormal extremals* and *singular trajectories* (see [10, 17, 32, 53]). Finally, even if one is able to find all the solutions of the PMP, it remains the problem of *selecting* among them the *optimal trajectories* (see [17, 46, 50, 88]). More details about this problem are given in Section 4.2. For these reasons, except for linear systems with quadratic cost, one can hope to find a complete solution to an optimal control problem in low dimension only. We just mention that in dimension two, most of the optimal control problems are very complicated (see for instance [8, 9, 17, 48, 87, 104] and Section 4.2) and in dimension 3 most of the optimal

[1] Minimizing this cost is very important in certain applications. Indeed if the energy transfered to the system is much more than the minimal energy necessary to induce the transition, undesirable phenomena like decoherence could happen. We recall that decoherence is the mechanism by which a large number of identical quantum systems that are all in the same state evolve to an incoherent mixture of states

[2] Minimizing this cost is important when the relaxation time is comparable with the minimum time necessary to induce the transition. We recall that the relaxation time is the mean time for the system to decay spontaneously from an excited level to the fundamental state, resulting in the creation of a photon. Notice that relaxation phenomena are not included in the Schrödinger equation (2).

control problems are open (see [30, 49, 97] for some results). However even in the cases in which one cannot find analytically the explicit expressions of the optimal trajectories, a deep geometric understanding of the optimal control problem is of great help for numerical optimization. This is what happened, for instance, with the paper [3] as explained below.

In the case of quantum mechanical systems, the invariance of the dynamics and of the costs **C1** and **C2** by right translations of the group, permits to push the analysis much further. For instance if one gets a right invariant problem on a Lie group of dimension 3, there is no the problem of integrability of the Hamiltonian systems associated to the PMP, since every invariant Hamiltonian system on a three-dimensional Lie group is always integrable (see for instance [69, 71]). If moreover the problem is driftless (as in those studied in the attached papers [1, 2, 4]) there are no abnormal extremals.

However the dimension of the state space grows quickly with the number of levels and one immediately gets in trouble, if a special form of the dynamics is not assumed. Indeed, for a quantum mechanical system with n levels, one gets a problem of dimension $2n-1$ for the problem downstairs and a problem of dimension n^2-1 for the problem upstairs. From the following table it should be clear that one can hope to find a complete solution to an optimal control problem in quantum mechanics, without assuming any special dynamics, only for a two level system.

number of energy levels (n)	$\dim(S^{2n-1})$	$\dim(SU(n)) = n^2-1$
2	3	3
3	5	8
4	7	15
5	9	24

The general two level problem is described in Section 3.3.5. In the next section we describe a class of systems for which the dynamics has a special form.

3.3.2 A driftless-class of systems

A very interesting class of systems on which it is possible to push the analysis much further is the one in which the controlled Hamiltonian reads

$$H(t) = \begin{pmatrix} E_1 & \mu_1\mathbf{\Omega}_1(t) & 0 & \cdots & 0 \\ \mu_1\mathbf{\Omega}_1^*(t) & E_2 & \mu_2\mathbf{\Omega}_2(t) & \ddots & \vdots \\ 0 & \mu_2\mathbf{\Omega}_2^*(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & E_{n-1} & \mu_{n-1}\mathbf{\Omega}_{n-1}(t) \\ 0 & \cdots & 0 & \mu_{n-1}\mathbf{\Omega}_{n-1}^*(t) & E_n \end{pmatrix} \tag{5}$$

Here $(^*)$ denotes the complex conjugation involution. The controls $\mathbf{\Omega}_1, \ldots, \mathbf{\Omega}_{n-1}$ are complex (they play the role of the real controls $u_1, \ldots, u_m$, with $m = 2(n-1)$) and $\mu_j > 0$, $(j = 1, \ldots, n-1)$ are real constants describing the couplings (intrinsic to the quantum system). In this model we have assumed that only adjacent levels j and $j+1$ can be coupled. In the following we call the problem *isotropic* if $\mu_1 = \mu_2 = \ldots = \mu_{n-1}$ otherwise we call the problem *nonisotropic.*

Justification of the model For $n = 2$ the dynamics (5) describes the evolution of a spin 1/2 particle driven by a magnetic field, that is constant along the z-axis and controlled both along the x and y axes. While for $n \geq 2$ it represents the first n levels of the spectrum of a molecule in the *rotating wave approximation* (see for instance [36]), and assuming that each external field couples only adjacent levels.

For this model the energy transfered by the external fields to the system in an interval of time $[0, T]$ is

$$\int_0^T \sum_{j=1}^{n-1} |\mathbf{\Omega}_j(t)|^2 \, dt, \tag{6}$$

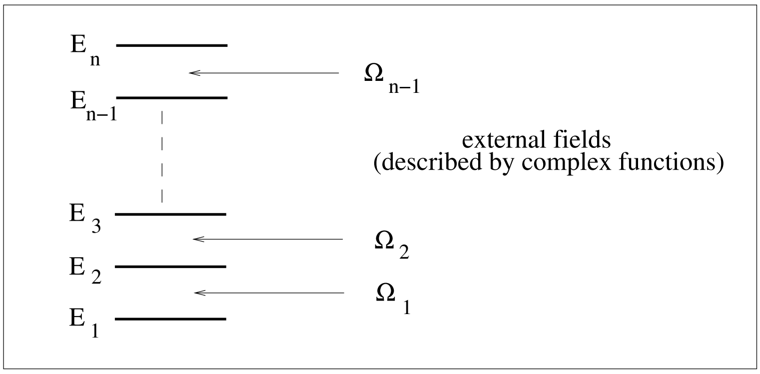

Figure 2:

while minimizing time with bounded controls means to minimize

$$\int_0^T 1\ dt, \quad \text{with } |\mathbf{\Omega}_j(t)| \leq M_j, \quad j = 1, \ldots, n-1 \tag{7}$$

where M_j are positive constants.

These kind of systems are very interesting since the drift term $H_0 = \text{diag}(E_1, \ldots, E_n)$ disappears in the so called *interaction picture*, i.e. after a unitary change of coordinates and a unitary change of controls. More precisely if in the equation (1) (resp. in the equation (3)), we make the transformation $\psi(t) \to e^{-iH_0 t}\psi(t)$ (resp. $g(t) \to e^{-iH_0 t} g(t)$) the new controlled Hamiltonian (that with abuse of notation we still call $H(t)$) reads

$$H(t) = \begin{pmatrix} 0 & \mu_1 \mathbf{\Omega}_1(t) e^{-i\,\omega_1 t} & 0 & \cdots & 0 \\ \mu_1 \mathbf{\Omega}_1^*(t) e^{i\,\omega_1 t} & 0 & \mu_2 \mathbf{\Omega}_2(t) e^{-i\,\omega_2 t} & \ddots & \vdots \\ 0 & \mu_2 \mathbf{\Omega}_2^*(t) e^{i\,\omega_2 t} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \mu_{n-1} \mathbf{\Omega}_{n-1}(t) e^{-i\,\omega_{n-1} t} \\ 0 & \cdots & 0 & \mu_{n-1} \mathbf{\Omega}_{n-1}^*(t) e^{i\,\omega_{n-1} t} & 0 \end{pmatrix}, \tag{8}$$

where $\omega_j = E_{j+1} - E_j$, $j = 1, 2, \ldots, n-1$. Hence setting

$$\Omega_j(t) := e^{i\,\omega_j t} \mathbf{\Omega}_j(t), \qquad j = 1, 2, \ldots, n-1, \tag{9}$$

we get a controlled Hamiltonian with no drift,

$$H(t) \quad = \quad \begin{pmatrix} 0 & \mu_1 \Omega_1(t) & 0 & \cdots & 0 \\ \mu_1 \Omega_1^*(t) & 0 & \mu_2 \Omega_2(t) & \ddots & \vdots \\ 0 & \mu_2 \Omega_2^*(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \mu_{n-1} \Omega_{n-1}(t) \\ 0 & \cdots & 0 & \mu_{n-1} \Omega_{n-1}^*(t) & 0 \end{pmatrix}. \tag{10}$$

Notice that after this transformation the cost (6) becomes

$$\int_0^T \sum_{j=1}^{n-1} |\Omega_j(t)|^2\ dt, \tag{11}$$

while minimizing time with bounded controls means to minimize

$$\int_0^T 1\ dt, \quad \text{with } |\Omega_j(t)| \leq M_j, \quad j = 1, \ldots, n-1.$$

Moreover this transformation leaves invariant the eigenstates. For more details about these transformations see the attached papers [1, 2, 3, 4].

Remark **1** After elimination of the drift, the problem of minimizing the energy (11) becomes a sub-Riemannian problem (see for instance [42, 65, 83]), i.e. an optimal control problem of the type

$$\dot{q} = \sum_{j=1}^{m} u_j F_j(q), \quad \int_0^T u_j^2 \, dt \to \min, \quad (T \text{ fixed}) \tag{12}$$

where q belongs to a N-dimensional smooth manifold, and F_j, $j = 1, 2, \ldots, m < N$ are smooth vector fields on M. For this cost the final time must be fixed otherwise there are no solutions to the minimization problem. We recall that a minimizer for this cost is parametrized with constant velocity ($\sum_{j=1}^m u_j^2 = const$). If the final time T is fixed in such a way the minimizer is parametrized by arc-length ($\sum_{j=1}^m u_j^2 = 1$), then minimizing the cost (12) is equivalent to minimize time with moduli of controls constrained in the set $\sum_{j=1}^m u_j^2 \leq 1$.

3.3.3 Description of the works for the driftless class of systems described in Section 3.3.2

The main purpose of the attached works [1, 2, 3, 4] is to find the explicit expression of optimal trajectories joining a couple of eigenstates and minimizing the time with bounded controls or the energy transfered to the system, during the transition. As explained in Section 3.3.1 this is in general a hard question due to the high dimensionality of the system. In the papers [1, 2, 3, 4] complete solutions for systems with two or three levels have been found.

As a byproduct of this research, one provides an answer to the following conjecture, well known in the community of quantum dynamics:

The resonance Conjecture: *Every optimal trajectory for the costs described above is in resonance i.e. it corresponds to a control of the form*

$$\mathbf{\Omega}_j(t) = U_j(t) e^{i[(E_{j+1} - E_j)t + \phi_j]}, \tag{13}$$

where $U_j(t)$ are real functions describing the amplitude of the external fields and $\phi_j \in \mathbb{R}$ are arbitrary phases. The quantities $(E_{j+1} - E_j)$ are called the resonance frequencies.

Roughly speaking to be in resonance means to use lasers oscillating with a frequency equal to the difference of energy between the levels that the laser is coupling.

Giving a positive answer to this conjecture si very important. Indeed if the conjecture were true, then, as explained in [3, 4], one could restrict to real variables, and the problem reduces from $SU(n)$ to $SO(n)$ (when working upstairs) or from $S^{2n-1} \subset \mathbb{C}^n$ to $S^{n-1} \in \mathbb{R}^n$ (when working downstairs).

The simplest case $n = 2$ has been studied in the attached work [1]. In this case the minimum time problem with bounded control and the minimum energy problem actually coincide and the controlled Hamiltonian is

$$H(t) = \begin{pmatrix} -E & \mathbf{\Omega}(t) \\ \mathbf{\Omega}^*(t) & E \end{pmatrix}. \tag{14}$$

Here we assume $|\mathbf{\Omega}| \leq M$. The coupling constant μ_1 has been eliminated redefining $\mathbf{\Omega}(t)$. In this case, one gets a subriemannian problem that is in fact an isoperimetric problem (in the sense of the calculus of variations). The optimal trajectories, steering the system from the first to the second eigenstate of $H_0 = diag(-E, E)$, correspond to controls in resonance with the energy gap $2E$, and with maximal amplitude i.e. $\mathbf{\Omega}(t) = Me^{i[(2E)t+\phi]}$, where $\phi \in \mathbb{R}$ is an arbitrary phase.

In this case, the time of transfer T_{tr} is proportional to the inverse of the laser amplitude. More precisely $T_{tr} = \pi/(2M)$. See [1], Theorem 1, p. 2118.

The case $n = 3$ is much more complicated. In [1] the minimum energy problem was solved in the isotropic case ($\mu_1 = \mu_2$) and assuming resonance as hypothesis. In this case the problem can be reduced to a problem of Riemannian geometry with singularities (called almost-Riemannian geometry) on the two dimensional sphere and can be solved completely.

Remark **2** More precisely we call almost-Riemannian geometry the generalization of Riemannian geometry that naturally arises in control theory in the following way. Let M be a two-dimensional smooth manifold and consider a pair of smooth vector fields F_1 and F_2 on M. If the pair F_1, F_2 is Lie bracket generating, i.e., if $\text{span}\{F_1(q), F_2(q), [F_1, F_2](q), [F_1, [F_1, F_2]](q), \ldots\}$ is full-dimensional at every $q \in M$, then the control system

$$\dot{q} = u_1 F_1(q) + u_2 F_2(q) \,, \quad q \in M \,, \tag{15}$$

is completely controllable and the cost

$$\int_0^1 (u_1^2(t) + u_2^2(t)) \, dt \to \min, \quad q(0) = q_0, \quad q(1) = q_1 \tag{16}$$

defines a continuous distance on M. When F_1 and F_2 are everywhere linear independent such distance is Riemannian and it corresponds to the metric for which (F_1, F_2) is an orthonormal moving frame. This distance gives to M the structure of metric space. If F_1 and F_2 become linearly dependent somewhere, then the corresponding Riemannian metric has singularities, but under generic conditions the metric structure is still well defined. Metric structures that can be defined locally in this way are called almost-Riemannian structures. Almost-Riemannian structures show interesting phenomena, in particular for what concerns the relation between curvature, presence of conjugate points, and topology of the manifold. Much more details about almost Riemannian geometry are given in Section 3.3.4 and in the attached paper [7].

In attached paper [2] the problem of minimizing the energy in the isotropic three-level problem was treated without assuming resonance as hypothesis. In that case, even if the optimal control problem lives in a space of big dimension ($dim(SU(3)) = 8$), it was possible to get explicit expressions of optimal controls and trajectories thanks to the special structure of the Lie algebra of the problem and to the fact that the cost is built with the Killing form, that renders the Hamiltonian system associated to the PMP Liouville integrable. In that paper resonance was obtained as consequence of the minimization process and explicit expressions of amplitudes were determined. See [2], Section 5.4, p. 564. This paper appeared also as Section 19.3 of the Book [32]. Some other details about this paper are given in Section 4.3.

In the attached paper [3], the possibility of restricting to resonant controls was generalized to n-level systems and to more general costs. More precisely consider a n-level system in which the controlled Hamiltonian has the form:

$$H(t) = H_0 + V(t) \tag{17}$$

where $H_0 = diag(E_1, ..., E_n)$ and $V(t)$ is an Hermitian matrix ($V(t)_{j,k} = V(t)^*_{k,j}$), whose elements are either identically zero or controls. More precisely $V_{j,k} \equiv 0$ or $V_{j,k} = \mu_{j,k} \boldsymbol{\Omega}_{j,k}$, for $j < k$, where the term $\boldsymbol{\Omega}_{j,k}$ is the external pulsed field couplings level j and level k, and $\mu_{j,k} > 0$ are the coupling constants.

This class of system (in which it is possible to eliminate the drift term H_0, with the same procedure described in Section 3.3.2), includes the systems described by the Hamiltonian (5).

In [3], for a convex cost depending only on the moduli of controls (i.e. amplitudes of the lasers), like minimum time with bounded controls or minimum energy, it was proved that there always exists a minimizer in resonance that connects a source and a target defined by conditions on the moduli of the components of the wave function (e.g. two eigenstates):

$$\boldsymbol{\Omega}_{j,k}(t) = U_{j,k}(t) e^{i[(E_j - E_k)t + \phi_{j,k}]}, \quad i < j = 1, ..., n,$$

where $\phi_{j,k} \in \mathbb{R}$ are some phases and $U_{j,k}(.) : [0, T] \to \mathbb{R}$ are the amplitudes of the lasers that should be determined. See [3], Theorem 1, p. 604.

This result gives essentially a positive answer to the resonance conjecture and permits to simplify considerably the difficulty of the problem passing to real variables. The following table shows the gain in terms of dimensions that one gets after to the proof of the resonance conjecture.

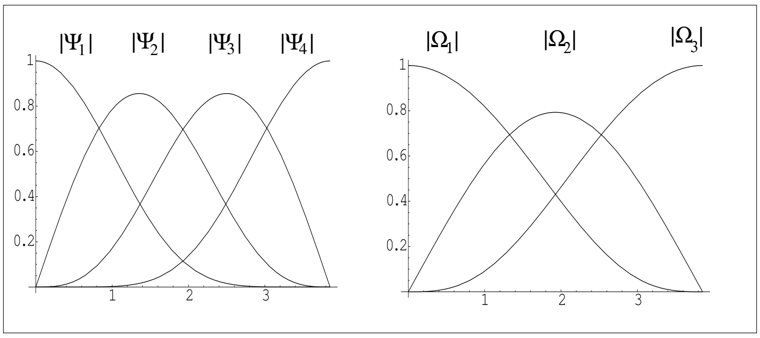

Figure 3:

number of energy levels (n)	$\dim(S^{2n-1})$	$\dim(SU(n)) = n^2 - 1$	$\dim(S^{n-1})$	$\dim(SO(n)) = \dfrac{n(n-1)}{2}$
2	3	3	1	1
3	5	8	2	3
4	7	15	3	6
5	9	24	4	10

Hence this reduction renders trivial the problem with $n = 2$. Moreover it is crucial to give a complete solution for nonisotropic systems with $n = 3$ for several costs interesting for applications (see the description of the paper [4] below). Finally this reduction gives some hope to find the time optimal trajectories joining eigenstates for systems with four levels and it is of great help in finding numerical solutions to problems with $n \geq 4$.[3] In Figure 3 a numerical integration of the PMP for an isotropic 4-level problem of type (5) (easily duable after elimination of the dirft) is presented.

Another result presented in [3] is the following. For the system reduced to real variable, we prove that close to any time t of the domain of a given minimizer, there exists an interval of time where the minimizer is not strictly abnormal (see [3], Theorem 3 p. 605). This result is a first step in the challenging problem of proving that for the reduced problems there are no strict abnormal minimizers. We recall that the presence of abnormal minimizers adds enormous difficulties to the optimal control problem.

In the paper [4], we take advantage of the reduction to real variables to find complete solutions for the nonisotropic three-level problem and the costs described above.

The minimum time problem with bounded controls is naturally studied downstairs. Originally this is a problem in dimension five. Thanks to the reduction to real variables this problem becomes a problem in dimension 2 of the form

$$\dot{q} = u_1 F_1 + u_2 F_2, \quad |u_j| \leq 1, \quad j = 1, 2 \tag{18}$$

where q belongs to a two dimensional manifold M and F_1 and F_2 are smooth vector fields on M.

The theory of time optimal syntheses for control systems of this type has been developed for the first time in Section 4.1 of the paper [4]. This is done in the same spirit of the theory developed in my book [17], written in collaboration with Benedetto Piccoli, for problems with one control and the drift. Optimal trajectories joining the state one to the state three are found explicitly.

The minimum energy is naturally studied upstairs, where it is a right invariant sub-Riemannian problem on $SO(3)$. Extremals (i.e. solution to the PMP) can be obtained in terms of Elliptic functions. The hardest

[3] For $n \geq 4$ there is no reason to believe that the Hamiltonian system associated to the PMP is integrable

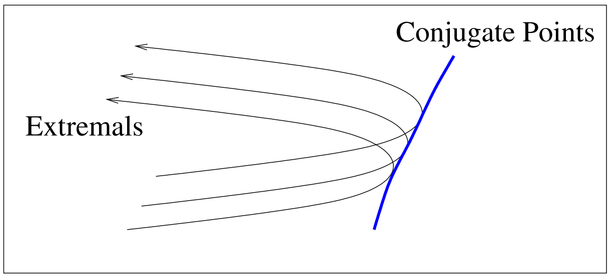

Figure 4:

part of the work is to compute the cut locus, i.e. the set of points where trajectories loose optimality (see [4], Section 5.3). Also for this cost, optimal trajectories steering the state one to the state three are found explicitly.

3.3.4 Conjugate points and curvature in almost-Riemannian geometry

An idea to try to prove optimality of extremals for the problem described just above was to write the problem downstairs, where it is a problem of almost-Riemannian geometry (cf. Remark 2), and try to show that there are no conjugate points. Roughly speaking conjugate points are the points at which extremals form an envelope (see Figure 4) losing local optimality. For a precise definition of conjugate points, see for instance [7], Section 3.1. Definition 12.

A classical technique to prove the absence of conjugate points is to prove that the Gaussian curvature is negative everywhere.

Surprisingly, this idea does not work in almost-Riemannian geometry and this is due to the presence of singularities in the Riemannian metric. Indeed even in the case where the Gaussian curvature is everywhere negative (where it is defined, i.e., where the vector fields defining the orthonormal base are linearly independent), geodesics may have conjugate points. We started to study this interesting phenomenon in the the attached paper [7], presenting a generalization of the Gauss-Bonnet theorem.

3.3.5 The two-level quantum system with drift

The problem of minimizing time with bounded controls or energy is even more difficult if it is not possible to eliminate the drift H_0. This happens, for instance, for a system in the form (5) with real controls $\mathbf{\Omega}_j(t) = \mathbf{\Omega}_j^*(t)$, $j = 1, ..., n-1$. In this case one can hope to find a solution to the optimal control problem only for $n = 2$.

The minimum energy problem for a two-level system, has been studied by D'alessandro in [57]. In that case, optimal solutions can be expressed in terms of Elliptic functions.

The minimum time problem with bounded controls, has been studied with my student Paolo Mason in [5] and it is much more complicated. After a suitable Hopf projection, it become a single input affine control problem on a two-dimensional sphere (the so called Bloch Sphere), and can be attacked with the techniques of optimal syntheses on 2-D manifolds developed by Sussmann, Bressan, Piccoli and myself, see for instance [8, 9, 10, 12, 47, 48, 86, 87, 103, 104] and recently gathered in the book [17], written in collaboration with Benedetto Piccoli. For some details on these techniques see Section 4.2.

Let $(-E, E)$ be the two energy levels, and $|\mathbf{\Omega}(t)| \leq M$ the bound on the field amplitude. For each couple of values E and M, we determine the time optimal synthesis starting from the level $-E$ and we provide the explicit expression of the time optimal trajectories steering the state one to the state two, in terms of a parameter that can be computed solving numerically a suitable equation.

For $M/E << 1$, every time optimal trajectory is bang-bang (i.e. concatenation of trajectories corresponding to $u = M$ and $u = -M$) and in particular the corresponding control is periodic with frequency of the order of the resonance frequency $\omega_R = 2E$. On the other side, for $M/E > 1$, the time optimal trajectory steering the state one to the state two is bang-bang with exactly one switching. Fixed E we also prove that for $M \to \infty$ the time needed to reach the state two tends to zero.

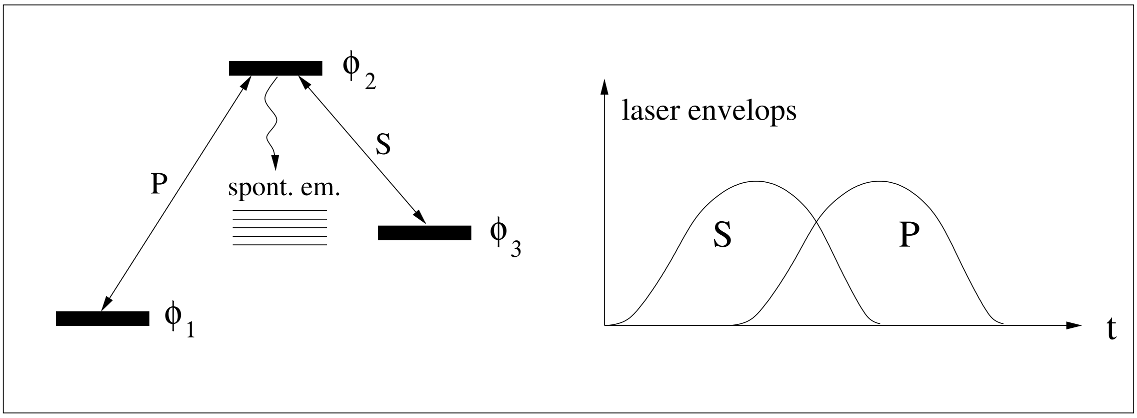

Figure 5:

In the case $M/E > 1$ a new interesting phenomenon appears: there are time optimal trajectories (not reaching the state two) containing a singular arc (i.e. corresponding to control zero).

Finally we compare these results with some known results of Khaneja, Brockett and Glaser and with those obtained by controlling the system with a complex control (i.e. the problem described in Section 3.3.3).

As byproduct we prove that the qualitative shape of the time optimal synthesis presents different patterns, that cyclically alternate as $M/E \to 0$, giving a partial proof of a conjecture formulated in the previous paper [14] written in collaboration with Yacine Chitour. In that paper the aim was to give an estimate on the maximum number of switchings for time optimal trajectories on $SO(3)$. More details about [14] are given in Section 4.3.

3.3.6 An Application of tracking techniques to the STIRAP precess

The Hamiltonian (5), with $n = 3$ is used also as a model for the Stimulated Raman Adiabatic Passage (STIRAP). It is a technique which permits an almost entire atomic or molecular population transfer from one specific quantum level to another, by a two step excitation process, using an intermediate level.

The use of two lasers coupling three states, rather than a single laser coupling two states may offers many advantages: the process can be made relatively robust (with respect to many of the experimental details), moreover one can produce excitation between states for which single photon transition are forbidden by parity. The most remarkable property of this techniques is that it is applicable to situations in which the intermediate atomic or molecular state is exposed to large loss rates.

In the simplest form the scheme involve a three states two-photon Raman process in which the interaction with a *pump* pulse P links the initial state ϕ_1 with an intermediate state ϕ_2, which interacts via a *Stokes* pulse S with a final target state ϕ_3. Figure 5 illustrates the connections. Typically state ϕ_1 might be a rotational level in the vibrational ground state of a molecule, and state ϕ_3 might be a highly excited vibrational state. In our model the laser frequencies are tuned to their respective resonance frequencies. States ϕ_1 and ϕ_3 must be long lived, with respect to the time in which the laser pulses are active, while state ϕ_2 will undergo spontaneous emission to states ϕ_1 and ϕ_3 and also to other states. The objective of STIRAP is to transfer all the population from state ϕ_1 into state ϕ_3 losing as few as possible by spontaneous emission from state ϕ_2.

In the STIRAP process the transfer is obtained by means of two overlapping laser pulses (usually Gaussian-shaped) in the so called counterintuitive order that is exposing the atom or molecule to the Stokes laser first and then to the pump laser (see Figure 5).

Initially the Stokes laser couple the two empty states and create a so called *trapped* state, then the use of the pump laser permits to populate directly the state ϕ_3. For a physical interpretation of the phenomenon see [43] and for a review on trapped states, see [39].

The shape (and in particular the shift between the two pulses) is designed, in the literature, using ideas coming from the adiabatic theory, examining the topology of eigen-energy surfaces, see for instance [66, 109].

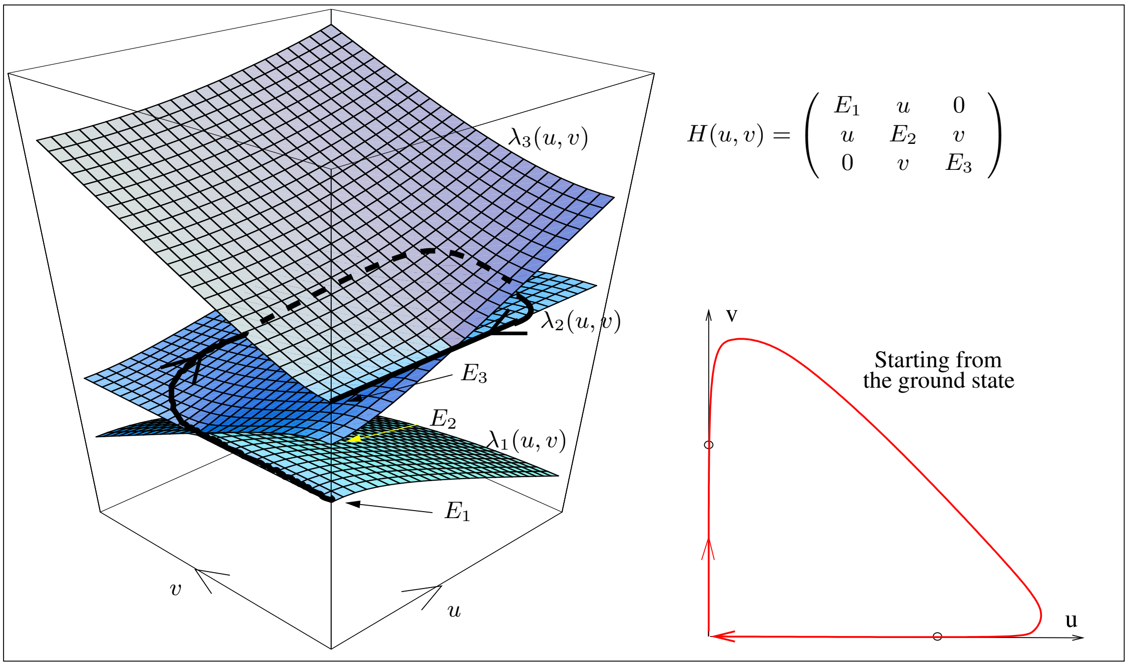

Figure 6: Eigenvalues intersections for a three-level Hamiltonian with two real controls u and v. The eigenvalues of $H(u,v)$ are indicated as $\lambda_j(u,v)$, $j = 1,2,3$. Notice that the designed path has fields acting in the counterintuitive order since first is active v (that in the Hamiltonian $H(u,v)$ couples level E_2 with level E_3 playing the role of the Stokes laser) and then is active u (that in the Hamiltonian $H(u,v)$ couples level E_1 with level E_2 playing the role of the pump laser)

Roughly speaking one consider the two controls in the Hamiltonian as two parameters and uses the intersections of the eigenvalues surfaces in this space to climb the energy levels. This is done with slow varying controls (adiabatic approximation). See Figure 6. The effectiveness of the transition is then verified numerically. [4]

This method is interesting (in particular for its possible generalizations to the infinite dimensional case, see Section 3.4) however a 100% transfer is obtained only as limit, using arbitrarily large laser magnitudes.

In the paper [1], using a tracking techniques, we develop a systematic way of designing laser pulses to improve the STIRAP process. In a three-level model, described by the Hamiltonian (5) (with complex controls), we get a 100% transfer from the state one to the state three. Moreover by increasing the energy of the laser pulses, we are able to reduce as much as we want the population of the state two during the process (being zero at the beginning and at the end). Hence we reduce as much as we want the lost from state two by spontaneous emission.

3.4 Infinite dimensional quantum systems

Despite all the results that have been obtained in the finite dimensional case, only few properties have been proved for the controlled Schrödinger equation as a PDE (with a finite number of controls appearing in the Hamiltonian[5]) and few satisfactory global controllability results are available. Negative controllability results can be found in [79, 93, 110]), while for positive controllability results we mention the result of K. Beauchard, and J.M. Coron [41] in which exact controllability between eigenstates (and suitable neighborhhod of them) is

[4] However a rigorous application of the adiabatic theory is not easy due the the lack of the spectral gap close to eigenvalues intersections. This problem has been solved in the attached paper [6], in collaboration with Riccardo Adami in the more general context of infinite dimensional systems.

[5] Some controllability results are known for problems in which the control is the wave function in a selected internal region. However this kind of controllability is almost impossible to implement experimentally.

proved for a well of potential driven by one external field.

Another positive controllability results is provided in the attached paper [6], in which we get controllability results for two models of controlled infinite dimensional systems, by climbing the eigenvalues surfaces in the space of controls, using the adiabatic approximation (see for instance [40, 72, 85, 108]), in the spirit of what described in Section 3.3.6, but in the infinite dimensional case. The crucial idea is that generically an Hamiltonian depending on two parameters present conical eigenvalues intersections, i.e, of dimension zero. Hence with two controls one can cross the eigenvalues intersections in one direction only, and climbing the energy levels, for instance as in Figure 6.

As mentioned in the footnote (4), the use of the adiabatic approximation to prove controllability results presents a difficulty. It requires that during the whole time evolution the eigenvalues remain separated by a non vanishing gap ("gap condition"). This condition cannot be satisfied by the spectrum of the Hamiltonian we are treating with, since we need eigenvalues intersections.

The main idea presented in [6] is that such a difficulty can be overcome by a decoupling between the levels other than the adiabatic one. In fact we prove that, *if an adiabatic path in the space of controls passes precisely through an eigenvalues intersection, then the system jumps to the next level at the same order of the adiabatic approximation.*

Our strategy, can be applied in many situations in which classical control theory would be too difficult or cumbersome. Besides, it provides explicit expressions of controls (motion planning), and most of all is very robust, in the sense that similar controls produce similar population transfers.

In the paper [6], we apply this method to two toy models. More specifically, given two arbitrary eigenstates of the uncontrolled system, we construct a path in the space of controls that steers the system from the first to the second; the target is reached only approximately, but the accuracy of the approximation can be arbitrarily improved slowing the process down and correspondingly raising its duration.

Let us describe our two models. The former is the simplest generalization to infinite dimension of the three-level model with real controls described in Figure 6. As in that cases, it is given in the representation of the eigenstates of the drift Hamiltonian, namely as an infinite dimensional matrix.

$$H(u,v)=\begin{pmatrix} E_0 & \alpha_0 u & 0 & 0 & 0 & \cdots \\ \alpha_0 u & E_1 & \beta_0 v & 0 & 0 & \cdots \\ 0 & \beta_0 v & E_2 & \alpha_1 u & 0 & \cdots \\ 0 & 0 & \alpha_1 u & E_3 & \beta_1 v & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{19}$$

Here the coefficients α_j's and β_j's (that we assume to be bigger than zero) are coupling constants.

For technical reasons we assume that the sequence of the E_j's diverges, and the quantities $\alpha_j/|E_{2j}|^\mu$ and $\beta_j/|E_{2j}|^\mu$ vanish as j goes to infinity for some $0<\mu<1$.

For this model, the decoupling responsable of the jump to next level, while crossing an an eigenvalue intersection, is due to the following fact. When $u=0$ (resp. $v=0$), then levels E_j and E_{j+1}, with j even (resp. odd) are decoupled (since H becomes a diagonal block matrix, in which each block is a 2×2 matrix).[6] Notice that all eigenvalues intersections lies on the axis $u=0$ or on the axis $v=0$. See Figure 7.

The second model consists of a quantum particle in a one-dimensional infinite potential well with some additional controlled external fields. In this case, the main obstacle to be overcome is that in a one dimensional quantum system the presence of degeneracies in the discrete spectrum is a highly nonstandard feature. In particular the non degeneracy of the ground state holds in any dimension for systems subject to a locally integrable potential field ([77]). Therefore our strategy consists in producing degeneracies by means of potentials with non integrable singularities. To this purpose we use point interaction potentials (Dirac δ and δ') with a strength to be sent to infinity. Let us recall that interactions like Dirac δ and δ' are widely used in modeling of quantum system, since Fermi's paper [60] up to contemporary applications [25, 34, 35].

[6] Indeed this decoupling permits to get immediatly controllability results using classical control theory. The interest of this toy model is to illustrate a method that can be used in more general situations.

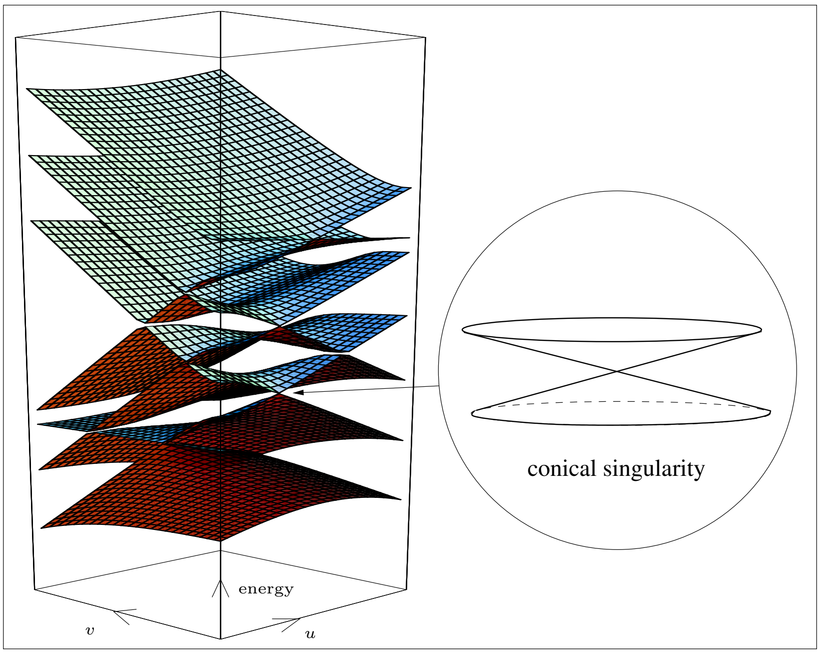

Figure 7: Eigenvalues intersections for the Hamiltonian (19), drawn for $u, v \geq 0$. Notice that each singularity is conical

.

We consider a particle confined to the interval $(0, \pi)$, whose Hamiltonian reads

$$\begin{aligned} H(u,v,w) \quad := \quad & -\partial_x^2 + u\delta(x-\pi/2) + v\delta'(x-\pi/2) \\ & +w\theta(x-\pi/2) \end{aligned} \tag{20}$$

and the drift coincides with $H(0,0,0) = -\partial_x^2$, with Dirichlet boundary conditions at 0 and π. See Figure 8.

Assume that at time zero the system lies in the ground state of the drift Hamiltonian and we switch a Dirac's delta interaction on, located at the center of the well, with strength $u(\varepsilon t)$. Then the energy of the ground state is slowly increasing with time, while the energy of the first excited level remains unchanged. In the limit $u(\varepsilon t) \to \infty$ the two energy levels coincide, but the associate eigenfunctions do not. We then use a Heaviside function in order to perform a rotation in the two dimensional eigenspace of the degenerate eigenvalue, and reach the eigenfunction of the first excited level of the drift Hamiltonian. In this way we obtain a transition from the ground level to the first excited.

In contrast with the previous model, here we exploit an intersection obtained letting the control u diverge; however the Hamiltonian (20) is well defined also for an infinite value of u and v.

Moreover, the gap condition is fulfilled because of the parity selection rule: the δ potential and the ground state of the drift are even, therefore even and odd levels are decoupled during the entire evolution and the effective gap is the one between the ground state and the second excited level.

Replacing the delta potential with a "delta prime" interaction, one can repeat this procedure and induce a transition from the first excited state to the second; more generally, alternating delta and delta prime one can reach any energy level. This control strategy can be generalized to any symmetric (coercive) potential (see Section 3.5)

It is worth mentioning that, for this model it seems extremely difficult to prove that it is possible to steer the system from two eigenstates using classical control theory (for instance using finite dimensional techniques on a Galerkin approximation of the system, and then passing to the limit).

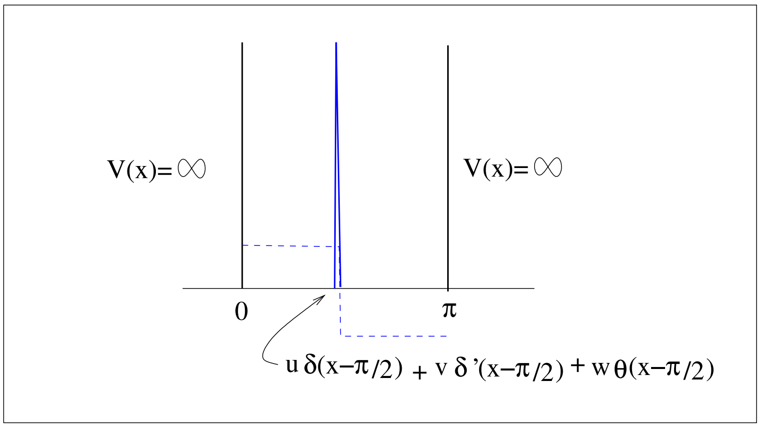

Figure 8:

3.5 Future research

The field of quantum control is nowadays very active and many things should be exploited in the future. In the next years I would like to work in the following directions.

3.5.1 Motion Planning for finite dimensional quantum systems

Even if the controllability problem is in a sense easy for finite dimensional quantum mechanical systems (cf. Section 3.3), the motion planning problem (i.e. making the system following approximatively a given trajectory) is always a highly nontrivial problem. Recently Jean-Paul Gauthier and Vladimir Zakalyukin (see [62, 63, 64]) developed a series of techniques that provide, in a wide range of situations, a *constructive* and in a sense *asymptotically optimal* solution, to the nonholonomic interpolation problem, in the driftless case.

I intend to extend their results to the *affine case*, to attack the motion planning problem for quantum systems with three or more levels and in which it is not possible to eliminate the drift. It is particularly interesting the case in which the number of controls is much less than the dimension of the space. In particular one would like to study the case with one control $u(t)$ and energy levels $E_1 < E_2 < \ldots < E_n$ satisfying $E_{j+1} - E_j > E_{j+2} - E_{j+1}$, $j = 1, \ldots n-2$, and prove the following fact. To steer the system from E_1 to E_n, requiring that $u(t)$ is small during the transition, then

$$u(t) = A(t)\sin(\omega(t)t + \phi),$$

where $\phi \in \mathbb{R}$, $A(t)$ is a slow varying real function, and $\omega(t)$ is an increasing function from $E_2 - E_1$ to $E_n - E_{n-1}$. This is what is expected by the physicists. This study should also give information useful to study the motion planning problem in the infinite dimensional case.

3.5.2 Controllability for infinite dimensional quantum systems

- Recently my collaborators A. Agrachev and Thomas Chambrion [29] gave an estimate on the minimum time (with one unbounded control) necessary to reach every point of the state space for a finite dimensional quantum system of the form $\dot{\psi} = (H_0 + uH_1)\psi$. Using this result, one can attack the controllability problem in the infinite dimensional case, when both the drift operator H_0 and the controlled operator H_1 have discrete spectrum. Since the cases interesting for applications are those in which H_1 is the operator of multiplication by a regular function (i.e. with H_1 having continuous spectrum), I will try to extend the controllability results to this case, using classical results of approximation of continuous spectrum operators with discrete spectrum ones.

- Following the ideas presented in the attached paper [6], in which controllability results are obtained with slowing varying controls, using the intersection of the eigenvalues in the space of controls, I would like to develop a theory working when the drift operator is the Laplacian plus a symmetric potential and the controlled Hamiltonian contains point interaction potentials.

4 Other Research Topics

4.1 Stability of Switched Systems for Arbitrary Switchings

This research concerns the papers [18, 19, 20] and several proceedings.

In recent years, the problem of stability and stabilizability of switched systems has attracted increasing attentions (see for instance [31, 38, 58, 75, 76] and the European project HYCON, http://www.ist-hycon.org), and still many questions remain unsolved.

By a switched system, we mean a family of continuous–time dynamical systems and a rule that determines at each time which dynamical system is responsible of the time evolution. More precisely, let $\{f_u : u \in U\}$ (where U is a subset of $\mathbb{R}^m$) be a finite or infinite set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems:

$$\dot{x} = f_u(x), \quad x \in M. \tag{21}$$

The rule is given by assigning the so-called switching function, i.e. a measurable function $u(.) : [0, \infty[\to U \subset \mathbb{R}^m$. Here, we consider the situation in which the switching function is not known a priori and represents some phenomenon (e.g. a disturbance) that is not possible to control. Therefore, the dynamics defined in (21) also fits into the framework of uncertain systems (cf. for instance [45]). For a discussion of various issues related to switched systems, we refer the reader to [44, 58, 76, 75].

A typical problem for switched systems goes as follows. Assume that, for every $u \in U$, the dynamical system $\dot{x} = f_u(x)$ satisfies a given property (P). Then one can investigate conditions under which property (P) still holds for $\dot{x} = f_{u(t)}(x)$, where $u(.)$ is an arbitrary switching function.

In [18, 19, 31, 58, 68], the case of linear switched systems was considered:

$$\dot{x}(t) = A_{u(t)}x(t), \quad x \in \mathbb{R}^n, \quad \{A_u\}_{u\in U} \subset \mathbb{R}^{n\times n}, \tag{22}$$

where $U \subset \mathbb{R}^m$ is a compact set, $u(.) : [0, \infty[\to U$ is a (measurable) switching function, and the map $u \mapsto A_u$ is continuous (so that $\{A_u\}_{u\in U}$ is a compact set of matrices). For these systems, the problem of asymptotic stability of the origin, uniformly with respect to switching functions was investigated.

In [31, 68], it is shown that the structure of the Lie algebra generated by the matrices A_u:

$$\mathbf{g} = \{A_u : u \in U\}_{L.A.},$$

is crucial for the stability of the system (22). The main result of [68] is the following:

Theorem 1 (Hespanha, Morse, Liberzon) *If* $\mathbf{g}$ *is a solvable Lie algebra, then the switched system (22) is GUAS.*

In [31] a generalization was given. Let $\mathbf{g} = \mathbf{r} \ltimes \mathbf{s}$ the Levi decomposition of $\mathbf{g}$ in its radical (i.e. the maximal solvable ideal of $\mathbf{g}$) and a semi–simple sub–algebra, where the symbol $\ltimes$ indicates the semidirect sum.

Theorem 2 (Agrachev, Liberzon) *If* $\mathbf{s}$ *is a compact Lie algebra then the switched system (22) is GUAS.*

Theorem 2 contains Theorem 1 as a special case. Anyway the converse of Theorem 2 is not true in general: if $\mathbf{s}$ is non compact, the system can be stable or unstable. This case was also investigated. In particular, if $\mathbf{g}$ has dimension at most 4 as Lie algebra, the authors were able to reduce the problem of the asymptotic stability of the system (22) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [31] for details. For this reason the bidimensional problem assumes particularly interest. In [18] (see also [19]) the single input case was investigated,

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \tag{23}$$

where A and B are two 2×2 real matrices with eigenvalues having strictly negative real part (Hurwitz in the following), $x \in \mathbb{R}^2$ and $u(.) : [0, \infty[\to \{0, 1\}$ is an arbitrary measurable switching function.

Under the assumption that A and B are both diagonalizable (in real or complex sense) a complete solution were found. More precisely a necessary and sufficient condition for GUAS were given in terms of three coordinate-invariant parameters: one depends on the eigenvalues of A, one on the eigenvalues of B and the last contains the interrelation among the two systems and it is in 1–1 correspondence with the cross ratio of the four eigenvectors of A and B in the projective line $\mathbb{C}P^1$.

In the proceeding [21] some non trivial examples of unstable systems for which the matrices A and B have real (and negative) eigenvalues are provided.

The stability conditions for (23) where obtained with a direct method without looking for a common Lyapunov function, but analyzing the locus in which the two vector fields are collinear, to build the "worst-trajectory".[7]

This methods was successful also to study a nonlinear generalization of this problem (see [20]). More precisely in [20] we study the stability of the time dependent nonlinear system $\dot{x}(t) = u(t)X(x(t)) + (1 - u(t))Y(x(t))$, where $x \in \mathbb{R}^2$, X and Y are two C^∞ vector fields, globally asymptotically stable at the origin and $u(.) : [0, \infty[\to \{0, 1\}$ is an arbitrary measurable function. Studying the topology of the set where X and Y are parallel, we give some sufficient and some necessary conditions for global asymptotic stability uniform with respect to switching and for global stability. Robustness of these conditions under small perturbations of the vector fields is studied.

4.1.1 Common Lyapunov functions for systems of type (22)

It is well known that the GUAS property for systems of type (22) is a consequence of the existence of a common Lyapunov function (LF, for short). However the concept of Lyapunov function is useful for practical purposes when one can prove that, for a certain class of systems, if a LF exists, then it is possible to find one of a certain type and possibly as simple as possible (e.g. polynomial with a bound on the degree, piecewise quadratic etc.). Typically one would like to work with a class of functions identified by a finite number of parameters. Once such a class of functions is identified, then in order to verify GUAS, one could use numerical algorithms to check (by varying the parameters) whether a LF exists (in which case the system is GUAS) or not (meaning that the system is not GUAS). For instance, a remarkable result for a given class C of systems of type (22) could be the following:

Claim: there exists a positive integer m (depending on n) such that, whenever a system of C admits a LF, then it admits one that is polynomial of degree less than or equal to m. In other words, the class of polynomials of degree at most m is sufficient to check GUAS for the class C.

Proving if this claim is true or false attracted some attention from the community.

For single input bidimensional systems of type (23), Shorten and K. Narendra provided in [99] a necessary and sufficient condition on the pair (A, B) to share a *quadratic* LF, but Dayawansa and Martin showed in [58] that there exist GUAS linear bidimensional systems not admitting a quadratic LF. Dayawansa and Martin also posed the problem of finding the minimal degree of a *polynomial* LF.

In the paper [19], we answer to this question. We prove that for systems of type (22) the GUAS property is equivalent to the existence of a *polynomial* LF, but the degree of that polynomial LF is not uniformly bounded over all the asymptotically stable systems. Hence the claim is false in general and polynomial LFs does not seem to be the most efficient tool to study the stability of linear switched systems. (Indeed for systems of type (23), stability conditions were obtained with another method.)

4.2 Time Optimal Syntheses on 2-D Manifolds

This research concerns the papers [4, 9, 8, 10, 12, 22, 23], the Lecture Notes [24], the book [17] and several proceedings.

[7] indeed Lyapunov functions does not seem to be the most efficient tool to study the stability of switching systems for arbitrary switchings as explained in Section 4.1.1.

4.2.1 Introduction

We consider optimal control problems in Bolza form:

$$\dot{x} = f(x, u), \tag{24}$$

$$\min \int_0^T L(x(t), u(t))\, dt, \qquad x(0) = x_0,\ x(T) = x_1, \tag{25}$$

where x belongs to some n–dimensional smooth manifold or, in particular, to $\mathbb{R}^n$ and $L : \mathbb{R}^n \times U \to \mathbb{R}$ is the *Lagrangian* or *running cost*. Fixing the initial point x_0 and letting the final condition x_1 vary in some domain of $\mathbb{R}^n$, we get a family of Optimal Control Problems.

One main issue is to introduce a concept of solution for this family of problems and, in this research, we focus on the concept of optimal synthesis. Roughly speaking, an *optimal synthesis* (see [17, 46, 50, 51, 52, 88]) is a collection of optimal trajectories starting from x_0, one for each final condition x_1, sharing certain regularity properties.

Geometric techniques provide a systematic method to attack the problem of building an optimal synthesis for nonlinear systems. Usually this approach consists in two steps:

1. Reduction procedure: it is based on the *Pontryagin Maximum, Principle* (PMP for short, see [32, 69, 89]) which is a first-order necessary condition for optimality. Roughly speaking, the PMP reduces the candidates for time optimality to the so called extremals, which are solutions of a pseudo-Hamiltonian system. This reduction procedure may be refined using higher order conditions, such as Clebsch-Legendre conditions, higher-order maximum principle, envelopes, conjugate points, index theory (cf. for instance [17, 26, 27, 28, 32, 69, 74, 95, 96, 100, 105, 106]);

2. Selection procedure: it consists of selecting the time optimal trajectories among the extremals that passed the test of Step **1.** (see for instance [17, 46, 51, 52, 88]).

Even though geometric techniques are powerful, Step **1.** is already non trivial and, in general, the second one is extremely difficult. For these reasons the construction of optimal syntheses is quite challenging and results are bounded to low dimensions.

We focus on a class of bidimensional Optimal Control Problems that is, at the same time, simple enough to permit the construction and complete classification of optimal syntheses, but, on the other side, sufficiently general to present a very rich variety of behaviors and to be used for several applications.

Geometric Problem: Minimum Time Motion on 2-D Manifolds
Given a smooth 2-D manifold M and two smooth vector fields X and Y on M, we want to steer a point $p \in M$ to a point $q \in M$ in minimum time using only integral curves of the two vector fields X and Y.

It may happen that the minimum time is obtained only by a trajectory γ whose velocity $\dot{\gamma}(t)$ belongs to the segment joining $X(\gamma(t))$ and $Y(\gamma(t))$ (not being an extremum of it). Thus, for existence purposes, we consider the set of velocities $\{vX(x) + (1 - v)Y(x) : 0 \le v \le 1\}$, that does not change the value of the infimum time.

The above geometric problem can be restated as the minimum time problem for the following control system. Defining $F = \frac{Y+X}{2}$ and $G = \frac{Y-X}{2}$ this control system can be written in local coordinates as:

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}, \quad |u| \le 1, \tag{26}$$

where F and G are $\mathcal{C}^\infty$ vector fields on M.

This problem was studied for $M = \mathbb{R}^2$ by Sussmann in the Analytic case (see [102, 104]) and by Sussmann, Bressan, Piccoli and myself in the $\mathcal{C}^\infty$ case (see [8, 9, 10, 12, 17, 47, 48, 86, 87, 102, 103, 104]). In the papers by Sussmann, Bressan and Piccoli, it was proved, under generic assumptions, the existence of an optimal synthesis in the sense of Boltyanskii-Brunovsky (see [46, 51, 52]). Moreover, their structural stability was showed and, in planar case, a complete classification via some topological graphs was given. My research in this field is presented in the next section.

4.2.2 Description of the Research

In the paper [9] we treat the problem of topological regularity of the minimum time function. We say that a continuous function, not necessarily smooth, is topologically a Morse function if its level sets are homeomorphic to the level sets of a Morse function. We prove that, generically, the minimum time function is a Morse function in topological sense giving a positive answer to a question of V.I. Arnold. This is very useful to characterize the topological properties of reachable sets and is sufficient, for example, to derive Morse inequalities.

Optimal trajectories satisfy the PMP, that provides a lift to the cotangent bundle that is a solution to a suitable pseudo–Hamiltonian system. The paper [8] is dedicated to analyze the set of extremals, i.e. trajectories satisfying PMP. The whole set of extremal in the cotangent bundle is called the *extremal synthesis*. We first prove that the support of extremals is a Whitney stratified set. Then in the cotangent bundle we give a topological classification of the singularities of the extremal synthesis and we study the projections of the support of extremals (regarded as a two dimensional object, after normalization) from $T^*M \times S^1$ to M. With respect to the Whitney classical singularities here we treat with a stratified set with "edges" and "corners" and beside cusps and folds we find other stable singularities. This investigation permits a deeper understanding of the relationships between synthesis singularities, minimum time function and Hamiltonian singularities.

In the construction of both optimal and extremal synthesis, a key role is played by abnormal extremals (that are trajectories with vanishing PMP's Hamiltonian). In particular the singularities of the synthesis involving abnormal extremals have some special features, that are studied in the paper [10].

The set of possible singularities along abnormal extremals is formed of 28 (equivalence classes of) singular points, but not all sequences of singularities can be realized. We prove that all possible sequences can be classified by a *set of words* recognizable by an *automaton* and, as a consequence, that all the 28 singularities can appear for some system.

In the paper [48], Bressan and Piccoli provided a topological classification via graphs of time-optimal flows on the plane. In [12] we generalize this result to the case of a two dimensional compact manifold. In the spirit of the classical work of Peixoto on classification of dynamical systems on two dimensional manifolds, we provide a topological classification via graphs of time-optimal flows, for generic control systems of the kind above on two dimensional orientable compact manifolds, also proving structural stability of generic optimal flows. More precisely, adding some additional structure to topological graphs, namely rotation systems, and thanks to a theorem of Heffter, dating back to 19^{th} century, we prove that there is a one to one correspondence between graphs with rotation systems and couples formed by a system and the 2-D manifold of minimal genus on which the system can be embedded.

The book [17] gathers all important results on this subject. It is suitable for graduate students in mathematics and in engineering, in the latter case with solid mathematical background, interested in optimal control and in particular in geometric methods.

An introductory chapter to geometric control is contained, while the rest of the book focuses on time optimal syntheses for systems of kind (26) on 2-D manifolds. The book contains exercises and bibliographical notes. We now describe each Chapter in more detail.

- In Chapter 1 we provide an introduction to some basic facts of Geometric Control Theory: controllability, optimal control, Pontryagin Maximum Principle, high order conditions, etc.

- Chapter 2 is dedicated to the construction of optimal synthesis. First, we provide a detailed study of the structure of optimal trajectories. Using this, the existence of an optimal synthesis, under generic assumptions, is proved. A complete classification of synthesis singularities is also given, presenting explicit examples for each equivalence class. The classification program is then completed by proving structural stability of optimal syntheses and associating to each of them a topological graph. Applications are given in the last section.

- In Chapter 3 we treat the problem of topological regularity of the minimum time function. While in Chapter 4 we study the extremal synthesis.

- Most of the results obtained so far are key for the following Chapter 5. There are four natural spaces where the mathematical objects previously defined live: the product $T^*M \times \mathbb{R}$ of triplets $(state, costate, time)$,

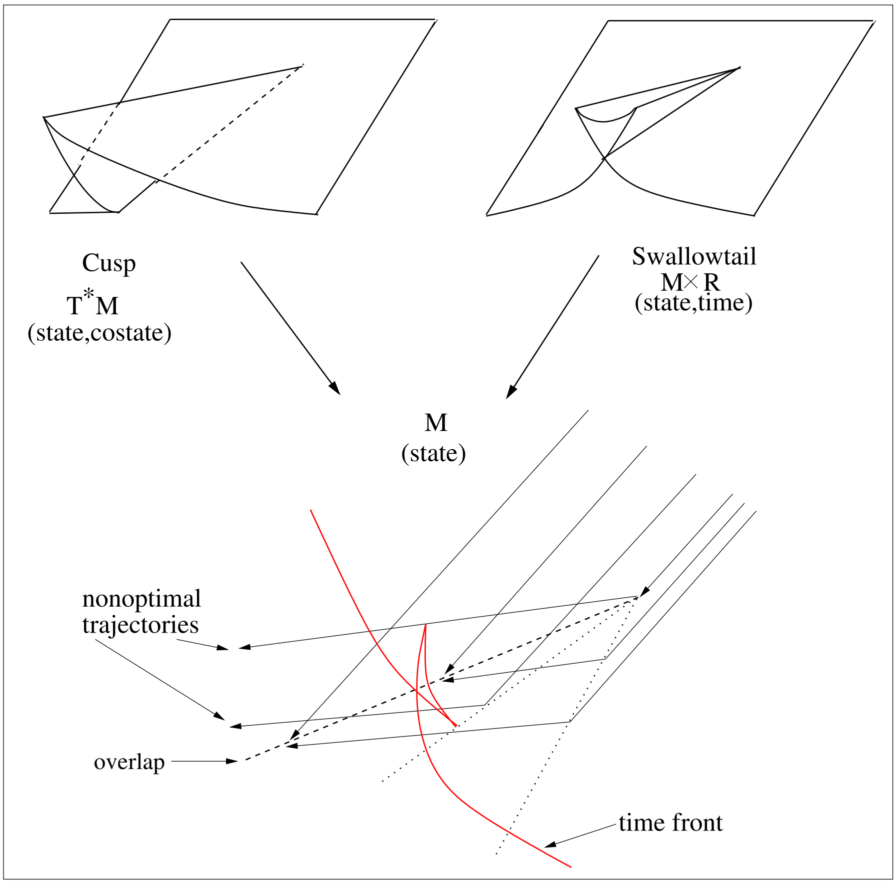

Figure 9: Example of a singularity of the optimal synthesis and related projection singularities.

the cotangent bundle T^*M where extremals evolve, the product $M \times \mathbb{R}$ of couples $(state, time)$, the base space M. The set of pairs $(extremals, time) = (state, costate, time)$ is a manifold in $T^*M \times \mathbb{R}$, while the set of extremals in T^*M is only a Whitney stratified set (no more a manifold). All projection singularities between these spaces are classified under generic conditions and the links with synthesis singularities are explained. For example, we show that curves reached optimally by two different trajectories, called *overlaps*, are generated at projection singularities (from T^*M to M) of *cusp* type. On the other side, these points correspond to swallowtails in $M \times \mathbb{R}$ and to singularities of the extremal time front on M. See Figure 9. Most of the material presented in Chapter 5 is not published elsewhere.

- Two appendices end the book. The first reports some technical proofs, while the second generalizes all results to the case of a *two dimensional source*. In particular, in the case $M = \mathbb{R}^2$, under a local controllability assumption, we can also prove the *semiconcavity* of the minimum time function. This property (that fails with a pointwise source) is typically encountered in calculus of variations problems or optimal control problems with set of velocities of the same dimension as the state space.

In the framework of quantum control, some of the results given in Chapter 2 are extended in the paper [4] to the "distributional" version of the same problem:

$$\min T, \quad \dot{x} = u_1 F_1 + u_2 F_2, \quad |u_1| \leq 1, |u_2| \leq 1. \tag{27}$$

4.3 Optimal Control and sub-Riemannian Geometry on Lie Groups

This research concerns the papers [2, 4, 5, 13, 14, 15] and several proceedings. It focus on optimal synthesis problems on Lie Groups and homogeneous spaces. These kind of problems are very interesting for applications to *mechanical Systems* (see for instance [16, 107]) and *quantum control*, (see Section 3).

As explained in Section 4.2.1, the problem of determine the optimal synthesis for a nonlinear system is extremely difficult. If the state space is two-dimensional, the problem of determining the time optimal synthesis for single-input control systems is now well-understood (see the previous section), however, for higher dimensions, very few examples of complete optimal syntheses for non linear control systems are available (see for instance [101]). Intermediate issues were thus deeply investigated: determining estimates for the number of switchings of optimal trajectories, describing the local structure of optimal trajectories, finding families of trajectories sufficient for optimality, cf. [30, 74, 82, 89, 96, 107], etc.

In the paper [13], using the *envelope theory* developed by Sussmann [105, 106], we investigate the structure of time-optimal trajectories for a driftless control system on $SO(3)$ of the type $\dot{x} = x(u_1 f_1 + u_2 f_2)$, $|u_1|, |u_2| \leq 1$, where $f_1, f_2 \in so(3)$ define two linearly independent left-invariant vector fields on $SO(3)$. This problem was first studied by Sussmann and Tang ([107]) in the case in which $\|f_1\| = \|f_2\|$. We show that every time-optimal trajectory is a finite concatenation of at most five (bang or singular) arcs. More precisely, a time-optimal trajectory is, on the one hand, bang-bang with at most either two consecutive switchings relative to the same input or three switchings alternating between two inputs, or, on the other hand, a concatenation of at most two bangs followed by a singular arc and then two other bangs. We end up finding a finite number of three-parameters trajectory types that are sufficient for time-optimality.

In the paper [14] we consider the control system (Σ) given by $\dot{x} = x(f + ug)$, where $x \in SO(3)$, $|u| \leq 1$ and $f, g \in so(3)$ define two perpendicular left-invariant vector fields normalized so that $\|f\| = \cos(\alpha)$ and $\|g\| = \sin(\alpha)$, $\alpha \in]0, \pi/4[$. Let $N(\alpha)$ be the maximum number of switchings for time-optimal trajectories of (Σ). This problem was first studied by Agrachev and Gamkrelidze that, using Index Theory, proved in [27] the following inequality:

$$N(\alpha) \leq \left[\frac{\pi}{\alpha}\right], \tag{28}$$

where $[\cdot]$ stands for the integer part. That result was not only an indirect indication that $N(\alpha)$ would tend to ∞ as α tends to zero, but it also provided a hint on the asymptotic of $N(\alpha)$ as α tends to zero.

The main result of the paper [14] is to confirm this fact i.e. $N(\alpha)$ tends to ∞ as α tends to zero. More precisely, we complete the inequality (28) as follows: $N_S(\alpha) \leq N(\alpha) \leq N_S(\alpha) + 5$, where $N_S(\alpha)$ is a suitable integer function of α such that $N_S(\alpha) \underset{\alpha \to 0}{\sim} \pi/(2\alpha)$. Our result improves (28) in two ways: **i)** for α small, it (essentially) divides the upper bound of $N(\alpha)$ by two with respect to (28); **ii)** it provides a lower bound of $N(\alpha)$ differing from the upper bound by a constant. The result is obtained by studying the time optimal synthesis of a projected control problem on S^2, where the projection is defined by an appropriate Hopf fibration.

The projected control problem on the unit sphere S^2 exhibits interesting features which are partly rigorously derived and partially described by numerical simulations. We also conjecture that the qualitative shape of the time optimal synthesis presents different patterns, that cyclically alternate as $\alpha \to 0$ depending on the remainder $r(\alpha) := \pi/2\alpha - [\pi/2\alpha]$.

The analysis has been pushed much further in [5, 15]. In [5], in the framework of quantum control, a partial proof of the conjecture has been provided, while in [15], the noncontrollability limit $\alpha \to 0$ has been studied. The main result of the paper [15] is that, for $\alpha \to 0$, the synthesis tends, in a suitable sense, to the synthesis of an harmonic oscillator, having as source a circle centered in the origin.

In the papers [2] and [4] we focus on sub-Riemannian problems on $SO(3)$ and $SU(3)$ for which the Hamiltonian system given by the PMP is completely integrable. These papers are written in the framework of quantum control (see Section 3).

More precisely in [2] we study the following problem. Let G be a real semisimple Lie group, with Lie algebra L, Cartan decomposition $L = K + P$, and consider the left invariant Subriemannian metric with distribution P and metric $Kill|_P$ (at identity). Here $Kill$ is the Killing form. This is the most natural left invariant Subriemannian problem on semisimple Lie groups. In particular we study the cases $G = SU(3)$ and $G = SO(3)$.

In [4] we study Subriemannian problem similar to the previous, but with cost built with a deformed killing form. On $SO(3)$ this problem does not have abnormal extremals (since it is contact) and the corresponding Hamiltonian system is completely integrable (since it leads to a left invariant Hamiltonian system on a Lie group of dimension 3, see for instance [69, 71]).

Then, using an appropriate Hopf fibration, we project the problem on the sphere S^2, getting an almost-Riemannian problem (see Remark 2 and Section 3.3.4). Most of the efforts are made to select the optimal trajectories.

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