

NONISOTROPIC 3-LEVEL QUANTUM SYSTEMS: COMPLETE SOLUTIONS FOR MINIMUM TIME AND MINIMUM ENERGY

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ABSTRACT. We apply techniques of subriemannian geometry on Lie groups and of optimal synthesis on 2-D manifolds to the population transfer problem in a three-level quantum system driven by two laser pulses, of arbitrary shape and frequency. In the rotating wave approximation, we consider a nonisotropic model, i.e., a model in which the two coupling constants of the lasers are different. The aim is to induce transitions from the first to the third level, minimizing **1)** the time of the transition (with bounded laser amplitudes), **2)** the energy transferred by lasers to the system (with fixed final time). After reducing the problem to real variables, for the purpose **1)** we develop a theory of time optimal syntheses for distributional problem on 2-D manifolds, while for the purpose **2)** we use techniques of subriemannian geometry on 3-D Lie groups. The complete optimal syntheses are computed.

1. Introduction.

1.1. Statement of the Problem. The problem of designing an efficient transfer of population between different atomic or molecular levels is crucial in many atomic-physics projects [5, 20, 26, 34]. Often excitation or ionization is accomplished by using a sequence of laser pulses to drive transitions from each state to the next state. The transfer should be as efficient as possible in order to minimize the effects of relaxation or decoherence that are always present. In the recent past years, people started to approach the design of laser pulses by using Geometric Control Techniques (see for instance [18, 19, 25, 31]). Finite dimensional closed quantum systems are in fact left (or right) invariant control systems on $SU(n)$, or on the corresponding Hilbert sphere $S^{2n-1} \subset \mathbb{C}^n$, where n is the number of atomic or molecular levels. For these kinds of systems very powerful techniques were developed both for what concerns controllability [3, 21, 24, 32] and optimal control [1, 9, 22].

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The most important and powerful tool for the study of optimal trajectories is the well known Pontryagin Maximum Principle (in the following PMP, see for instance [1, 22, 30]). It is a first order necessary condition for optimality and generalizes the Weierstraß conditions of Calculus of Variations to problems with non-holonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle that is a solution to a suitable pseudo-Hamiltonian system. Anyway, giving a complete solution to an optimization problem (that for us means to give an optimal synthesis, see for instance [6, 9, 16, 29]) remains extremely difficult for several reasons. First, one is faced with the problem of integrating a Hamiltonian system (that generically is not integrable excepted for very special costs). Second, one should manage with “non Hamiltonian solutions” of the PMP, the so called abnormal extremals. Finally, even if one is able to find all the solutions of the PMP, it remains the problem of selecting among them the optimal trajectories. For these reasons, usually, one can hope to find a complete solution to an optimal control problem in low dimension only ([2, 9, 15, 33]).

In this paper we apply techniques of subriemannian geometry on Lie groups and of optimal synthesis on 2-D manifolds to the population transfer problem in a three-level quantum system driven by two external fields (in the rotating wave approximation) of arbitrary shape and frequency. The dynamics is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$)

$$i \frac{d\psi(t)}{dt} = H\psi(t), \quad (1)$$

where $\psi(\cdot) = (\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot))^T : [0, T] \rightarrow \mathbb{C}^3$, $\sum_{j=1}^3 |\psi_j(t)|^2 = 1$ (i.e., $\psi(t)$ belongs to the sphere $S^5 \subset \mathbb{C}^3$), and

$$H = \begin{pmatrix} E_1 & \mu_1 \mathcal{F}_1 & 0 \\ \mu_1 \mathcal{F}_1^* & E_2 & \mu_2 \mathcal{F}_2 \\ 0 & \mu_2 \mathcal{F}_2^* & E_3 \end{pmatrix}, \quad (2)$$

where E_1 , E_2 and E_3 are real numbers representing the energy levels. Here $(*)$ denotes the complex conjugation involution. The controls $\mathcal{F}_1(\cdot), \mathcal{F}_2(\cdot)$, that we assume to be measurable functions, different from zero only in a fixed interval, are the external pulsed field, while $\mu_j > 0$, ($j = 1, 2$) are the couplings (intrinsic to the quantum system) that we have restricted to couple only levels j and $j + 1$ by pairs.

Using standard arguments of controllability on compact Lie groups and corresponding homogeneous spaces, one gets the following (see the recent survey [32] or the papers [3, 10, 11, 21, 24]):

Proposition 1. *Assume that there exists constants $M_1, M_2 > 0$ such that $|u_1(t)| \leq M_1$ and $|u_2(t)| \leq M_2$ for a.e. $t > 0$. Then there exists a time $\tau(M_1, M_2)$ such that the control system (1), (2) is completely controllable in time $\tau(M_1, M_2)$.*

The model (1), (2) belongs to a class of systems on which it is possible to eliminate the so called drift term (i.e., the term $\text{diag}(E_1, E_2, E_3)$) by a unitary change of coordinates and a change of controls (see Section 1.2).

The aim is to induce complete population transfer from the state one ($|\psi_1|^2 = 1$), called source, to the state three ($|\psi_3|^2 = 1$), called target, minimizing the criteria described in the following.

Energy in fixed time

$$\int_0^T (|\mathcal{F}_1(t)|^2 + |\mathcal{F}_2(t)|^2) dt, \quad (3)$$

This cost is the energy of the laser pulses. After elimination of the drift and reduction to a real problem (see below), the problem of minimizing this cost becomes a singular-Riemannian problem (or a subriemannian problem when lifted on the group $SO(3)$, see Section 1.2) and was studied in [11], in the “nongeneric” case in which $\mu_1 = \mu_2$. In the following we call the problem (1), (2), (3) isotropic if $\mu_1 = \mu_2$ otherwise we call the problem nonisotropic. For the cost (3), to guarantee the existence of minimizers, the final time T should be fixed, and such minimizers are parameterized with constant velocity ($|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 = \text{const}$). Moreover the controls are not assumed a priori bounded. Anyway if the final time T is fixed in such a way the minimizer is parameterized by arclength ($|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 = 1$), then minimizing the cost (3) is equivalent to minimize the time, with moduli of controls constrained in the closed set

$$|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 \leq 1. \quad (4)$$

For this cost we can normalize $\mu_1 = \mu_2 = 1$ in the isotropic case, while $\mu_1 = 1$, $\mu_2 = \alpha > 0$ for the nonisotropic case. Notice that this change of notation modifies the costs only by a multiplicative constant.

Time with bounded controls

Another interesting cost is the time of transfer under the conditions

$$|\mathcal{F}_1(t)| \leq \nu_1 \text{ and } |\mathcal{F}_2(t)| \leq \nu_2. \quad (5)$$

In this case we call the problem isotropic if $\mu_1\nu_1 = \mu_2\nu_2$. Otherwise we call the problem nonisotropic. Let us notice that if we do not require the constraints (5), then there is no minimizer.

For this cost we can normalize $\mu_1 = \nu_1 = \mu_2 = \nu_2 = 1$ in the isotropic case, while $\mu_1 = \nu_1 = 1$, $\mu_2 = \alpha > 0$, $\nu_2 = 1$ for the nonisotropic case.

After these normalizations the Hamiltonian and the costs read

$$H = \begin{pmatrix} E_1 & \mathcal{F}_1 & 0 \\ \mathcal{F}_1^* & E_2 & \alpha\mathcal{F}_2 \\ 0 & \alpha\mathcal{F}_2^* & E_3 \end{pmatrix}, \quad (6)$$

$$\text{energy in fixed time: } \int_0^T (|\mathcal{F}_1(t)|^2 + |\mathcal{F}_2(t)|^2) dt, \quad (7)$$

$$\text{time with bounded controls: } T = \int_0^T 1 dt, \text{ with } |\mathcal{F}_1(t)| \leq 1 \text{ and } |\mathcal{F}_2(t)| \leq \alpha \quad (8)$$

In the following we call the parameter α the nonisotropy factor.

Remark 1. The problem of inducing a transition from the first to the third eigenstate, can be formulated, as usual, at the level of the wave function $\psi(t)$, but also at the level of the time evolution operator (the resolvent), denoted here by $g(t)$. We have $\psi(t) = g(t)\psi(0)$, $g(t) \in U(3)$, $g(0) = id$. For $g(t)$ the Schrödinger equation (1) reads $\dot{g}(t) = -iHg(t)$. In the following we call the optimal control problem for $\psi(t)$ and for $g(t)$ respectively the problem downstairs and the problem upstairs. In this

kind of problems one can take advantages of working both upstairs and downstairs depending on the specific task. This approach happened to be successful in some other problems of optimal control on Lie groups, see for instance [14]. Notice that the evolution of the trace part is decoupled from the rest. It follows that we can always assume $E_1 + E_2 + E_3 = 0$ and $g(t) \in SU(3)$.

This paper is the continuation of a series of papers on optimal control of finite dimensional quantum systems [10, 11, 12]. In [10], the problem of minimizing the energy of the lasers pulses was studied for a two-level system and for an isotropic three-level system. The two-level system (that is a problem on $SU(2)$) was completely solved (in this case minimizing energy is equivalent to minimize time with bounded controls), while the three-level problem (that upstairs is a problem on $SU(3)$) was solved assuming resonance as hypothesis (i.e., controls oscillating with a frequency equal to the difference of the energy levels that the laser is coupling, see formula (9) below). Thanks to this assumption, the problem (downstairs) could be reduced to a two-dimensional problem and solved completely. In [11] the problem of minimizing the isotropic energy in the three-level problem was treated without assuming resonance as hypothesis. In that case, even if the optimal control problem lives in a space of big dimension ($\dim(SU(3)) = 8$), it was possible to get explicit expressions of optimal controls and trajectories thanks to the special structure of the Lie algebra of the problem and to the fact that the cost is built with the Killing form, that renders the Hamiltonian system associated to the PMP Liouville integrable. In that paper resonance was obtained as consequence of the minimization process and explicit expressions of amplitudes were provided. In [12], the possibility of restricting to resonant controls was generalized to n -level systems and to more general costs. More precisely consider a n -level system of the kind

$$\begin{cases} i \frac{d\psi(t)}{dt} = (D + V(t))\psi \\ \psi(\cdot) := (\psi_1(\cdot), \dots, \psi_n(\cdot))^T : [0, T] \rightarrow \mathbb{C}^n, \sum_i |\psi_i|^2 = 1, \end{cases}$$

where $D = \text{diag}(E_1, \dots, E_n)$ and $V(t)$ is an Hermitian matrix ($V(t)_{j,k} = V(t)_{k,j}^*$), whose elements are either identically zero or controls. (i.e., $V_{i,j} \equiv 0$ or $V_{i,j} = \mu_{i,j} F_{i,j}$, for $i < j$, the term $F_{i,j}$ being the external pulsed field coupling level i and level j). Then for a convex cost depending only on the moduli of controls (i.e., amplitudes of the lasers), like those described above, it was proved that there always exists a minimizer in resonance that connects a source and a target defined by conditions on the moduli of the components of the wave function (e.g. two eigenstates): $\mathcal{F}_{j,k}(t) = u_{j,k}(t) e^{i[(E_j - E_k)t + \xi_{j,k}]}$, $i < j = 1, \dots, n$, where $\xi_{j,k} \in [-\pi, \pi]$ are some phases and $u_{j,k}(\cdot) : [0, T] \rightarrow \mathbb{R}$ are the amplitudes of the lasers that should be determined. This result permits to reduce the problem to real variables (i.e., from $SU(n)$ to $SO(n)$ upstairs, or from $S^{2n-1} \subset \mathbb{C}^n$ to $S^{n-1} \subset \mathbb{R}^n$ downstairs), and therefore to simplify considerably the difficulty of the problem (see below for the reduction to real variables in the 3-level case).

In the present paper we take advantage of this procedure to find complete solutions to the optimal control problem for the three level quantum system (1), (6) and the costs (7), (8). More precisely:

- A:** the minimum time problem with bounded controls (downstairs) is a problem in dimension five. In this case, since the dimension of the state space is big, the problem of finding extremals and selecting optimal trajectories can be

extremely hard. The fact that one can restrict to minimizers that are in resonance permits to reduce the problem to a two-dimensional problem, that can be solved with techniques similar to those used in [9]. This is the goal of Section 4;

B: the minimum energy problem is naturally lifted to a right invariant subriemannian problem on the group $SU(3)$ (as explained in Remark 1). This problem cannot be solved with the techniques used in [11] for the isotropic case, because now the cost is built with a “deformed Killing form”. Anyway since we can restrict to resonant minimizers the problem is reduced to a contact subriemannian problem on $SO(3)$, that does not have abnormal extremals (since it is contact) and the corresponding Hamiltonian system is completely integrable (since it leads to a right invariant Hamiltonian system on a Lie group of dimension 3, see for instance [22, 23]). The complete solutions can be found in terms of Elliptic functions. This is the aim of Section 5.

Remark 2. Thanks to the reduction given in [12], a minimum energy problem for a n -level system, with levels coupled by pairs, is a singular-Riemannian problem on S^{n-1} or can be lifted to a right invariant subriemannian problem on $SO(n)$. Anyway there is no reason to believe that the Hamiltonian system associated to the PMP is Liouville integrable for $n \geq 4$. See [13] for some numerical solutions to this problem for $n = 4$.

1.2. Reduction to Real Variables. In the following, for the 3-level problem (1), (6) we recall how to reduce the problem to real variables using the fact that we can restrict to resonant controls, i.e., controls of the form

$$\mathcal{F}_j(t) = u_j(t)e^{i[(E_{j+1}-E_j)t+\xi_j]}, \quad j = 1, 2, \quad (9)$$

where $\xi_j \in [-\pi, \pi]$ are some phases and $u_j(\cdot) : [0, T] \rightarrow \mathbb{R}$ are the amplitudes of the lasers that should be determined. Using the so called interaction picture, i.e., making the unitary transformation

$$\psi(t) \rightarrow U^{-1}(t)\psi(t), \quad \text{where } U(t) := \text{diag}(e^{-iE_1}, e^{-iE_2}, e^{-iE_3}), \quad (10)$$

and next making the transformation (to kill the phases)

$$\psi(t) \rightarrow V^{-1}\psi(t), \quad \text{where } V := \text{diag}(1, e^{i(-\pi/2-\xi_1)}, e^{i(-\pi-\xi_1-\xi_2)}), \quad (11)$$

the Schrödinger equation becomes

$$\frac{d\psi(t)}{dt} = H(t)\psi(t). \quad (12)$$

Notice that these transformations leave invariant the source $|\psi_1|^2 = 1$ and the target $|\psi_3|^2 = 1$. Now, one can restrict equation (12) to reals (i.e., to the sphere S^2 , i.e., $\psi(\cdot) : [0, T] \rightarrow \mathbb{R}^3$, $\psi_1^2 + \psi_2^2 + \psi_3^2 = 1$) by taking a real initial condition (the initial phase is arbitrary). Thus the drift is eliminated and the Hamiltonian belongs to $\mathfrak{so}(3)$

$$H = \begin{pmatrix} 0 & -u_1 & 0 \\ u_1 & 0 & -\alpha u_2 \\ 0 & \alpha u_2 & 0 \end{pmatrix}. \quad (13)$$

For more details on these transformations, see [10, 11, 12]. After reduction to real variables, the states one and three are represented respectively by the couples of

points $(\pm 1, 0, 0)^T$ and $(0, 0, \pm 1)^T$. And the costs in which we are interested become:

$$\text{energy in fixed time: } \int_0^T (u_1(t)^2 + u_2(t)^2) dt, \tag{14}$$

$$\text{time with bounded controls: } \begin{cases} T = \int_0^T 1 dt, \text{ under the conditions:} \\ |u_1(t)| \leq 1 \text{ and } |u_2(t)| \leq 1. \end{cases} \tag{15}$$

Remark 3. Beside the proof that there always exist minimizers in resonance, another useful result given in [12] is that (after reduction to real variables), there always exists a minimizer corresponding to positive coordinates (and it is also given the description of how to reconstruct all other resonant minimizers from this last one). Hence we can assume $\psi \in S^+$ where S^+ is the subset of the sphere S^2 corresponding to positive coordinates

$$S^+ = \{\psi = (\psi_1, \psi_2, \psi_3)^T \in \mathbb{R}^3 : \psi_1^2 + \psi_2^2 + \psi_3^2 = 1, \psi_i \geq 0\}. \tag{16}$$

Let $Lip([0, T], S^+)$ be the set of Lipschitz functions from $[0, T]$ to S^+ . Our minimization problem, in which we precise the functional spaces, is:

Problem downstairs (PD) *Consider the control system (12), (13), where $u_i(\cdot) \in L^\infty([0, T], \mathbb{R})$ ($i = 1, 2$), $\psi(\cdot) \in Lip([0, T], S^+)$. Find the optimal trajectory and control steering the point $\psi = (1, 0, 0)^T$ (still called source or state one), to the point $\psi = (0, 0, 1)^T$ (still called target or state three), that minimizes the cost (14) or (15).*

Remark 4. We attack the problem by computing all trajectories in S^+ , starting from $(1, 0, 0)^T$, and satisfying the PMP (the so called extremals). Then we have to select, among all extremal trajectories reaching the target, those having smallest cost. Since we will find that all extremals passing through a point of $S^+ \setminus \{(1, 0, 0)^T\}$ coincide before this point, we get a stronger result than a solution to the problem **(PD)**. Indeed, we obtain the complete optimal synthesis, i.e., a set of optimal trajectories starting from $(1, 0, 0)^T$ and reaching every point of S^+ . For a more sophisticated definition of optimal synthesis, see [9, 29]. The optimal synthesis can be useful for applications, when one needs to reach a final state that is not an eigenstate.

For what concerns the problem upstairs, after the transformations (10), (11) the equation for the operator of temporal evolution is

$$\dot{g}(t) = Hg(t), \tag{17}$$

where now $g(t) \in SO(3)$ and H is given by (13). To pass from the problem upstairs to the problem downstairs, one should take the first column of $g(t)$, i.e., should use the projection

$$\begin{aligned} \Pi : SO(3) &\rightarrow S^2 \\ g &\mapsto g\psi(0) = g(1, 0, 0)^T. \end{aligned} \tag{18}$$

Upstairs the source and the target are not points, but are respectively the sets \mathcal{S} and \mathcal{T} of matrices of $SO(3)$ that projected with (18) give rise to $(1, 0, 0)^T$ and $(0, 0, 1)^T$.

More precisely we have

$$\mathcal{S} := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & SO(2) \end{array} \right) = SO(2) \subset SO(3), \quad (19)$$

$$\mathcal{T} := g_0 \mathcal{S}, \text{ where } g_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3), \text{ i.e.,}$$

$$\mathcal{T} = \left(\begin{array}{c|c} 0 & SO(2) \\ \hline 1 & 0 \end{array} \right). \quad (20)$$

Notice that \mathcal{S} is a subgroup of $SO(3)$ while \mathcal{T} is not (indeed it is a translation of a subgroup). The problem upstairs is finally:

Problem upstairs (PU) Consider the control system (13), (17) where $u_i(\cdot) \in L^\infty([0, T], \mathbb{R})$ ($i = 1, 2$), $g(\cdot) \in Lip([0, T], SO(3))$. Find the optimal trajectories and controls connecting the source \mathcal{S} to the target \mathcal{T} that minimize the cost (14) or (15).

Remark 5. Notice that after elimination of the drift, both control systems (12), (13), and (13), (17) can be rewritten in the form (called distributional)

$$\dot{y} = u_1 F_1(y) + u_2 F_2(y), \quad (21)$$

where $y \in S^+$ downstairs, or $y \in SO(3)$ upstairs and

$$F_1(y) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y, \quad F_2(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} y.$$

A problem with a dynamics (21) and cost (14) belongs to the frameworks of subriemannian geometry and singular-Riemannian geometry (see [4, 27]).

Using standard arguments of controllability on compact Lie groups and corresponding homogeneous spaces, one gets the following:

Proposition 2. After elimination of the drift term and reduction to real variables, the control system upstairs (13), (17) (resp. the control system downstairs (12), (13)) is completely controllable on $SO(3)$ (resp. on S^+).

Remark 6. Notice that both problems (PD) and (PU) can be seen as minimum time problems on a compact manifold with velocities in a convex compact set (and there is complete controllability). Then, by Filippov theorem, they have solutions (see for instance [17]).

1.3. Structure of the Paper. In Section 2 we state PMP, our main tool, and we show that, for our minimization problems, there are no abnormal extremals. In Section 3 we state our main results. In Section 4 we treat the minimum time problem downstairs. In Subsection 4.1 we develop a theory of time optimal syntheses on 2-D manifolds for distributional systems with bounded controls (this subsection is written in such a way to be as self-consistent as possible) and in the next subsection we build explicitly the time optimal synthesis. In Section 5, attacking the problem upstairs, we treat the minimum energy problem. We give explicit expressions of the

optimal controls, for which the corresponding trajectories form an optimal synthesis on S^+ .

2. Pontryagin Maximum Principle. Our main tool is the Pontryagin Maximum Principle (PMP), stated below for the two cases in which the final time is free or fixed. For the proof see for instance [1, 30]. In this version the source and the target are not just points, but more generally smooth submanifolds. This is useful while working upstairs (where the source and the target are defined respectively by (19) and (20)).

Theorem (Pontryagin Maximum Principle) *Consider the control system $\dot{x} = f(x, u)$ with a cost of the form $\int_0^T f^0(x(t), u(t)) dt$, where the final time T is free, and initial and final conditions given by $x(0) \in M_{in}$, $x(T) \in M_{fin}$, where x belongs to a manifold M and $u(\cdot) : [0, \infty[\rightarrow U \subset \mathbb{R}^m$. Assume moreover that M , f , f^0 are smooth and that M_{in} and M_{fin} are smooth submanifolds of M .*

*Define for every $(x, \lambda, u) \in T^*M \times U$*

$$\mathcal{H}(x, \lambda, u) := \langle \lambda, f(x, u) \rangle + \lambda_0 f^0(x, u). \tag{22}$$

*If the couple $(x(\cdot), u(\cdot)) : [0, T] \rightarrow M \times U$ (with $u(\cdot)$ measurable and essentially bounded, and $x(\cdot)$ Lipschitz) is optimal, then there exists a never vanishing Lipschitz function $(\lambda(\cdot), \lambda_0) : t \in [0, T] \mapsto (\lambda(t), \lambda_0) \in T_{x(t)}^*M \times \mathbb{R}$ (where $\lambda_0 \leq 0$ is a constant) such that for a.e. $t \in [0, T]$ we have:*

- i:** $\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \lambda}(x(t), \lambda(t), u(t))$,
- ii:** $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), \lambda(t), u(t))$,
- iii:** $\mathcal{H}(x(t), \lambda(t), u(t)) = \mathcal{H}_M(x(t), \lambda(t))$, where $\mathcal{H}_M(x, \lambda) := \max_{v \in U} \mathcal{H}(x, \lambda, v)$.
- iiii:** $\mathcal{H}_M(x(t), \lambda(t)) = 0$.
- v:** $\langle \lambda(0), T_{x(0)}M_{in} \rangle = \langle \lambda(T), T_{x(T)}M_{fin} \rangle = 0$ (transversality conditions).

*If the final time T is fixed by hypothesis, then condition **iiii** must be replaced by*

iiibis: $\mathcal{H}_M(x(t), \lambda(t)) = k \geq 0$, for some constant k .

Definition 1. The map $\lambda(\cdot) : [0, T] \rightarrow T_{x(t)}^*M$ is called covector. The real-valued map on $T^*M \times U$, defined in (22) is called PMP-Hamiltonian. A trajectory $x(\cdot)$ (resp. a couple $(x(\cdot), \lambda(\cdot))$) satisfying conditions **i**, **ii**, **iii** and **iiii** (or **iiibis**) is called an extremal (resp. an extremal pair). If $(x(\cdot), \lambda(\cdot))$ satisfies **i**, **ii**, **iii** and **iiii** (or **iiibis**) with $\lambda_0 = 0$ (resp. $\lambda_0 < 0$), then it is called an abnormal extremal (resp. a normal extremal).

Remark 7. Notice that the definition of abnormal extremal does not depend on the cost but only on the dynamics (in fact if $\lambda_0 = 0$, the cost disappears in (22)). For general properties of abnormal extremals, we refer to [7, 27].

Proposition 3. *For the control systems downstairs (12), (13), there are no abnormal extremals. The same holds for the control system upstairs (13), (17).*

Proof. For the problems (12), (13) and (13), (17), the set of admissible velocities at a point x writes $\{u_1 F_1(x) + u_2 F_2(x) \mid (u_1, u_2) \in U\}$, where the set U is a subset of \mathbb{R}^2 containing 0 in its interior. This implies that, if there were an abnormal extremal, its covector $\lambda(\cdot)$ should annihilate F_1 and F_2 all along the trajectory, but also, using PMP, their bracket $[F_1, F_2]$. But for both problems **(PD)** and **(PU)**, $(F_1, F_2, [F_1, F_2])$ forms a generating family of the tangent space which implies that $\lambda(\cdot) \equiv 0$, all along the trajectory. This is forbidden by the PMP. \square

3. Main Results. In this section we present explicitly the optimal controls steering state one to state three for our two costs. The optimal syntheses are presented in Sections 4 and 5 and pictured in Figures 3 and 6. In both cases, all the extremals issued from $(1, 0, 0)^T$ given by the PMP are optimal in S^+ and they leave S^+ in finite time. For the minimum energy problem, there is no cut-locus, i.e., two optimal trajectories starting from the source at time zero, cannot intersect for positive time. On the other side, for the minimum time problem, two optimal trajectories can intersect for $t > 0$, but in this case they coincide before the intersection.

Remark 8. Notice that due to the symmetries of the problem, for both costs, if $(u_1(\cdot), u_2(\cdot)) : [0, T] \rightarrow \mathbb{R} \times \mathbb{R}$ are the optimal controls steering the state one to the state three and corresponding to a nonisotropy factor α , then $(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) : [0, \alpha T] \rightarrow \mathbb{R} \times \mathbb{R}$, defined by $(\bar{u}_1(t), \bar{u}_2(t)) := (u_2(T - t/\alpha), u_1(T - t/\alpha))$, are the optimal controls steering the state one to the state three and corresponding to a nonisotropy factor $1/\alpha$.

3.1. Minimum Time. For the minimum time problem, optimal controls are a.e. piecewise constant, and the corresponding trajectory is a concatenation of arcs of circle. There are three different cases depending on the value of α :

Case $0 < \alpha < 1$: In this case the minimizer corresponds to the control:

- $u_1(t) = u_2(t) = 1$ for $t \in [0, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}]$. The corresponding trajectory steers the point $(1, 0, 0)^T$ to the point $(0, \sqrt{1-\alpha^2}, \alpha)^T$.
- $u_1(t) = 0, u_2(t) = 1$ for $t \in [\frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}} + \frac{\arccos \alpha}{\alpha}]$. The corresponding trajectory steers the point $(0, \sqrt{1-\alpha^2}, \alpha)^T$ to the point $(0, 0, 1)^T$.

Case $\alpha = 1$: In this case the minimizer corresponds to $u_1(t) = u_2(t) = 1$ for $t \in [0, \frac{\pi}{\sqrt{2}}]$ and steers the state one to the state three.

Case $\alpha > 1$: In this case the minimizer corresponds to:

- $u_1(t) = 1, u_2(t) = 0$ for $t \in [0, \arccos(\frac{1}{\alpha})]$. The corresponding trajectory steers the point $(1, 0, 0)^T$ to the point $(\frac{1}{\alpha}, \sqrt{1 - \frac{1}{\alpha^2}}, 0)^T$.
- $u_1(t) = u_2(t) = 1$ for $t \in [\arccos(\frac{1}{\alpha}), \arccos(\frac{1}{\alpha}) + \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})]$. The corresponding trajectory steers the point $(\frac{1}{\alpha}, \sqrt{1 - \frac{1}{\alpha^2}}, 0)^T$ to the point $(0, 0, 1)^T$.

3.2. Minimum Energy. For the minimum energy problem, there exists a unique strictly positive number $m_3(0)$ such that the trajectory, corresponding to the controls given below, steers the state $(1, 0, 0)^T$ to the state $(0, 0, 1)^T$ and is parameterized by arclength. The parameter $m_3(0)$, appearing in these formulas, should be computed numerically, for instance by the dichotomy method presented in Section 5.3.4.

Case $0 < \alpha \leq 1$: $u_1(t) = cn(\alpha m_3(0) t; k), u_2(t) = sn(\alpha m_3(0) t; k)$, where we have defined the modulus of the elliptic functions cn and sn to be equal to $k = \frac{\sqrt{1-\alpha^2}}{\alpha m_3(0)}$.

Case $\alpha > 1$: $u_1(t) = cd(\frac{\sqrt{\alpha^2-1}}{k} t; k), u_2(t) = \sqrt{1-k^2} sd(\frac{\sqrt{\alpha^2-1}}{k} t; k)$, where we have defined the modulus of the elliptic functions cd and sd to be equal to $k = \sqrt{\frac{\alpha^2-1}{\alpha^2 m_3(0)^2 + \alpha^2 - 1}}$.

For the parameter $m_3(0)$ we have the following estimate: $\sqrt{\frac{1-\alpha^2}{\alpha^2}} < m_3(0) \leq \sqrt{\frac{4}{3\alpha^2} - 1}$, if $0 < \alpha \leq 1$, and $0 < m_3(0) \leq \frac{1}{\sqrt{3}}$, if $\alpha > 1$. The time needed to reach the target is given by formula (65) if $\alpha \leq 1$ and by formula (67) if $\alpha > 1$. In the particular case $\alpha = 1$, we recover the results of [11].

4. Minimum Time.

4.1. Minimum Time for Distributional Systems on 2-D Manifolds. In this section, we develop a theory of time optimal syntheses on 2-D manifolds for distributional systems with bounded controls. For this purpose we use ideas similar to those used by Sussmann, Bressan, Piccoli and the first author in [8, 15, 28, 35, 36] for the minimum time stabilization to the origin for the “control affine version” of the same problem ($\dot{x} = F(x) + uG(x)$, $u \in [-1, 1]$, where x belongs to a 2-D-manifold) and recently rewritten in [9]. This section is written to be as self-consistent as possible.

4.1.1. *Basic Definitions and PMP.* We focus on the following:

Problem (P) Consider the control system

$$\dot{x} = u_1 F_1(x) + u_2 F_2(x), \quad x \in M, \quad |u_i| \leq 1, \quad i = 1, 2, \tag{23}$$

where

H0: M is a smooth 2-D manifold. The vector fields F_1 and F_2 are C^∞ and the control system (23) is complete on M .

We are interested in the problem of reaching every point of M in minimum time from a source M_{in} that is a smooth submanifold of M .

The theory developed next is then applied to **(PD)**:

$$\left\{ \begin{array}{l} \dot{\psi} = u_1 F_1(\psi) + u_2 F_2(\psi), \quad \psi \in S^+, \quad |u_i| \leq 1, \quad i = 1, 2, \\ F_1(\psi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} -\psi_2 \\ \psi_1 \\ 0 \end{pmatrix} = -\psi_2 \frac{\partial}{\partial \psi_1} + \psi_1 \frac{\partial}{\partial \psi_2}, \\ F_2(\psi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -\psi_3 \\ \psi_2 \end{pmatrix} = \alpha \left(-\psi_3 \frac{\partial}{\partial \psi_2} + \psi_2 \frac{\partial}{\partial \psi_3} \right). \end{array} \right. \tag{24}$$

Definition 2. A control for the system (23) is a measurable function $u(\cdot) = (u_1(\cdot), u_2(\cdot)) : [a_1, a_2] \rightarrow [-1, 1]^2$. The corresponding trajectory is a Lipschitz continuous map $x(\cdot) : [a_1, a_2] \rightarrow M$ such that $\dot{x}(t) = u_1(t)F_1(x(t)) + u_2(t)F_2(x(t))$ for almost every $t \in [a_1, a_2]$. Since the system is autonomous we can always assume that $[a_1, a_2] = [0, T]$.

For us, a solution to the problem **(P)** is an optimal synthesis that is a collection $\{(x_{\bar{x}}(\cdot), u_{\bar{x}}(\cdot))$ defined on $[0, T_{\bar{x}}], \bar{x} \in M\}$ of trajectory–control pairs such that $x_{\bar{x}}(0) \in M_{in}$, $x_{\bar{x}}(T_{\bar{x}}) = \bar{x}$, and $x_{\bar{x}}(\cdot)$ is time optimal.

In the following we use the notation $u = (u_1, u_2)$ and (in a local chart) $x = (x_1, x_2)$, $F_1 = ((F_1)_1, (F_1)_2)$, $F_2 = ((F_2)_1, (F_2)_2)$. Let us introduce a definition to describe different types of controls.

Definition 3. Let $u(\cdot) = (u_1(\cdot), u_2(\cdot)) : [a_1, a_2] \subset [0, T] \rightarrow [-1, 1]^2$ be a control for the control system (23).

- $u(\cdot)$ is said to be a bang control if for almost every $t \in [a_1, a_2]$, $u(t) = \bar{u} \in \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$. Similarly $u(\cdot)$ is said to be a u_i -bang control if for almost every $t \in [a_1, a_2]$, $u_i(t) = \underline{u} \in \{\pm 1\}$.
- A switching time of $u(\cdot)$ is a time $t \in [a_1, a_2]$ such that for every $\varepsilon > 0$, $u(\cdot)$ is not bang on $(t - \varepsilon, t + \varepsilon) \cap [a_1, a_2]$. Similarly a u_i -switching time of $u(\cdot)$ is a time $t \in [a_1, a_2]$ such that for every $\varepsilon > 0$, $u(\cdot)$ is not u_i -bang on $(t - \varepsilon, t + \varepsilon) \cap [a_1, a_2]$.
- If $u_A : [a_1, a_2] \rightarrow [-1, 1]^2$ and $u_B : [a_2, a_3] \rightarrow [-1, 1]^2$ are controls, their concatenation $u_B * u_A$ is the control

$$(u_B * u_A)(t) := \begin{cases} u_A(t) & \text{for } t \in [a_1, a_2], \\ u_B(t) & \text{for } t \in [a_2, a_3]. \end{cases}$$

The control $u(\cdot)$ is called bang-bang if it is a finite concatenation of bang arcs. Similarly one defines u_i -bang-bang controls.

- A trajectory of (23) is a bang trajectory, (resp. bang-bang trajectory), if it corresponds to a bang control, (resp. bang-bang control). Similarly one defines u_i -bang and u_i -bang-bang trajectories.

A key role is played by the following three functions defined on M

$$\begin{aligned} \Delta_A(x) &:= \text{Det}(F_1(x), F_2(x)) = (F_1)_1(F_2)_2 - (F_2)_1(F_1)_2, & (25) \\ \Delta_{B_1}(x) &:= \text{Det}(F_1(x), [F_1, F_2](x)) = (F_1)_1([F_1, F_2])_2 - ([F_1, F_2])_1(F_1)_2, & (26) \\ \Delta_{B_2}(x) &:= \text{Det}(F_2(x), [F_1, F_2](x)) = (F_2)_1([F_1, F_2])_2 - ([F_1, F_2])_1(F_2)_2. & (27) \end{aligned}$$

Notice that these definitions depend on the choice of the coordinate system, but not the sets $\Delta_A^{-1}(0)$, $\Delta_{B_1}^{-1}(0)$, $\Delta_{B_2}^{-1}(0)$ of their zeros, that are respectively the set of points where F_1 and F_2 are parallel, the set of points where F_1 is parallel to $[F_1, F_2]$ and the set of points where F_2 is parallel to $[F_1, F_2]$. Using PMP it turns out (see Section 4.1.2) that these loci are fundamental in the construction of the optimal synthesis. In fact, assuming that they are embedded one-dimensional submanifolds of M , we have the following:

- in each connected region of $M \setminus (\Delta_A^{-1}(0) \cup \Delta_{B_1}^{-1}(0) \cup \Delta_{B_2}^{-1}(0))$, every extremal trajectory is bang-bang with at most two switchings (one of the control u_1 and one of the control u_2). Moreover the possible switchings are determined. More precisely for every $x \in M \setminus (\Delta_A^{-1}(0) \cup \Delta_{B_1}^{-1}(0) \cup \Delta_{B_2}^{-1}(0))$ define

$$f_i(x) := -\frac{\Delta_{B_i}(x)}{\Delta_A(x)}. \tag{28}$$

If $f_i > 0$ (resp. $f_i < 0$), we have that u_i can only switch from -1 to $+1$ (resp. from $+1$ to -1);

- the support of u_i -singular trajectories (that are trajectories for which the u_i -switching function identically vanishes, and for which u_i can assume values different from ± 1 , see Definition 4 below) is always contained in the set $\Delta_{B_i}^{-1}(0)$.

For the problem **(P)**, the PMP says the following:

Corollary (Pontryagin Maximum Principle for the problem (P)) *Consider the control system (23) subject to **H0**. For every $(x, \lambda, u) \in T^*M \times [-1, 1]^2$, define*

$$\mathcal{H}(x, \lambda, u) := u_1 < \lambda, F_1(x) > + u_2 < \lambda, F_2(x) > + \lambda_0,$$

If the couple $(x(\cdot), u(\cdot)) : [0, T] \rightarrow M \times [-1, 1] \times [-1, 1]$ is time optimal then there exist a never vanishing Lipschitz continuous covector $\lambda(\cdot) : t \in [0, T] \mapsto \lambda(t) \in T_{x(t)}^*M$ and a constant $\lambda_0 \leq 0$ such that for a.e. $t \in [0, T]$:

- i: $\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \lambda}(x(t), \lambda(t), u(t))$,
- ii: $\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), \lambda(t), u(t)) = -\langle \lambda(t), (u_1(t)\nabla F_1 + u_2(t)\nabla F_2)(x(t)) \rangle$,
- iii: $\mathcal{H}(x(t), \lambda(t), u(t)) = \mathcal{H}_M(x(t), \lambda(t))$, where $\mathcal{H}_M(x, \lambda) := \max\{\mathcal{H}(x, \lambda, u) : u \in [-1, 1]^2\}$,
- iiii: $\mathcal{H}_M(x(t), \lambda(t)) = 0$,
- v: $\langle \lambda(0), T_{x(0)}M_{in} \rangle = 0$ (transversality condition).

Remark 9. In this version of PMP, $\lambda(\cdot)$ is always different from zero otherwise the conditions **iii**, **iiii** would imply $\lambda_0 = 0$ (cf. PMP in Section 2). An extremal is said to be nontrivial if it does not correspond to controls a.e. vanishing. Notice that a trivial extremal is an abnormal extremal. In the following we often refer to nontrivial abnormal extremal (NTAE, for short).

4.1.2. *Switching Functions, Singular Trajectories and Predicting Switchings.* In this section we are interested in determining when the controls switch from +1 to -1 or viceversa and when they may assume values in $] - 1, +1[$. Moreover we would like to predict which kind of switchings can happen, using properties of the vector fields F_1 and F_2 . A key role is played by the following:

Definition 4. (Switching Functions) Let $(x(\cdot), \lambda(\cdot))$ be an extremal pair. The corresponding switching functions are defined as $\phi_i(t) := \langle \lambda(t), F_i(x(t)) \rangle$, $i = 1, 2$.

Remark 10. Notice that $\phi_i(\cdot)$ are at least Lipschitz continuous. Moreover using the switching functions, conditions **iii** and **iiii** imply:

$$\mathcal{H}(x(t), \lambda(t), u(t)) = u_1(t)\phi_1(t) + u_2(t)\phi_2(t) + \lambda_0 = 0 \text{ a.e..} \tag{29}$$

The following lemma characterizes abnormal extremals.

Lemma 1. Let $(x(\cdot), \lambda(\cdot))$ (defined on $[a_1, a_2]$) be an extremal pair. We have:

1. $(x(\cdot), \lambda(\cdot))$ is an abnormal extremal if and only if $\phi_1(\cdot) \equiv \phi_2(\cdot) \equiv 0$ on $[a_1, a_2]$;
2. if $(x(\cdot), \lambda(\cdot))$ is an abnormal extremal, then $\text{Supp}(x(\cdot)) \subset \Delta_A^{-1}(0)$;
3. if $(x(\cdot), \lambda(\cdot))$ is not an abnormal extremal, then $\phi_1(\cdot)$ and $\phi_2(\cdot)$ never vanish at the same time;
4. let $\bar{t} \in [a_1, a_2]$ be a time such that $\phi_1(\bar{t}) = 0$, $x(\bar{t}) \in \Delta_A^{-1}(0)$, $F_1(x(\bar{t})) \neq 0$. Then $(x(\cdot), \lambda(\cdot))$ is an abnormal extremal. The same holds if $\phi_2(\bar{t}) = 0$, $x(\bar{t}) \in \Delta_A^{-1}(0)$, $F_2(x(\bar{t})) \neq 0$.

Proof. Let us prove **1**. The sufficiency is obvious. Let us prove the necessity. Equation (29), with $\lambda_0 = 0$ and **iii** of PMP, imply that both controls vanish (trivial abnormal extremals) or that $\phi_1(\cdot) \equiv \phi_2(\cdot) \equiv 0$ on $[a_1, a_2]$. Let us prove **2**. From $\phi_1(\cdot) \equiv \phi_2(\cdot) \equiv 0$ we have that $\lambda(\cdot)$ is orthogonal to $F_1(x(\cdot))$ and $F_2(x(\cdot))$. Since $\lambda(\cdot)$ is never vanishing, it follows that $F_1(x(\cdot))$ is parallel to $F_2(x(\cdot))$. To prove **3**, assume by contradiction that $(x(\cdot), \lambda(\cdot))$ is not an abnormal extremal and that there exists a time \bar{t} for which $\phi_1(\bar{t}) = \phi_2(\bar{t}) = 0$. From (29) there exists a sequence $t_m \nearrow \bar{t}$ such that $u_1(t_m)$ and $u_2(t_m)$ are defined and $u_1(t_m)\phi_1(t_m) + u_2(t_m)\phi_2(t_m) + \lambda_0 = 0$. Since $\phi_i(t_m) \rightarrow 0$, $i = 1, 2$, it follows $\lambda_0 = 0$. Contradiction. To prove **4.**, observe that $\phi_1(\bar{t}) = 0$ implies that $\lambda(\bar{t})$ annihilates $F_1(x(\bar{t}))$. From $x(\bar{t}) \in \Delta_A^{-1}(0)$ we have that $F_2(x(\bar{t}))$ is parallel to $F_1(x(\bar{t}))$, hence $\lambda(\bar{t})$ annihilates also $F_2(x(\bar{t}))$. It follows

$\phi_2(\bar{t}) = 0$. From (29) we have $\lambda_0 = 0$. The same proof is valid exchanging the two indexes 1 and 2. \square

The following lemma restricts the set where the support of abnormal extremals can live.

Lemma 2. *Let $(x(\cdot), \lambda(\cdot))$ be an abnormal extremal. Then $\text{Supp}(x(\cdot)) \subset \Delta_A^{-1}(0) \cap \Delta_{B_1}^{-1}(0) \cap \Delta_{B_2}^{-1}(0)$.*

Proof. Let $x(\cdot)$ defined on $[a_1, a_2]$ be an abnormal extremal. Then condition **2.** of Lemma 1 implies that F_1 and F_2 are parallel along $\text{Supp}(x(\cdot))$. It follows that the distribution restricted to $\text{Supp}(x(\cdot))$ is integrable in the Frobenius sense. Hence also $[F_1, F_2]$ is parallel to F_1 and F_2 along $\text{Supp}(x(\cdot))$. By definition of Δ_{B_i} it follows that $\Delta_A, \Delta_{B_1}, \Delta_{B_2}$ are zero along $x(\cdot)$. \square

We recall that for the problem (24), there are no abnormal extremals (cf. Proposition 3). Anyway, as a consequence of Lemma 2, the presence of NTAE is highly nongeneric also for the general system (23) subject to **H0**, since generically, points of $\Delta_A^{-1}(0) \cap \Delta_{B_1}^{-1}(0) \cap \Delta_{B_2}^{-1}(0)$ are isolated.

The switching functions determine when the controls switch from +1 to -1 and viceversa. In fact, from the maximization condition **iii**, one immediately gets:

Lemma 3. *Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, T]$ be an extremal pair and $\phi_i(\cdot)$ the corresponding switching functions. If $\phi_i(t) \neq 0$ for some $t \in]0, T[$, then there exists $\varepsilon > 0$ such that $x(\cdot)$ corresponds to a constant control $u_i = \text{sgn}(\phi_i)$ on $]t - \varepsilon, t + \varepsilon[$. Moreover if $\phi_i(\cdot)$ has a zero at t , and if $\dot{\phi}_i(t)$ exists and is strictly larger than zero (resp. strictly smaller than zero) then there exists $\varepsilon > 0$ such that $x(\cdot)$ corresponds to constant control $u_i = -1$ on $]t - \varepsilon, t[$ and to constant control $u_i = +1$ on $]t, t + \varepsilon[$ (resp. to constant control $u_i = +1$ on $]t - \varepsilon, t[$ and to constant control $u_i = -1$ on $]t, t + \varepsilon[$).*

Notice that on every interval where $\phi_i(\cdot)$ has no zero (resp. finitely many zeroes) the corresponding control is u_i -bang (resp. u_i -bang-bang). We are then interested in differentiating ϕ_i . One immediately gets:

Lemma 4. *Let $(x(\cdot), \lambda(\cdot))$, defined on $[0, T]$ be an extremal pair and $\phi_i(\cdot)$ the corresponding switching functions. Then it holds a.e. $\dot{\phi}_1(t) = u_2(t) < \lambda(t), [F_2, F_1](x(t)) >$, $\dot{\phi}_2(t) = u_1(t) < \lambda(t), [F_1, F_2](x(t)) >$.*

From Lemma 3 it follows that u_i can assume values different from ± 1 on some interval $[a_1, a_2]$ only if the corresponding switching function vanishes identically there.

Remark 11. Lemma 1 asserts that, if there are no NTAE (as for the system (24)), then u_1 and u_2 never switch at the same time. In this case, from Lemma 4 it follows that in a neighborhood of a u_1 -switching, $\phi_1(\cdot)$ is a \mathcal{C}^1 function. A similar statement holds for $\phi_2(\cdot)$.

Definition 5. A nontrivial extremal trajectory $x(\cdot)$ defined on $[a_1, a_2]$ is said to be u_i -singular if the corresponding switching function $\phi_i(\cdot)$ vanishes identically on $[a_1, a_2]$.

Remark 12. From Lemma 1 it follows that $(x(\cdot), \lambda(\cdot))$ is a NTAE if and only if it is a u_1 - u_2 -singular trajectory.

Next we show that an extremal trajectory, between two u_i -switchings, intersects the set $\Delta_A^{-1}(0) \cup \Delta_{B_1}^{-1}(0) \cup \Delta_{B_2}^{-1}(0)$. Moreover we prove that u_i -singular trajectories must run on the set $\Delta_{B_i}^{-1}(0)$. Let us define two crucial functions.

Definition 6. On the set of points $x \in M \setminus \Delta_A^{-1}(0)$, the vector fields $F_1(x), F_2(x)$ form a basis of T_xM and we define the scalar functions $f_1(x), f_2(x)$ to be the coefficients of the linear combination: $[F_1, F_2](x) = f_2(x)F_1(x) - f_1(x)F_2(x)$.

The following Lemma gives a relation between f_1, f_2 , and the functions $\Delta_A, \Delta_{B_1}, \Delta_{B_2}$.

Lemma 5. *Let $x \in M \setminus \Delta_A^{-1}(0)$ then $f_i(x) = -\frac{\Delta_{B_i}(x)}{\Delta_A(x)}, i = 1, 2$.*

Proof. We have $\Delta_{B_1}(x) = Det(F_1(x), [F_1, F_2](x)) = Det(F_1(x), f_2(x)F_1(x) - f_1(x)F_2(x)) = -f_1(x)Det(F_1(x), F_2(x)) = -f_1(x)\Delta_A(x)$, and similarly for Δ_{B_2} . □

The functions f_1 and f_2 are crucial in studying which kind of switchings can happen near u_i -ordinary points defined next:

Definition 7. A point $x \in M$ is called a u_i -ordinary point if $x \notin \Delta_A^{-1}(0) \cup \Delta_{B_i}^{-1}(0)$.

On the set of u_i -ordinary points the structure of optimal trajectories is particularly simple:

Proposition 4. *Let $\Omega \subset M$ be an open connected set made of u_i -ordinary points. Then all extremal trajectories $x(\cdot) : [a_1, a_2] \rightarrow \Omega$, are u_i -bang-bang with at most one u_i -switching. Moreover if $f_i > 0$ (resp. $f_i < 0$) in Ω then $x(\cdot)$ corresponds to control u_i that is:*

- a.e. constantly equal to $+1$, or
- a.e. constantly equal to -1 or
- has a $-1 \rightarrow +1$ switching (resp. has a $+1 \rightarrow -1$ switching).

Proof. Let $x(\cdot) :]a_1, a_2[\rightarrow \Omega$ be an extremal trajectory and $\phi_1(\cdot)$ be the corresponding u_1 -switching function. If $\phi_1(\cdot)$ has no zero, then $x(\cdot)$ is a u_1 -bang and the conclusion follows. Let t_1 be a zero of $\phi_1(\cdot)$. The time t_1 cannot be a zero of $\phi_2(\cdot)$ otherwise $x(t_1)$ could not be a u_i -ordinary point (by Lemma 1 we would have $\Delta_A(x(t_1)) = 0$). From Lemma 4 it follows that $\phi_1(\cdot)$ is C^1 in a neighborhood of t_1 . Moreover t_1 cannot be a zero of $\dot{\phi}_1(\cdot)$ otherwise $x(t_1)$ could not be a u_1 -ordinary point (we would have $\Delta_{B_1}(x(t_1)) = 0$). Since in a neighborhood of t_1, u_2 is a.e. constantly equal to $+1$ or -1 , we can assume u_2 constant in this neighborhood, and we have

$$\begin{aligned} \dot{\phi}_1(t_1) &= u_2(t_1) \langle \lambda(t_1), [F_2, F_1](x(t_1)) \rangle = u_2(t_1) \langle \lambda(t_1), (-f_2F_1 + f_1F_2)(x(t_1)) \rangle \\ &= u_2(t_1)f_1(x(t_1)) \langle \lambda(t_1), F_2(x(t_1)) \rangle = u_2(t_1)f_1(x(t_1)) \phi_2(t_1). \end{aligned}$$

Now from Lemma 3 we have that $sgn(u_2(t_1)) = sgn(\phi_2(t_1))$, and since $\phi_2(t_1) \neq 0$ it follows $sgn(\dot{\phi}_1(t_1)) = sgn(f_1(x(t_1)))$. Using again Lemma 3, it follows that if $f_1 > 0$ (resp. $f_1 < 0$) then we can have only a $-1 \rightarrow +1$ switching (resp. a $+1 \rightarrow -1$ switching). A similar proof can be done for a zero of ϕ_2 . □

We are now interested in properties of u_i -singular trajectories.

Lemma 6. *Let $(x(\cdot), \lambda(\cdot))$, defined on $[0, T]$, be a u_i -singular trajectory on $[a_1, a_2] \subset [0, T]$, then $Supp(x(\cdot)|_{[a_1, a_2]}) \subset \Delta_{B_i}^{-1}(0)$.*

Proof. If $(x(\cdot), \lambda(\cdot))$ is a NTAE, the conclusion follows from Lemma 2. Assume now that $(x(\cdot), \lambda(\cdot))$ is not a NTAE. To simplify the notation assume $i = 1$, the case $i = 2$ being similar. From $\phi_1(\cdot) = \langle \lambda(\cdot), F_1(x(\cdot)) \rangle \equiv 0$ on $[a_1, a_2]$ we have that $\dot{\phi}_1(\cdot) = u_2(\cdot) \langle \lambda(\cdot), [F_2, F_1](x(\cdot)) \rangle \equiv 0$ a.e. in $[a_1, a_2]$. Since $(x(\cdot), \lambda(\cdot))$ is not a NTAE, it follows that u_2 is a.e. equal to $+1$ or -1 in $[a_1, a_2]$. Therefore $\lambda(\cdot)$ is orthogonal both to $F_1(x(\cdot))$ and $[F_1, F_2](x(\cdot))$. Being $\lambda(\cdot)$ nontrivial then $F_1(x(\cdot))$ is parallel to $[F_1, F_2](x(\cdot))$. The conclusion follows. \square

Lemma 7. *Let $(x(\cdot), \lambda(\cdot))$ be a u_1 -singular trajectory (resp. a u_2 -singular trajectory) on $[a_1, a_2]$.*

Assume $\langle \nabla \Delta_{B_1}(x(t)), F_1(x(t)) \rangle \neq 0$ (resp. $\langle \nabla \Delta_{B_2}(x(t)), F_2(x(t)) \rangle \neq 0$) and define respectively

$$\varphi_1(t) := -u_2(t) \frac{\langle \nabla \Delta_{B_1}(x(t)), F_2(x(t)) \rangle}{\langle \nabla \Delta_{B_1}(x(t)), F_1(x(t)) \rangle},$$

$$\varphi_2(t) := -u_1(t) \frac{\langle \nabla \Delta_{B_2}(x(t)), F_1(x(t)) \rangle}{\langle \nabla \Delta_{B_2}(x(t)), F_2(x(t)) \rangle}.$$

Then $\varphi_1(\cdot)$ (resp. $\varphi_2(\cdot)$) satisfies $|\varphi_1(t)| \leq 1$ (resp. $|\varphi_2(t)| \leq 1$) a.e. in $[a_1, a_2]$. Moreover $x(\cdot)$ corresponds to the control $(\varphi_1(\cdot), u_2(\cdot))$ (resp. $(u_1(\cdot), \varphi_2(\cdot))$) in $[a_1, a_2]$.

Proof. Assume $i = 1$, the case $i = 2$ being similar. Let $u_1(\cdot), u_2(\cdot)$ be the controls corresponding to $x(\cdot)$, that is $\dot{x}(t) = u_1(t)F_1(x(t)) + u_2(t)F_2(x(t))$, for almost every t . From $\Delta_{B_1}(x(t)) = 0$, we have for a.e. t : $0 = \frac{d}{dt} \Delta_{B_1}(x(t)) = \nabla \Delta_{B_1}(x(t)) \cdot (u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)))$. This means that $u_1(t) = \varphi_1(t)$. The condition $|\varphi_1(t)| \leq 1$ is simply a consequence of the fact that, by assumption, $x(\cdot)$ is an admissible trajectory. \square

Remark 13. Notice that the three functions $\Delta_A, \Delta_{B_1}, \Delta_{B_2}$ are not invariant by change of coordinates and/or orientation of M . Anyway, the sets of their zeros, the functions f_i (see Definition 6) and the functions φ_i (see Lemma 7) do not depend on the coordinate system.

4.1.3. More on Singular Trajectories. From Lemma 6 it follows that the condition $\Delta_{B_i}^{-1}(0) \neq \emptyset$ is necessary for the presence of a u_i -singular trajectory. Anyway in general it is not sufficient. It may very well happen that $\Delta_{B_i}^{-1}(0) \neq \emptyset$ and no singular trajectory is running on it. Another common situation is that a singular trajectory is running just a subset of $\Delta_{B_i}^{-1}(0)$.

The description of singular trajectories is very intricate in general (see for instance [9] for the control affine case). Here we describe what happens in the simplest case (that is enough for what follows), i.e., when we restrict our attention to a subset S_1 of $\Delta_{B_1}^{-1}(0)$ satisfying the following conditions:

- C1:** S_1 is a smooth one-dimensional connected embedded submanifold of M ,
- C2:** S_1 does not intersect $\Delta_A^{-1}(0)$ and $\Delta_{B_2}^{-1}(0)$,
- C3:** S_1 is not tangent to the vectors $F_1 + F_2$ and $-F_1 + F_2$,
- C4:** f_1 changes sign on S_1 .

See Figure 1 A. Similarly, we restrict our attention to a subset S_2 of $\Delta_{B_2}^{-1}(0)$ satisfying the same conditions **C1–C4**, but with the indexes 1 and 2 exchanged.

Let us treat the case of S_1 , being the case of S_2 similar. First observe that, if a u_1 -singular trajectory defined in $[a_1, a_2]$ is running on S_1 , then u_2 cannot switch on $[a_1, a_2]$, otherwise, by Lemma 1, $x(\cdot)$ would be a NTAE (and this is not the case

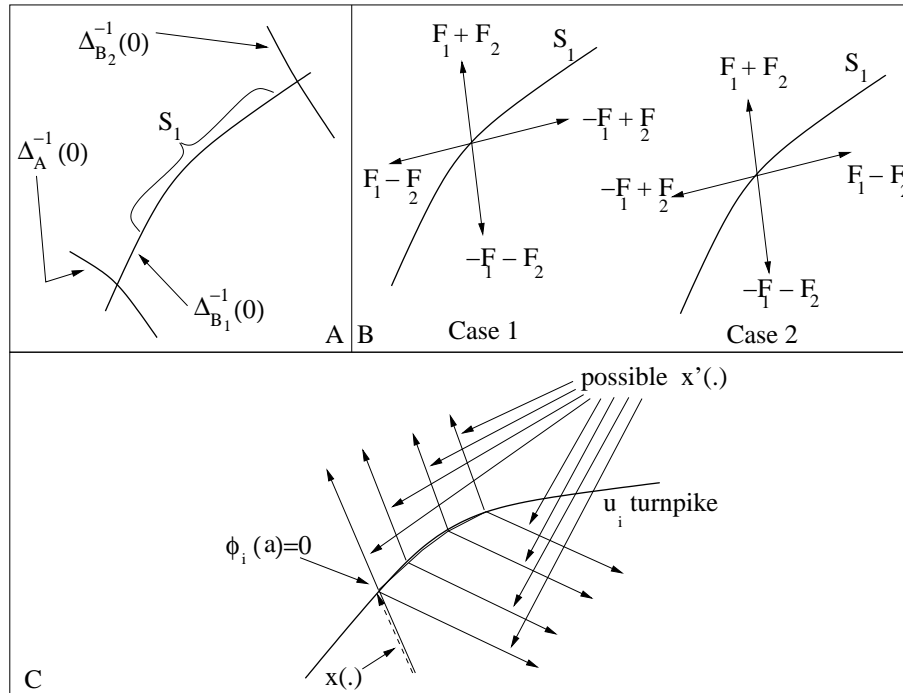


FIGURE 1. Hypotheses on $\Delta_{B_1}^{-1}(0)$ (Figure A). Possible positions of the vector fields F_1 and F_2 on $\Delta_{B_1}^{-1}(0)$ (Figure B). Shape of the synthesis in a neighborhood of a point where a trajectory is becoming u_i -singular (Figure C)

thanks to **C2**). Assume, for instance, $u_2 = 1$. Due to **C3**, $F_1 + F_2$ and $-F_1 + F_2$ point either on the opposite, or on the same sides of S_1 (see Figure 1 B, cases 1 and 2). Since the admissible velocities belong to the segment joining $F_1 + F_2$ and $-F_1 + F_2$ it follows that, only in case 1 of Figure 1 B, it is possible to run on S_1 . Moreover, using a similar argument to that of Lemma 12 p. 47 of [9], one can prove that if $f_1 < 0$ on the side where $F_1 + F_2$ points, then a trajectory running on S_1 cannot be optimal. These facts are collected in the following:

Lemma 8. *Let S_1 (resp. S_2) be a subset of M satisfying conditions **C1**, **C2**, **C3**, **C4** above. Then on S_1 (resp. on S_2) cannot run an optimal u_1 -singular trajectory (resp. an optimal u_2 -singular trajectory) if (at least) one of the two conditions is satisfied.*

OO1: $F_1 + F_2$ and $-F_1 + F_2$ (resp. $F_1 + F_2$ and $F_1 - F_2$) point on the same side of S_1 (resp. S_2);

OO2: $F_1 + F_2$ and $-F_1 + F_2$ point on opposite sides of S_1 (resp. $F_1 + F_2$ and $F_1 - F_2$ point on opposite sides of S_2), and in a neighborhood of S_1 (resp. S_2) we have the following. On the side where points $F_1 + F_2$, we have $f_1 < 0$ (resp. $f_2 < 0$).

If on S_1 (resp. on S_2) **OO1** and **OO2** are not satisfied (that means that S_1 is a candidate support for an optimal trajectory) then, following [9], we call S_1 a u_1 -turnpike (resp. a u_2 -turnpike).

An extremal trajectory $x(\cdot)$, corresponding to control u_1 constantly equal to $+1$ or -1 and reaching a u_1 -turnpike at a point $\bar{x} = x(\bar{t})$, can enter it only if the corresponding u_1 -switching function is vanishing at \bar{t} , otherwise, after \bar{t} , it will correspond to the same control. Moreover, using an argument similar to that of Lemma 11 p. 46 of [9] one can prove that if an extremal trajectory enters a turnpike, then it can exit it with control $+1$ or -1 at every successive time. These facts are stated in the following Lemma, and illustrated in Figure 1 C.

Lemma 9. *Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, a]$, $a > 0$, be a bang extremal pair that verifies $x(a) = \bar{x}$, $\bar{x} \in S_i$ where S_i is a u_i -turnpike, and assume that $\phi_i(a) = \langle \lambda(a), F_i(x(a)) \rangle = 0$. Moreover let b, c be two real numbers, sufficiently close to a , such that $a \leq b < c$, and let $x'(\cdot) : [0, c] \rightarrow M$ be a trajectory, corresponding to controls $u'_1(\cdot)$ and $u'_2(\cdot)$ such that:*

- $x'(\cdot)|_{[0, a]} = x(\cdot)$,
- $\text{Supp}(x'(\cdot)|_{[a, b]}) \subset S_i$, and $u'_j(\cdot)$, $j \neq i$, is bang on $[0, c]$,
- $x'(\cdot)|_{[b, c]}$ is bang.

Then $x'(\cdot)$ is extremal. Moreover, if $\phi'_i(\cdot)$ is the u_i -switching function corresponding to $x'(\cdot)$, then $\phi'_i|_{[a, b]} \equiv 0$.

Remark 14. For the problem **(P)**, under generic conditions on the vector fields F_1 and F_2 , one can classify synthesis singularities, stable syntheses, singularities of the minimum time wave fronts, with the same techniques used in [9].

4.2. Time Optimal Synthesis for the 3-Level Quantum System. In this section, we apply the theory developed in Section 4.1 to the system (24). Here our source is the point $(1, 0, 0)^T$ and the task is to reach the target $(0, 0, 1)^T$ in minimum time. As explained in Remark 4, the first step is to compute all extremal trajectories in S^+ . Then we have to select, among all extremal trajectories reaching the target, the one having smallest cost. Since it turns out that all these trajectories are optimal (because all extremals passing through a point of $S^+ \setminus \{(1, 0, 0)^T\}$ coincide before this point) it follows global optimality, and we get the complete time optimal synthesis in S^+ .

When one is dealing with a minimum time problem for a linear system $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^2$, $u \in [-1, 1]$, the equation for $\lambda(\cdot)$, given by PMP, is decoupled from the equation for $x(\cdot)$ and does not contain the control, hence for every value of $\lambda(0)$ one can compute $\lambda(t)$ and the corresponding switching function $\langle \lambda(t), B \rangle$ from which one determines the value of the control. For our problem (24), as consequence of bilinearity, the equation for $\lambda(\cdot)$ does not depend on $x(\cdot)$, but still contains the control. For this reason we have to use the theory developed before, that uses the functions Δ_A , Δ_{B_i} , f_i .

One of the most important points is to prove that there is no chattering (i.e., an infinite concatenation of bang and/or singular arcs in finite time). In synthesis theory this step is called the finite dimensional reduction because it permits to restrict the set of candidate optimal trajectories to a family of trajectories parameterized by a finite number of parameters (the switching times). The absence of chattering is proved in Proposition 8. Thanks to that, we will be able to prove that every optimal control is piecewise constant and this permits to solve explicitly the equation for λ (see Section 4.2.2).

4.2.1. *Preliminary Computations and Facts.* From (24) we have in the standard coordinates of \mathbb{R}^3

$$[F_1, F_2](\psi) = \alpha \left(-\psi_3 \frac{\partial}{\partial \psi_1} + \psi_1 \frac{\partial}{\partial \psi_3} \right), \quad \Delta_A = \alpha \psi_2, \quad \Delta_{B_1} = \alpha \psi_1, \quad \Delta_{B_2} = -\alpha^2 \psi_3.$$

Hence

$$\begin{aligned} \Delta_A^{-1}(0) &= \{\psi \in S^+ \mid \psi_2 = 0\}, \\ \Delta_{B_1}^{-1}(0) &= \{\psi \in S^+ \mid \psi_1 = 0\}, \\ \Delta_{B_2}^{-1}(0) &= \{\psi \in S^+ \mid \psi_3 = 0\}. \end{aligned}$$

They form the boundary of S^+ . Using Lemma 6, the two sets $\Delta_{B_1}^{-1}(0)$, $\Delta_{B_2}^{-1}(0)$ permit to locate singular trajectories (if they exist). Now we want to compute the values of the singular controls, i.e., the functions φ_1 and φ_2 defined in Lemma 7. They are identically equal to zero where they are defined. Indeed $\Delta_{B_1} = \alpha \psi_1$, hence F_2 is orthogonal to $\nabla \Delta_{B_1}$ on $\Delta_{B_1}^{-1}(0)$. Notice that φ_1 is not defined when F_1 is also orthogonal to the gradient of Δ_{B_1} , (hence when F_1 and F_2 are parallel, that is along $\Delta_A^{-1}(0)$), anyway at these points φ_1 can be defined by continuity. A similar computation works for φ_2 . These facts are collected in the following:

Proposition 5. *If an extremal trajectory $x(\cdot)$ is u_1 -singular (resp. u_2 -singular) in $[a_1, a_2] \subset [0, T]$, then, in $[a_1, a_2]$, it corresponds to control u_1 (resp. u_2) equal to zero and its support belongs to $\Delta_{B_1}^{-1}(0)$ (resp. $\Delta_{B_2}^{-1}(0)$), i.e., the circle of equation $\psi_1 = 0$ (resp. $\psi_3 = 0$).*

Now let us study which switchings are admitted. In S^+ we have

$$\begin{aligned} f_1(\psi) &= -\frac{\psi_1}{\psi_2} < 0, \quad \text{where } \psi_2 \neq 0, \psi_1 \neq 0, \\ f_2(\psi) &= \alpha \frac{\psi_3}{\psi_2} > 0, \quad \text{where } \psi_2 \neq 0, \psi_3 \neq 0. \end{aligned}$$

Remark 15. Using the sign of these two functions one immediately checks that in S^+ the set $\Delta_{B_1}^{-1}(0) \setminus \{(0, 1, 0)^T, (0, 0, 1)^T\}$ is a u_1 -turnpike and the set $\Delta_{B_2}^{-1}(0) \setminus \{(1, 0, 0)^T, (0, 1, 0)^T\}$ is a u_2 -turnpike.

Hence using Proposition 4 we get:

Proposition 6. *In $S^+ \setminus (\Delta_A^{-1}(0) \cup \Delta_{B_1}^{-1}(0))$ (resp. $S^+ \setminus (\Delta_A^{-1}(0) \cup \Delta_{B_2}^{-1}(0))$) every extremal trajectory has at most one u_1 -switching (resp. one u_2 -switching) according to the following rules: $u_1 : +1 \longrightarrow -1$, (resp. $u_2 : -1 \longrightarrow +1$).*

Since there are no abnormal extremals, using 4. of Lemma 1, one gets the next proposition, that is useful in the following to prove that there is no chattering.

Proposition 7. *Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, T]$ be an extremal pair. Let $\bar{t} \in [0, T]$ be such that $x(\bar{t}) \in \Delta_{B_1}^{-1}(0) \setminus \{(0, 0, 1)^T\}$ (resp. $x(\bar{t}) \in \Delta_{B_2}^{-1}(0) \setminus \{(1, 0, 0)^T\}$). Then $\phi_1(x(\bar{t})) \neq 0$ (resp. $\phi_2(x(\bar{t})) \neq 0$).*

The next proposition shows that every extremal is a finite concatenation of bang and singular arcs.

Definition 8. Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, T]$ be an extremal pair. A time $\bar{t} \in [0, T]$ is said to be a chattering time if there exists a nontrivial sequence t_n , tending to \bar{t} , with u_1 or u_2 being not bang or singular on any neighborhood of t_n .

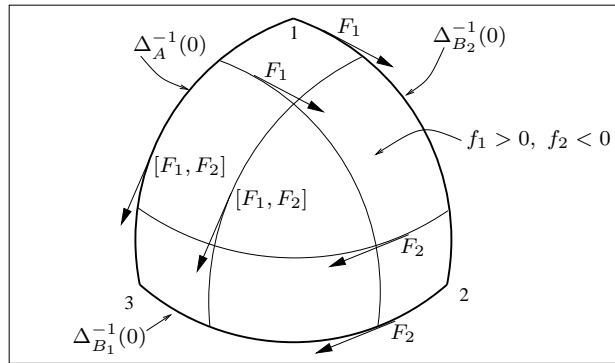


FIGURE 2. The vector fields $F_1, F_2, [F_1, F_2]$, the sets $\Delta_A^{-1}(0), \Delta_{B_1}^{-1}(0), \Delta_{B_2}^{-1}(0)$, and the functions f_1 and f_2 on S^+ .

Proposition 8. *Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, T]$ be an extremal pair. Then no $\bar{t} \in [0, T]$ is a chattering time.*

Proof. Let us prove the proposition for u_2 , being the proof for u_1 equivalent. Assume by contradiction that there exists a time $\bar{t} \in [0, T]$ of u_2 -chattering and that the chattering is on the left of \bar{t} , being the opposite case equivalent. By Proposition 6, $x(\bar{t})$ is not in $S^+ \setminus (\Delta_A^{-1}(0) \cup \Delta_{B_2}^{-1}(0))$. Moreover since $\phi_2(\bar{t}) = 0$, by Proposition 7 we have that $x(\bar{t})$ does not belong to $\Delta_A^{-1}(0) \setminus \{(1, 0, 0)^T\}$. Hence $x(\bar{t}) \in \Delta_{B_2}^{-1}(0)$.

We claim that there are two times τ_1 and τ_2 such that $0 < \tau_1 < \tau_2 < \bar{t}$, $x(\tau_1) \in \Delta_{B_2}^{-1}(0)$ and $x(\tau_2) \notin \Delta_{B_2}^{-1}(0)$. Indeed, if it is not the case, then on some interval $]\bar{t} - \varepsilon, \bar{t}[$ ($\varepsilon > 0$), the support of $x(\cdot)$ is included in $\Delta_{B_2}^{-1}(0)$ or in $S^+ \setminus \Delta_{B_2}^{-1}(0)$ and in both cases there is no chattering. As a consequence of the form of the dynamics (see Figure 3), in S^+ , an extremal can leave and come back to $\Delta_{B_2}^{-1}(0)$ only having a $+1 \rightarrow -1$ switching of u_2 in $S^+ \setminus \Delta_{B_2}^{-1}(0)$ which is forbidden by Proposition 6. \square

Remark 16. Thanks to Propositions 5 and 8, every extremal trajectory corresponds to piecewise constant controls, i.e., it is a finite concatenation of arcs of circle.

Remark 17. Once the control u_1 (resp. u_2) takes value -1 (resp. 1) on a non-trivial interval of time, it will not switch anymore in S^+ . Indeed, outside $\Delta_{B_1}^{-1}(0)$ (resp. $\Delta_{B_2}^{-1}(0)$) it cannot switch (see Proposition 6), and the trajectory cannot reach $\Delta_{B_1}^{-1}(0)$ (resp. $\Delta_{B_2}^{-1}(0)$) with control $u_1 = -1$ (resp. $u_2 = 1$).

4.2.2. *Construction of the Synthesis.* In the following we construct the complete time optimal synthesis. First we write a (linear) system of equations for the switching functions and its solutions along a bang or singular arc, that can be easily verified:

Proposition 9. *Let $(x(\cdot), \lambda(\cdot))$ defined on $[0, T]$ be an extremal pair. Recall that $\phi_i(\cdot) = \langle \lambda(\cdot), F_i(x(\cdot)) \rangle$, $i = 1, 2$ and define $\phi_3(\cdot) := \langle \lambda(\cdot), [F_1, F_2](x(\cdot)) \rangle$. Then, along a bang or a singular arc, the triplet (ϕ_1, ϕ_2, ϕ_3) is solution of the time invariant ODE*

$$\begin{cases} \dot{\phi}_1 = -u_2 \phi_3 \\ \dot{\phi}_2 = u_1 \phi_3 \\ \dot{\phi}_3 = \alpha^2 u_2 \phi_1 - u_1 \phi_2, \end{cases} \tag{30}$$

i.e., we have

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{pmatrix} = R(t) \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix}, \tag{31}$$

where

$$R(t) = \begin{pmatrix} \frac{\omega \cos(t\sqrt{\omega})}{\omega} & -\frac{u_1 u_2 (-1+\cos(t\sqrt{\omega}))}{\omega} & -\frac{u_2 \sin(t\sqrt{\omega})}{\sqrt{\omega}} \\ -\frac{\alpha^2 u_1 u_2 (-1+\cos(t\sqrt{\omega}))}{u_1^2+\alpha^2 u_2^2} & \frac{\alpha^2 u_2^2+u_1^2 \cos(t\sqrt{\omega})}{\omega} & \frac{u_1 \sin(t\sqrt{\omega})}{\sqrt{\omega}} \\ \frac{\alpha^2 u_2 \sin(t\sqrt{\omega})}{\sqrt{\omega}} & -\frac{u_1 \sin(t\sqrt{\omega})}{\sqrt{\omega}} & \cos(t\sqrt{\omega}) \end{pmatrix}. \tag{32}$$

and $\omega = u_1^2 + \alpha^2 u_2^2$.

Since there is no chattering, we have:

Proposition 10. *For every extremal trajectory $x(\cdot)$ such that $x(0) = (1, 0, 0)^T$, there exists $\varepsilon > 0$ such that it corresponds to controls $(u_1, u_2) = (1, 0)$, or to control $(u_1, u_2) = (1, 1)$, in $[0, \varepsilon]$.*

Proof. Since $(1, 0, 0)^T \notin \Delta_{B_1}^{-1}(0)$, u_1 cannot be zero for small times. Moreover $u_1 = -1$ or $u_2 = -1$ do not permit to $x(\cdot)$ to enter S^+ . \square

Remark 18. Notice that, since F_1, F_2 and $[F_1, F_2]$ is a generating family of the tangent space, the knowledge of λ is equivalent to the knowledge of (ϕ_1, ϕ_2, ϕ_3) . Since the initial covector $\lambda(0)$ is free, one have some freedom on the choice of $(\phi_1(0), \phi_2(0), \phi_3(0))$. More precisely, starting from the point $(1, 0, 0)^T$, we have $\phi_2(0) = 0$ (because $F_2 = 0$), $\phi_1(0) > 0$ (otherwise u_1 would not be 1 for small time) and $\phi_3(0)$ is free. Moreover, since the covector is defined up to a positive constant, we can normalize $\phi_1(0) = 1$.

Notice also that a u_1 -singular (resp. u_2 -singular) extremal defined on $[a, b]$ has $\phi_1(\cdot) = \phi_3(\cdot) = 0$ (resp. $\phi_2(\cdot) \equiv \phi_3(\cdot) \equiv 0$) on $[a, b]$.

In the following propositions, we study in details the two extremals starting from $(1, 0, 0)^T$ and corresponding to controls $(1, 0)$ and $(1, 1)$ and the possible switchings along them. Notice that if $\alpha < 1$ then the trajectory corresponding to controls $u_1 = u_2 = 1$ intersects the boundary of S^+ on $\Delta_{B_1}^{-1}(0)$ while if $\alpha > 1$ it intersects the boundary of S^+ on $\Delta_A^{-1}(0)$.

Proposition 11. *The trajectory starting from $(1, 0, 0)^T$ and corresponding to control $u_1 = +1, u_2 = 0$ is extremal up to time $\pi/2$ (i.e., until it leaves S^+). Moreover for every $a \in]0, \pi/2[$ there exists $\varepsilon > 0$ such that the trajectory starting from $(1, 0, 0)^T$ and corresponding to control $(1, 0)$ in $[0, a]$ and to control $(1, 1)$ in $]a, a + \varepsilon[$ is extremal.*

Proof. The first claim is immediately checked by plugging $u_1 = 1, u_2 = 0$ inside the equations (31) and (32) for time $t \in [0, \frac{\pi}{2}]$, and choosing initial covector in such a way that $\phi_1(0) = 1$ and $\phi_2(0) = \phi_3(0) = 0$. Since along this extremal the $\phi_i(\cdot)$ are constant it follows that the trajectory starting from $(1, 0, 0)^T$ and corresponding to control $u_1 = +1, u_2 = 0$ is extremal until it leaves S^+ .

The second claim can be seen as a consequence of Lemma 9. Anyway, it can be checked directly by setting $\phi_1(0) = 1, \phi_2(0) = \phi_3(0) = 0$ and plugging $u_1 = 1, u_2 = 0$ inside the equation (31) and (32) for time $t \in [0, a]$ and $u_1 = 1, u_2 = 1$ for time $t \in]a, a + \varepsilon[$. Finally one have to check that $\phi_1(\cdot)$ and $\phi_2(\cdot)$ remain positive for an ε small enough. \square

Similarly one gets the following:

Proposition 12. *The trajectory starting from $(1, 0, 0)^T$ and corresponding to control $u_1 = +1, u_2 = +1$ is extremal up to time T_α where*

$$T_\alpha = \begin{cases} \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}} & \text{if } \alpha \leq 1 \\ \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2}) & \text{if } \alpha \geq 1. \end{cases} \tag{33}$$

(Notice that if $\alpha \leq 1, T_\alpha$ is the time in which the curve reaches $\Delta_{B_1}^{-1}(0)$ and leaves S^+ .) Moreover for every $0 < a < T_\alpha$ there exists $\varepsilon > 0$ sufficiently small such that the trajectory starting from $(1, 0, 0)^T$ and corresponding to controls $(u_1, u_2) = (1, 1)$ in $[0, a]$ and to controls $(u_1, u_2) = (-1, 1)$ in $]a, a + \varepsilon]$ is extremal. The same happens if $a = T_\alpha$ and $\alpha \neq 1$.

Next we call γ^{++} defined on $[0, T_\alpha]$, the extremal trajectory starting from $(1, 0, 0)^T$ and corresponding to controls $(u_1, u_2) = (1, 1)$. In the two following propositions we prolong most of the extremals built in Propositions 11 and 12.

Proposition 13. *Let $(x(\cdot), \lambda(\cdot))$ be an extremal pair starting from $(1, 0, 0)^T$, corresponding to controls $(u_1, u_2) = (1, 0)$ in the interval $[0, a]$ ($0 < a < \frac{\pi}{2}$) and switching to controls $(1, 1)$ at time a . Then:*

- A.** *If $\alpha \leq 1$, there is no other switching before leaving S^+ .*
- B.** *If $\alpha > 1$ and $a \geq \arccos(\frac{1}{\alpha})$, there is no other switching before leaving S^+ .*
- C.** *If $\alpha > 1$ and $a < \arccos(\frac{1}{\alpha})$, then the control u_1 switches to -1 at time $a + T_\alpha$ and there is no other switching before leaving S^+ .*

Proof. First remember that after time a , the control u_2 does not switch (see Remark 17). Since $x(\cdot)$ is u_2 -singular in $[0, a]$ and corresponds to control $u_1 = 1$, we have $\phi_1(a) = 1$ and $\phi_2(a) = \phi_3(a) = 0$. Using equation (31) and (32) one easily checks when ϕ_1 changes of sign. This allows to check if there is a u_1 -switching before leaving S^+ . Moreover remember that once $(u_1, u_2) = (-1, 1)$ then there is no additional switching (see Remark 17). This allows to conclude for the last claim. \square

Remark 19. Notice that in case **C**. the set of points where the extremal trajectories are switching from controls $(1, 1)$ to $(-1, 1)$ describes a curve that is a great arc of circle between the points $\gamma^{++}(T_\alpha)$ and the target $(0, 0, 1)^T$. Indeed this is a consequence of the fact that the time between $\Delta_{B_2}^{-1}(0)$ (which is a great arc of circle) and the u_1 -switching is constant.

Similarly one gets:

Proposition 14. *Let $(x(\cdot), \lambda(\cdot))$ be an extremal pair, starting from $(1, 0, 0)^T$, corresponding to controls $(1, 1)$ in $[0, a]$ with $a \leq T_\alpha$ and switching to controls $(-1, 1)$ at time a . Then it is extremal until it leaves S^+ by reaching $\Delta_A^{-1}(0)$.*

The following proposition concludes the construction of all extremal trajectories. The proof is similar to the proofs above and it is left to the reader.

Proposition 15. *In the case $\alpha < 1$, for every $a \in [T_\alpha, T_\alpha + T'_\alpha]$ where $T'_\alpha = \frac{\arccos(\alpha)}{\alpha}$ the trajectory starting from $(1, 0, 0)^T$ and corresponding to controls:*

- $(u_1, u_2) = (1, 1)$ for $t \in [0, T_\alpha]$,
- $(u_1, u_2) = (1, 0)$ for $t \in]T_\alpha, a]$,
- $(u_1, u_2) = (1, -1)$ for $t \in [a, b]$, where b is the last time in which the trajectory stays in S^+ ,
is extremal.

The previous propositions describe all the possible extremals starting from $(1, 0, 0)^T$. They are pictured in Figure 3. One immediately checks the following:

Claim: Each point of S^+ is reached by an extremal trajectory. Moreover, all the extremal trajectories reaching a fixed point, coincide before that point. This allows to conclude that all these extremals are in fact optimal and we have built the complete time optimal synthesis for the system (24) starting from $(1, 0, 0)^T$.

Notice that the only points reached by more than one extremal are on $Supp(\gamma^{++})$ and on the support of singular trajectories. These facts are collected in the following:

Theorem 1. *The set of trajectories starting from $(1, 0, 0)^T$ and corresponding to the controls given below is a time optimal synthesis for the system (24).*

In the sequel we denote by T the last time in which the control is defined, that is the last time in which the corresponding trajectory stays in S^+ .

Case $\alpha < 1$

$$(u_1, u_2) = \begin{cases} (1, 0) \text{ for } t \in [0, a] \text{ with } a \in [0, \frac{\pi}{2}], \\ (1, 1) \text{ for } t \in]a, T]. \end{cases}$$

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, a] \text{ with } a \in]0, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}], \\ (-1, 1) \text{ for } t \in]a, T]. \end{cases}$$

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}], \\ (0, 1) \text{ for } t \in]\frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}, a] \\ \text{with } a \in]\frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}} + \frac{\arccos(\alpha)}{\alpha}], \\ (-1, 1) \text{ for } t \in]a, T]. \end{cases}$$

The curve reaching the target $(0, 0, 1)^T$ corresponds to following controls:

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}], \\ (0, 1) \text{ for } t \in]\frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}}, \frac{\arccos(-\alpha^2)}{\sqrt{1+\alpha^2}} + \frac{\arccos(\alpha)}{\alpha}]. \end{cases}$$

Case $\alpha = 1$

$$(u_1, u_2) = \begin{cases} (1, 0) \text{ for } t \in [0, a] \text{ with } a \in [0, \frac{\pi}{2}], \\ (1, 1) \text{ for } t \in]a, T]. \end{cases}$$

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, a] \text{ with } a \in [0, \frac{\pi}{\sqrt{2}}], \\ (-1, 1) \text{ for } t \in]a, T]. \end{cases}$$

The curve reaching the target $(0, 0, 1)^T$ corresponds to following controls:

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, \frac{\pi}{\sqrt{2}}]. \end{cases}$$

Case $\alpha > 1$

$$(u_1, u_2) = \begin{cases} (1, 0) \text{ for } t \in [0, a] \text{ with } a \in [\arccos(\frac{1}{\alpha}), \frac{\pi}{2}], \\ (1, 1) \text{ for } t \in]a, T]. \end{cases}$$

$$(u_1, u_2) = \begin{cases} (1, 0) \text{ for } t \in [0, a] \text{ with } a \in]0, \arccos(\frac{1}{\alpha})[, \\ (1, 1) \text{ for } t \in]a, a + T_\alpha], \\ (-1, 1) \text{ for } t \in]a + T_\alpha, T]. \end{cases}$$

$$(u_1, u_2) = \begin{cases} (1, 1) \text{ for } t \in [0, a] \text{ with } a \in]0, T_\alpha], \\ (-1, 1) \text{ for } t \in]a, T]. \end{cases}$$

The curve reaching the target $(0, 0, 1)^T$ corresponds to following controls:

$$(u_1, u_2) = \begin{cases} (1, 0) \text{ for } t \in [0, \arccos(\frac{1}{\alpha})], \\ (1, 1) \text{ for } t \in]\arccos(\frac{1}{\alpha}), \frac{1}{\sqrt{1+\alpha^2}} \arccos(-\frac{1}{\alpha^2})]. \end{cases}$$

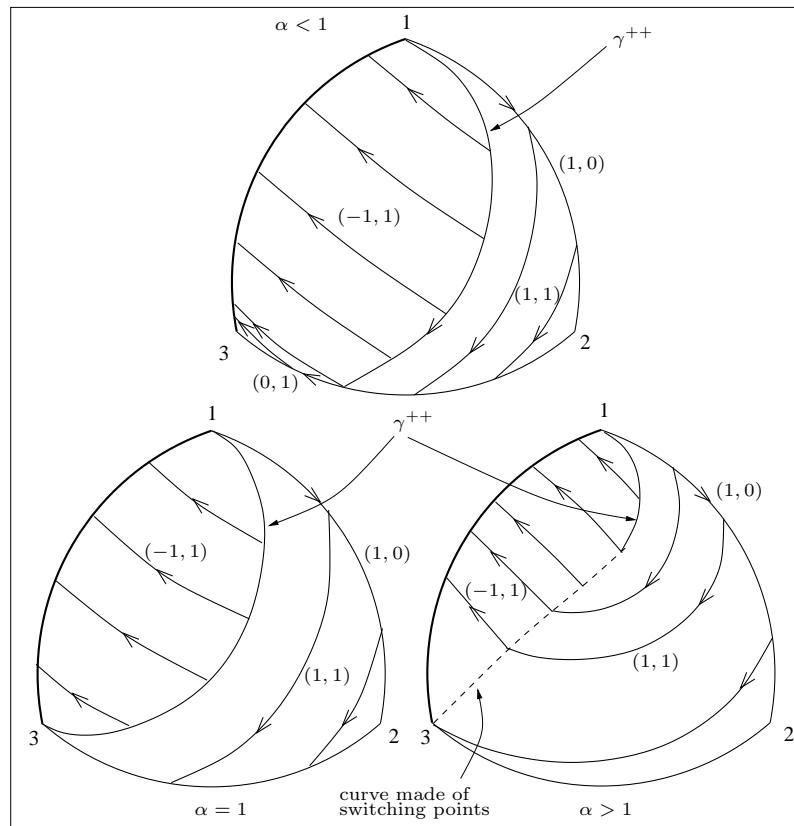


FIGURE 3. Image of the synthesis for minimum time with bounded controls in the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$

From the controls given above, one can easily get the explicit expression of the corresponding optimal trajectories that are concatenations of at most three arcs of circle that are bang or singular. More details about optimal trajectories reaching the target $(0, 0, 1)^T$ are given in Section 3.1.

5. Minimum Energy. In this section, for the minimum energy problem, we provide explicit expressions (in terms of elliptic functions) for optimal controls linking the first and the third level, in terms of a parameter that will be determined numerically. In the following we restrict our attention to extremal trajectories that are parametrized by arclength. This is possible thanks to the fact that, in this special case, the PMP-Hamiltonian system is Liouville integrable (since it is a right invariant Hamiltonian system on $SO(3)$, see Section 5.2). Some efforts are required to prove that all extremals are in fact optimal (see Section 5.3). Similarly to Section 4, as byproduct we get the complete optimal synthesis (cf. Remark 4). In the sequel, for convenience of computations, we make a change of variables on the controls in such a way that the nonisotropy factor α does not appear in the dynamics, but in the cost.

5.1. Change of Variables. A problem whose dynamics verifies an equation of the form of (13), (17) is said to be right invariant on the matrix group $SO(3)$. This

group is endowed with a very special structure of real semi-simple compact Lie group.

In the following it is more natural to present the problem upstairs (i.e., at the level of the group), and in a slightly different way. Actually, we make the change of variables: $v_1(\cdot) = u_1(\cdot)$, $v_2(\cdot) = \alpha u_2(\cdot)$. In this way, at the level of the group $SO(3)$, the minimization problem becomes:

Problem \mathcal{P} :

$$\frac{dg}{dt}(t) = \begin{pmatrix} 0 & -v_1 & 0 \\ v_1 & 0 & -v_2 \\ 0 & v_2 & 0 \end{pmatrix} g(t) = dR_{g(t)} \begin{pmatrix} 0 & -v_1 & 0 \\ v_1 & 0 & -v_2 \\ 0 & v_2 & 0 \end{pmatrix}, \quad (34)$$

$$\text{minimize } J(v_1(\cdot), v_2(\cdot)) = \int_0^T (v_1^2(t) + \frac{1}{\alpha^2} v_2^2(t)) dt. \quad (35)$$

$$g(0) \in \mathcal{S}, \quad g(T) \in \mathcal{T}. \quad (36)$$

Here $v_i : [0, T] \rightarrow \mathbb{R}$, for $i = 1, 2$ are measurable essentially bounded functions, and dRg denotes the differential of R_g , where R_g is the right translation by g , i.e., $R_g g' = g'g$, where $g, g' \in SO(3)$. Notice that dRg is still a right translation by g . The cost function depending only on the controls (i.e., it is invariant under the group action) is called right invariant. The source \mathcal{S} (i.e., the lift of the state one), and the target \mathcal{T} (i.e., the lift of the state three), are defined in (19), (20). Recall that the first column $(\psi_1(t), \psi_2(t), \psi_3(t))^T$ of $g(t)$ represents the position (on the sphere S^2) of the system downstairs (cf. Formula (18)). Recall moreover that, in the minimization (35), to guarantee the existence of minimizers, the final time T should be fixed. In the following, we fix T in such a way that the trajectories are parameterized by arclength.

5.2. Integration of the Hamiltonian System Given by the PMP. For the problem \mathcal{P} the Lie group structure allows a nice formulation of the Pontryagin Maximum Principle. To take advantage of this formulation, we need some extra notions, presented in the next paragraph. What follows can be found in a more general setting with proofs and further discussions in [1] or [22].

5.2.1. Trivialization of the co-Tangent Bundle. We denote by $\mathfrak{so}(3)$ the tangent space at identity of the smooth manifold $SO(3)$. Its dual space (the vector space of all linear functions on $\mathfrak{so}(3)$) will be denoted by $\mathfrak{so}(3)^*$. It is a classical fact that the cotangent bundle of $SO(3)$ is globally diffeomorphic to the product $SO(3) \times \mathfrak{so}(3)^*$. That is, with each pair (g, P) in $SO(3) \times \mathfrak{so}(3)^*$, we may associate a linear form p on $T_g SO(3)$ defined by $p(v) = P(dR_g^{-1}(v))$. Using exactly the same idea, the tangent bundle of $SO(3)$ is diffeomorphic to $SO(3) \times \mathfrak{so}(3)$ by identification of $dR_g^{-1}(T_g SO(3))$ with $\mathfrak{so}(3)$. Using this formalism, the canonical symplectic form on $T^*SO(3)$ has a nice expression (see [22]), and we may rewrite the PMP in a compact form.

5.2.2. Hamiltonian of the System. For the problem \mathcal{P} , the PMP-Hamiltonian \mathcal{H} (see (22)) can be thought as a map $H_v : \mathfrak{so}(3)^* \rightarrow \mathbb{R}$ that for each $\lambda_0 \in \mathbb{R}$, and each control $v = (v_1, v_2) \in \mathbb{R}^2$ is defined by

$$H_v(P) = P \left[\begin{pmatrix} 0 & -v_1 & 0 \\ v_1 & 0 & -v_2 \\ 0 & v_2 & 0 \end{pmatrix} \right] + \lambda_0 (v_1^2 + \frac{1}{\alpha^2} v_2^2). \quad (37)$$

Using this notation, the Pontryagin Maximum Principle writes as follows. Notice that, as proved in [11], it is enough to require the transversality condition at the source (and at identity).

Corollary (Pontryagin Maximum Principle for Right Invariant Problems)
 Let $(v(\cdot), g(\cdot)) \in L^\infty([0, T], \mathbb{R}^2) \times Lip([0, T], SO(3))$ be a control-trajectory pair of the system (34). If $(v(\cdot), g(\cdot))$ is a solution to the problem \mathcal{P} then there exists a constant $\lambda_0 \leq 0$ and a Lipschitzian curve $P(\cdot) \in Lip([0, T], \mathfrak{so}(3)^*)$ such that the pair $(P(\cdot), \lambda_0)$ never vanishes and for almost every t in $[0, T]$ we have

$$\begin{cases} \frac{dg(t)}{dt} &= dH_{v(t)}(P(t))g(t), \\ \frac{dP(t)}{dt} &= -ad_{dH_{v(t)}(P(t))}^*P(t). \end{cases} \tag{38}$$

Moreover

$$H_{v(t)}(P(t)) = \max_{\bar{v} \in \mathbb{R}^2} H_{\bar{v}}(P(t)) = \text{const} \tag{39}$$

$$P(t)(T_{id}g(t)\mathcal{S}g^{-1}(t)) = 0 \text{ (Transversality condition)}. \tag{40}$$

In the statement of the PMP, $ad_{dH_{v(t)}(P(t))}^*P(t)$ is defined by $ad_{dH_{v(t)}(P(t))}^*P(t)(v) = P(t)([dH_{v(t)}(P(t)), v])$. In this case, since there are no abnormal extremals, (see Proposition 3) we can normalize $\lambda_0 = -1/2$. An easy computation leads to

$$v_1(\cdot) = \phi_1(\cdot) \quad \text{and} \quad v_2(\cdot) = \alpha \phi_2(\cdot).$$

where

$$\phi_1(t) = P(t) \left[\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \quad \text{and} \quad \phi_2(t) = \alpha P(t) \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \right].$$

Notice that $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are defined as in Definition 4.

Since the group $SO(3)$ is semi-simple and compact, then the canonical Killing form $Kill(x, y) = tr(xy)$ is negative definite on the Lie algebra $\mathfrak{so}(3)$. Hence $-\frac{1}{2}Kill(\cdot, \cdot)$ provides an isomorphism between $\mathfrak{so}(3)$ and its dual space $\mathfrak{so}(3)^*$. Using this identification, to each linear form $P : \mathfrak{so}(3) \rightarrow \mathbb{R}$ in $\mathfrak{so}(3)^*$, we can associate a skew symmetric matrix M_P of $\mathfrak{so}(3)$ in the following way: by definition of M_P , for each matrix N in $\mathfrak{so}(3)$, we have $-\frac{1}{2}Kill(M_P, N) = P(N)$. Equation (38) for P can be rewritten in the famous Lax-Poincaré form (see [23] for a proof and further discussions)

$$\frac{dM_P(t)}{dt} = [dH_v(P(t)), M_P(t)], \tag{41}$$

where we have identified $\mathfrak{so}(3)^{**}$ with $\mathfrak{so}(3)$.

5.2.3. *Expressions of the Controls.* Let us define $m_1(\cdot), m_2(\cdot), m_3(\cdot)$ three Lipschitzian functions from $[0, T]$ to \mathbb{R} by

$$M_P(t) =: \begin{pmatrix} 0 & -m_1(t) & -m_3(t) \\ m_1(t) & 0 & -m_2(t) \\ m_3(t) & m_2(t) & 0 \end{pmatrix}, \tag{42}$$

for every time t in $[0, T]$. From the maximality condition (39), we deduce that for almost every t in $[0, T]$, we have $m_1(t) = v_1(t)$ and $m_2(t) = \frac{1}{\alpha^2}v_2(t)$. From the transversality condition (40), we deduce $v_2(0) = 0$. Equation (41) writes

$$\begin{cases} \frac{dv_1(t)}{dt} &= -m_3(t)v_2(t) \\ \frac{dv_2(t)}{dt} &= \alpha^2 m_3(t)v_1(t) \\ \frac{dm_3(t)}{dt} &= -\frac{1-\alpha^2}{\alpha^2} v_1(t)v_2(t). \end{cases} \tag{43}$$

Remark 20. In the case in which $\alpha = 1$ (see [11]), the system (43) admits the solution $v_1(t) = \cos(Kt)$, $v_2(t) = \sin(Kt)$, $m_3 = K$, where K is a real constant. If $\alpha \neq 1$ and $m_3(0) \neq 0$, the function $m_3(\cdot)$ is no more constant.

In the case in which $\alpha \neq 1$, it is easy to check that the two functions $K_1(\cdot)$ and $K_2(\cdot)$ defined by

$$\begin{aligned} K_1 : t \in [0, T] &\mapsto \frac{1}{2} \left(v_1(t)^2 + \frac{1}{\alpha^2}v_2(t)^2 \right), \\ K_2 : t \in [0, T] &\mapsto \frac{1}{2} \left(v_1(t)^2 - \frac{\alpha^2}{1-\alpha^2}m_3(t)^2 \right), \end{aligned} \tag{44}$$

have zero derivative and hence are constant on the time interval $[0, T]$. Notice that $K_1(\cdot)$ is just the maximized Hamiltonian (39). In the following we normalize $K_1 = \frac{1}{2}$, that corresponds to parametrize the extremals by arclength. In this way our problem is equivalent to a minimum time problem with controls constrained in an ellipse (cf. Section 1.1).

As already said in Section 4, we restrict our study to trajectories that remain in the first positive octant S^+ . For such trajectories, we want to prove that $m_3(0) \geq 0$. Since $\psi_2(\cdot) \geq 0$, we must have $v_1(0) \geq 0$. We already know (for transversality reasons) that $|v_1(0)| = 1$. Hence $v_1(0) = 1$. Proceed now by contradiction and assume $m_3(0) < 0$. Equations (43) insure that $v_2(\cdot)$ is strictly negative for (strictly positive) small times. This implies that $\psi_3(\cdot) < 0$ for small times, which contradicts the fact that $\psi(\cdot)$ takes values in S^+ . As a corollary, we may state that $v_2(\cdot)$ is positive for small times. Using now the classical theory of elliptic functions (see [37] chapter XXII for example), we can express $v_1(\cdot)$, $v_2(\cdot)$ and $m_3(\cdot)$ as elliptic functions of order 2. Indeed, writing, for all t small enough

$$v_2(t) = \alpha\sqrt{1 - v_1(t)^2}, \quad \text{and} \quad m_3(t) = \sqrt{m_3(0)^2 - \frac{1-\alpha^2}{\alpha^2}(1 - v_1(t)^2)}, \tag{45}$$

we can express $v_1(\cdot)$ as the solution of

$$\frac{dv_1(t)}{dt} = -\alpha\sqrt{1 - v_1(t)^2}\sqrt{m_3(0)^2 - \frac{1-\alpha^2}{\alpha^2}(1 - v_1(t)^2)}, \tag{46}$$

with

$$v_1(0) = 1. \tag{47}$$

The solution of (46) with initial condition (47) is known for small times (see [37], chapter XXII). Hence we have the following:

Proposition 16. *For every extremal trajectory, starting from $(1, 0, 0)^T$ downstairs (or from \mathcal{S} upstairs), there exists $\varepsilon > 0$ such that, for $t \in [0, \varepsilon]$, it corresponds to controls a.e. given by:*

- If $m_3(0) = 0$, then

$$v_1(t) = 1, \tag{48}$$

$$v_2(t) = 0, \tag{49}$$

$$m_3(t) = 0. \tag{50}$$

- If $\alpha < 1$ and $m_3(0)^2 < \frac{1-\alpha^2}{\alpha^2}$, then

$$v_1(t) = \operatorname{dn}(\sqrt{1-\alpha^2} t; k), \tag{51}$$

$$v_2(t) = \alpha k \operatorname{sn}(\sqrt{1-\alpha^2} t; k), \tag{52}$$

$$m_3(t) = m_3(0) \operatorname{cn}(\sqrt{1-\alpha^2} t; k), \tag{53}$$

where we have defined the modulus of the elliptic functions dn , sn and cn to be equal to $k = \frac{\alpha m_3(0)}{\sqrt{1-\alpha^2}}$.

- If $\alpha \leq 1$ and $m_3(0)^2 = \frac{1-\alpha^2}{\alpha^2}$, then

$$v_1(t) = \frac{1}{\cosh(\sqrt{1-\alpha^2} t)}, \tag{54}$$

$$v_2(t) = \alpha \tanh(\sqrt{1-\alpha^2} t), \tag{55}$$

$$m_3(t) = \frac{m_3(0)}{\cosh(\sqrt{1-\alpha^2} t)}. \tag{56}$$

- If $\alpha \leq 1$ and $m_3(0)^2 > \frac{1-\alpha^2}{\alpha^2}$, then

$$v_1(t) = \operatorname{cn}(\alpha m_3(0) t; k), \tag{57}$$

$$v_2(t) = \alpha \operatorname{sn}(\alpha m_3(0) t; k), \tag{58}$$

$$m_3(t) = m_3(0) \operatorname{dn}(\alpha m_3(0) t; k), \tag{59}$$

where we have defined the modulus of the elliptic functions cn , sn and dn to be equal to

$$k = \frac{\sqrt{1-\alpha^2}}{\alpha m_3(0)}. \tag{60}$$

- If $\alpha > 1$, then

$$v_1(t) = \operatorname{cd}\left(\frac{\sqrt{\alpha^2-1}}{k} t; k\right), \tag{61}$$

$$v_2(t) = \alpha \sqrt{1-k^2} \operatorname{sd}\left(\frac{\sqrt{\alpha^2-1}}{k} t; k\right), \tag{62}$$

$$m_3(t) = m_3(0) \operatorname{nd}\left(\frac{\sqrt{\alpha^2-1}}{k} t; k\right), \tag{63}$$

where we have defined the modulus of the elliptic functions cd , sd and nd to be equal to

$$k = \sqrt{\frac{\alpha^2-1}{\alpha^2 m_3(0)^2 + \alpha^2 - 1}}. \tag{64}$$

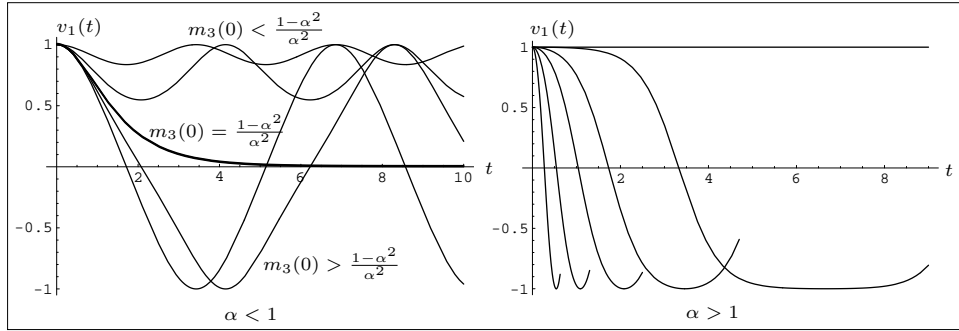


FIGURE 4. Decreasing of $v_1(t)$ as function of $m_3(0)$ and t

It happens that the expressions for $v_1(\cdot)$, $v_2(\cdot)$ and $m_3(\cdot)$ given in Proposition 16 make sense for every time in \mathbb{R}^+ . Plugging them into (43), one checks that they are in fact solutions for every time in \mathbb{R}^+ . Therefore we have:

Proposition 17. *The result of Proposition 16 holds for every $t \in \mathbb{R}^+$.*

Remark 21. Notice that, in Propositions 16 and 17, we do not require that the trajectory reaches the target. To find the optimal trajectory reaching the target (it happens to be unique), we first prove that all these extremals are in fact optimal, and then we determine numerically the value of $m_3(0)$ corresponding to it. In this way, we also provide the complete optimal synthesis.

It is easy to check that the map $(t, m_3(0), \alpha) \mapsto (v_1(t), v_2(t))$ is continuous. It is also possible to prove that this map is actually analytic. Consequently, the map $(t, m_3(0), \alpha) \mapsto \psi(t)$ is itself analytic. Equations (48) to (64) give the form of the optimal controls in the sense that for every optimal trajectory $\psi(\cdot)$ starting from point $(1, 0, 0)^T$, there exists one $m_3(0)$ such that $\psi(\cdot)$ corresponds to controls $v_1(\cdot)$ and $v_2(\cdot)$ as given by equations (48) to (64). What remains to do is to prove that all these trajectories are in fact optimal in S^+ . This is the content of the next section. First we need some preliminary results collected in the following proposition that is illustrated in Figure 4.

Proposition 18. *If $(\alpha, m_3(0))$ satisfies $\alpha \leq 1$ and $m_3(0)^2 = \frac{1-\alpha^2}{\alpha^2}$, then $v_1(\cdot)$ is a decreasing function with limit equal to 0 at infinity. If $(\alpha, m_3(0))$ satisfies $\alpha > 1$ or $m_3(0)^2 \neq \frac{1-\alpha^2}{\alpha^2}$, then $v_1(\cdot)$ is a periodic function with period $\Theta_{\alpha, m_3(0)}$ and it is decreasing until $\frac{\Theta_{\alpha, m_3(0)}}{2}$. Let us extend the definition of $\Theta_{\alpha, m_3(0)}$ to couples $(\alpha, m_3(0))$ such that $\alpha \leq 1$ and $m_3(0)^2 = \frac{1-\alpha^2}{\alpha^2}$ by setting $\Theta_{\alpha, m_3(0)} = +\infty$. If $t, m_3(0)$ and $\tilde{m}_3(0)$ are such that $m_3(0) < \tilde{m}_3(0)$ and $t \leq \frac{1}{2} \min\{\Theta_{\alpha, m_3(0)}, \Theta_{\alpha, \tilde{m}_3(0)}\}$ then their corresponding controls satisfy $v_1(t) > \tilde{v}_1(t)$.*

Proof. The proof of the first claims is a direct consequence of formulas (48), (51), (54), (57) and (61) for control $v_1(\cdot)$. Let us prove the last claim. For $t \in [0, \Theta_{\alpha, m_3(0)}/2]$, $\dot{v}_1(t)$ is given by equation (46). Since the right hand side of that equation is strictly decreasing with respect to $m_3(0)$ for $m_3(0) > 0$, one concludes, using standard comparison arguments in ODEs. \square

5.3. Optimal Synthesis. In this section, we prove that all the extremals given by Propositions 16 and 17 are actually optimal as long as they stay in S^+ .

Remark 22. Recall that in S^+ , our minimization problem is a singular-Riemannian problem. In $S^+ \setminus \Delta_A^{-1}(0)$, the singular-Riemannian structure is in fact a Riemannian one, and its curvature is

$$c(\psi_1, \psi_2, \psi_3) = -1 - \alpha^2 - \frac{2\psi_1^2}{\psi_2^2} - \frac{2\alpha^2\psi_3^2}{\psi_2^2} < 0.$$

If we were exactly in the context of Riemannian geometry, since the topology of S^+ is trivial, this would be enough to conclude the non existence of cut-locus or conjugate locus, i.e., that every extremal trajectory is in fact optimal. Here, the structure is singular along $\Delta_A^{-1}(0)$ and a correct proof of optimality via curvature requires some care and leads to tedious details. For this reason in what follows we prefer to prove optimality without using the concept of curvature.

The first step is to prove that there is no cut-locus.

5.3.1. Non Existence of Cut-Locus in S^+ . Recall (see Section 4.2.1) that in S^+ , the functions $f_1(\cdot)$ and $f_2(\cdot)$ are defined by: $f_1(\psi) = -\frac{\psi_1}{\psi_2}$, $f_2(\psi) = \alpha\frac{\psi_3}{\psi_2}$. With the same techniques used in Section 4, (cf. Proposition 4) one can easily prove that, since $f_1(\cdot) < 0$ (resp. $f_2(\cdot) > 0$) in $Int(S^+)$, then $v_1(\cdot)$ (resp. $v_2(\cdot)$) can change of sign only once from positive to negative (resp. from negative to positive). Since for small values of t , $v_2(t)$ is positive, we have that $v_2(\cdot)$ remains positive (resp. strictly positive) as long as the corresponding trajectory belongs to S^+ (resp. the interior of S^+). In particular, this implies that for every extremal trajectory, $\psi_3(\cdot)$ is an increasing function of the time. Moreover since $m_3(\cdot)$ is positive (see equations (50), (53), (56), (59) and (63)), the system (43) insures that $v_1(\cdot)$ is decreasing as long as the extremal trajectory stays in S^+ . Hence we have:

Proposition 19. *Let $(\psi(\cdot), v(\cdot))$, be a trajectory-control pair given by Propositions 16, 17, starting from $(1, 0, 0)^T$. Then $\psi(\cdot)$ leaves S^+ in finite time. More precisely setting $\psi(\cdot) = (\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot))$ there exists $\bar{t} \in \mathbb{R}^+$ such that $\psi_1(\bar{t}) = 0$ or $\psi_2(\bar{t}) = 0$ and $\psi(t) \in S^+$ in $[0, \bar{t}]$. Moreover in $[0, \bar{t}]$ we have:*

- $v_2(t) \geq 0$.
- $\psi_3(\cdot)$ is an increasing function.
- $v_1(\cdot)$ is decreasing.

Finally every extremal trajectory $\psi(\cdot)$ cannot self-intersect in S^+ . In other words, for every $s < t < \bar{t}$ we have $\psi(s) \neq \psi(t)$.

Now, let us investigate the structure of extremal curves for small time. Using the Hamiltonian equations (43), it is easy to compute the 3-jet of the trajectory $\psi(\cdot)$ corresponding to a given $m_3(0)$. One gets that $\psi(0)$, $\dot{\psi}(0)$ and $\ddot{\psi}(0)$ do not depend on $m_3(0)$, but one has $\ddot{\psi}_1(0) = 0$, $\ddot{\psi}_2(0) = -\alpha^2 m_3(0)^2 - 1$ and $\ddot{\psi}_3(0) = 2\alpha^2 m_3(0)$. It follows that, two curves, corresponding to two different values of $m_3(0)$, do not intersect for strictly positive (small enough) time.

Proposition 20. *There is no cut locus in S^+ . More precisely, for every positive t such that $\psi(t)$ and $\tilde{\psi}(t)$ belong to S^+ and such that $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ (with $\psi(\cdot) \neq \tilde{\psi}(\cdot)$) are both optimal up to time t , we have $\psi(t) \neq \tilde{\psi}(t)$.*

Proof. Consider two optimal curves $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$, corresponding respectively to $m_3(0)$, $v_1(\cdot)$, $v_2(\cdot)$ and $\tilde{m}_3(0)$, $\tilde{v}_1(\cdot)$, $\tilde{v}_2(\cdot)$. Since the jets of $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ are

different at time zero, it follows that there exists $\varepsilon > 0$ such that $Supp(\psi|_{]0,\varepsilon[}) \cap Supp(\tilde{\psi}|_{]0,\varepsilon[}) = \emptyset$.

If the supports of the two curves do not intersect in S^+ , then there is nothing to prove. Assume now that the supports of the two curves intersect in S^+ . Let t be the first positive time such that $Supp(\psi|_{]0,t[}) \cap Supp(\tilde{\psi}|_{]0,t[}) \neq \emptyset$. Since the curves are parametrized by arclength, and they are optimal, it follows that $\psi(t) = \tilde{\psi}(t)$.

Let us first consider the case where $\psi(t) \in S^+ \setminus \Delta_A^{-1}(0)$. We may assume that $m_3(0) < \tilde{m}_3(0)$. Since $\tilde{\psi}(\cdot)$ is optimal, it does not self intersect and reaches the boundary of S^+ in finite time. Hence, the support of $\tilde{\psi}(\cdot)$ delimits two simply connected compact domains in S^+ . In the following we call S_R^+ , the region containing points of $\Delta_A^{-1}(0)$ arbitrarily close to $(1, 0, 0)^T$, and S_L^+ the other one. Let us prove that $\dot{\psi}(t)$ cannot be collinear to $\dot{\tilde{\psi}}(t)$. Indeed, on one hand, if these vectors are collinear with the same versus at t , then $v_1(t) = \tilde{v}_1(t)$. But, thanks to Proposition 19, since the two curves are optimal and hence extremal in S^+ , their controls $v_1(\cdot)$ and $\tilde{v}_1(\cdot)$ are decreasing until t . Hence, thanks to Proposition 18, t is less than their half periods and $v_1(t) = \tilde{v}_1(t)$ implies $m_3(0) = \tilde{m}_3(0)$. On the other hand, if these vectors are collinear with opposite versus, then $v_2(t) = -\tilde{v}_2(t) = 0$ which implies that $\psi(t)$ and $\tilde{\psi}(t)$ belong to $\Delta_{B_2}^{-1}(0)$. But one can easily check that the only optimal trajectory with points in $\Delta_{B_2}^{-1}(0)$, for $t > 0$, corresponds to $m_3(0) = 0$. Let us now prove that $\psi(t)$ cannot be transverse to $\tilde{\psi}(t)$. Indeed, since $m_3(0) < \tilde{m}_3(0)$ then, for small time s , $\psi(s)$ belongs to S_L^+ . Moreover, since t is the first time at which $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ intersect, then $Supp(\psi)|_{]0,t[} \subset S_L^+$. Hence the only way for $\dot{\psi}(t)$ to be transverse to $\dot{\tilde{\psi}}(t)$ is to point toward S_R^+ . But this implies that $v_1(t) < \tilde{v}_1(t)$, which is in contradiction with Proposition 18. Hence we got the contradiction for the existence of cut-points in $S^+ \setminus \Delta_A^{-1}(0)$.

If $\psi(t) \in \Delta_A^{-1}(0)$, the previous argument can be adapted in the following way. Computing the 3-jets of $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ at time t , one can prove that $Supp(\psi|_{]0,t[}) \subset S_R^+$. Moreover, computing the 3-jets of $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ at time zero, one can prove that $Supp(\psi|_{]0,t[}) \subset S_L^+$. It follows the contradiction. \square

5.3.2. *Optimality of Extremals.* Proposition 20 allows to prove:

Proposition 21. *Each extremal trajectory issued from $(1, 0, 0)^T$ is optimal as long as it stays in S^+ . As a consequence, by each point of S^+ passes only one extremal trajectory such that its support is entirely included in S^+ .*

Proof. In S^+ , we define the set \mathcal{D} by $\mathcal{D} = \Delta_{B_1}^{-1}(0) \cup \Delta_A^{-1}(0) \setminus \{(1, 0, 0)^T\}$. On \mathcal{D} , we define the complete order \ll by:

- $(0, \psi_2, \psi_3) \ll (0, \tilde{\psi}_2, \tilde{\psi}_3)$ iff $\psi_2 > \tilde{\psi}_2$,
- $(0, \psi_2, \psi_3) \ll (\tilde{\psi}_1, 0, \tilde{\psi}_3)$ if $\psi_2 > 0$,
- $(\psi_1, 0, \psi_3) \ll (\tilde{\psi}_1, 0, \tilde{\psi}_3)$ iff $\psi_1 < \tilde{\psi}_1$.

From Proposition 20, we know that for every point x in \mathcal{D} , there is only one optimal trajectory issued from $(1, 0, 0)^T$ that reaches x . We define the function $\Phi(\cdot)$ which, to each point in \mathcal{D} , associates the value $m_3(0)$ of the corresponding optimal trajectory. Our goal is to show that $\Phi(\cdot)$ is an homeomorphism between \mathcal{D} and \mathbb{R}^+ .

Optimal trajectories cannot intersect, hence $\Phi(\cdot)$ is an increasing function. Now, we claim that $\Phi(\cdot)$ is actually continuous. Indeed since it is increasing, then, for every $x \in \mathcal{D}$, $\Phi(\cdot)$ has an upper limit $\ell(x)$ on the left and a lower limit $L(x)$ on the

right at x . If $\Phi(\cdot)$ were not continuous at x , then $\ell(x) < L(x)$. But both extremals corresponding to $m_3(0) = \ell(x)$ and $m_3(0) = L(x)$ reach x and are optimal as limit of equicontinuous optimal curves. This is in contradiction with Proposition 20. Hence Φ is continuous.

As a consequence, since Φ is a continuous increasing function, it is one-to-one from \mathcal{D} to a certain interval I of \mathbb{R}^+ . Since $\psi_3(\cdot)$ is a strictly increasing function in the interior of S^+ , the trajectory associated to $m_3(0) = 0$, $v_1(\cdot) = 1$, $v_2(\cdot) = 0$ is the only extremal reaching the point $(0, 1, 0)^T$ and hence it is optimal. This proves that $0 \in I$. Assume now that I is bounded. The knowledge of the jets of the extremals corresponding to $m_3(0)$ in I shows that there exists an $\varepsilon > 0$ such that every extremal corresponding to $m_3(0) \in I$ stays in S^+ until ε . But thanks to the continuity of the distance function to $(1, 0, 0)^T$, there are points in $\Delta_A^{-1}(0) \setminus \{(1, 0, 0)^T\}$ whose subriemannian distance to $(1, 0, 0)^T$ is less than ε . Hence these points are not reached by curves with $m_3(0) \in I$ which is a contradiction. Hence I is not bounded.

Hence, to each value of $m_3(0)$, it corresponds one trajectory $\psi(\cdot)$ in S^+ and one point x in \mathcal{D} such that $\psi(\cdot)$ is the only extremal reaching x . We conclude that $\psi(\cdot)$ is optimal in S^+ . □

5.3.3. Minimum Time of Transfer. In this section we provide explicit expressions of the minimum time needed to reach the target from the source, as function of the nonisotropy factor α and of the value of $m_3(0)$, that will be determined numerically in the next section.

Recall that Proposition 21 insures that there is only one extremal, reaching the target $(0, 0, 1)^T$ that remains in the positive octant. Since, at $(0, 0, 1)^T$, we have $F_1 = 0$, then $v_1(T) = 0$.

Actually, thanks to Proposition 18, T is the *first* root of $v_1(\cdot)$. Moreover, it can be seen from equations (48), (51) and (54), that, if $m_3(0) = 0$, or if $\alpha \leq 1$ and $m_3(0)^2 \leq \frac{1-\alpha^2}{\alpha^2}$, then $v_1(\cdot)$ has no root. Hence, we have the following expressions for the minimum time of transfer as function of α and $m_3(0)$. In the case $\alpha = 1$, $m_3(0)$ and the minimum time of transfer have been computed in [11]. Set $\mathbf{K}(k) := \int_0^{\frac{\pi}{2}} \frac{ds}{\sqrt{1-k^2 \sin^2(s)}}$, i.e., the complete elliptic integral of the second kind. We have:

$$\text{Case } \alpha < 1. T = \frac{\mathbf{K}(k)}{\alpha m_3(0)}, \text{ where } k \text{ is given by (60).} \tag{65}$$

$$\text{Case } \alpha = 1. m_3(0) = \frac{1}{\sqrt{3}} \text{ and : } T = \sqrt{3} \frac{\pi}{2}. \tag{66}$$

$$\text{Case } \alpha > 1. T = \frac{\mathbf{K}(k)}{\sqrt{\alpha^3 m_3(0)^2 + \alpha^2 - 1}}, \text{ where } k \text{ is given by (64).} \tag{67}$$

5.3.4. Upper Bound and Numerical Estimation of $m_3(0)$. In this section, we provide a numerical estimation of the initial condition $m_3(0)$ for the optimal trajectory reaching the target $(0, 0, 1)^T$. The method consists in using a dichotomy algorithm in the following way.

Thanks to Proposition 21, optimal trajectories associated to $\tilde{m}_3(0) > m_3(0)$ leave S^+ at a point of the set $\{\psi_2 = 0\}$ and optimal trajectories associated to $\tilde{m}_3(0) < m_3(0)$ leaves S^+ at a point of the set $\{\psi_1 = 0\}$. Hence, starting from an arbitrary value of $m_3(0)$, at each step, the next value will be chosen smaller if the corresponding optimal trajectory leaves S^+ at a point of the set $\{\psi_2 = 0\}$ and larger

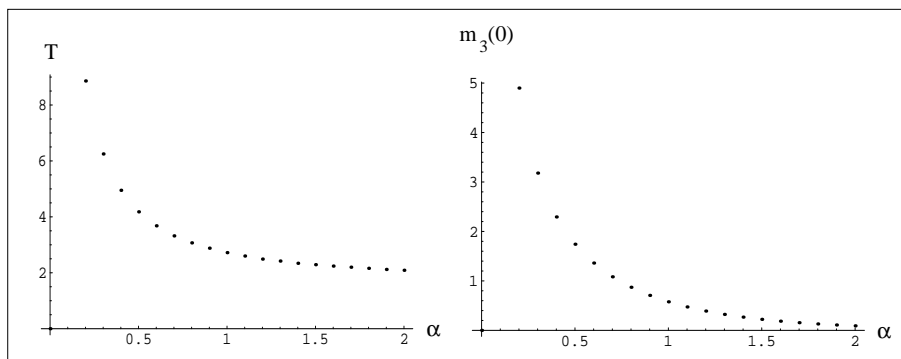


FIGURE 5. Time needed to reach the target and corresponding value of $m_3(0)$ as function of the nonisotropy factor α .

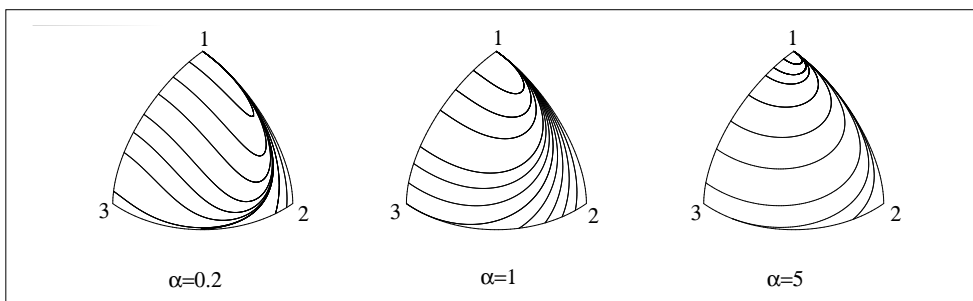


FIGURE 6. Images of the optimal synthesis for the minimum energy problem for $\alpha = 0.2$, $\alpha = 1$ and $\alpha = 5$

in the other case. This algorithm is of course much more efficient if we provide an a priori upper bound of $m_3(0)$.

Recall that the problem of minimizing energy in fixed time is equivalent to minimize time with controls constrained in the set $V_\alpha := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2/\alpha^2 \leq 1\}$. Moreover recall that for the isotropic case $\alpha = 1$, the time to reach the target $(0, 0, 1)^T$ is $\frac{\sqrt{3}\pi}{2}$. More in general, for an isotropic case with controls constrained in the dilated set $W_{\alpha'}^{\text{isotr}} := \{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 + v_2^2 \leq \alpha'^2\}$, $\alpha' > 0$, the time to reach the target is $\frac{\sqrt{3}\pi}{2\alpha'}$. In the following we use the fact that if $V_\alpha \subset W_{\alpha'}^{\text{isotr}}$, then the time to reach the target T is bigger than $\frac{\sqrt{3}\pi}{2\alpha'}$. We have:

Case $\alpha < 1$. From equation (65), we deduce $T \leq \frac{\pi}{2} \frac{1}{\alpha m_3(0) \sqrt{1-k^2}}$, where k is given by formula (60). Moreover since in this case $V_\alpha \subset W_1^{\text{isotr}}$ we get $m_3(0) \leq \frac{1}{\alpha\sqrt{3}} \frac{1}{\sqrt{1-k^2}} \leq \sqrt{\frac{4}{3\alpha^2} - 1}$.

Case $\alpha = 1$. $m_3(0) = \frac{1}{\sqrt{3}}$

Case $\alpha > 1$. From equation (67), we deduce $\sqrt{\alpha^2 m_3(0)^2 + \alpha^2 - 1} T \leq \frac{\pi}{2} \frac{1}{\sqrt{1-k^2}}$, where k is given by formula (64). Since in this case, $V_\alpha \subset W_\alpha^{\text{isotr}}$ we get $T \geq \frac{\sqrt{3}\pi}{2\alpha}$ and hence $m_3(0) \leq \frac{1}{\sqrt{3}}$. Figure 5 depicts the time needed to reach the target and the corresponding value of $m_3(0)$ as functions of the nonisotropy factor α .

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