Introduction to geodesics in sub-Riemannian geometry

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Contents

Preface

These lecture notes contain the first part of the lectures about sub-Riemannian geodesics given by the first author at the IHP Trimester “Geometry, Analysis and Dynamics on sub-Riemannian manifolds”, Paris, Sept - Dec 2014. The point of view is the one of geometric control and Hamiltonian systems.

In Chapter 1, we recall some preliminaries of differential geometry, with special attention to vector fields and, Lie brackets and vector bundles. This material is classical, but it is presented for self-containedness and to introduce the notation used in the following chapters.

Chapter 2 is devoted to sub-Riemannian structures. We introduce the general framework and we prove three fundamental results: the finiteness and the continuity of the sub-Riemannian distance (under the bracket generating condition); the existence of length-minimizers; the infinitesimal characterization of length-minimizers. The first is the classical Chow-Rashevski theorem, the second is a version of the Filippov existence theorem and the third is the Pontryagin maximum principle proved for the special case of systems that in linear the control with quadratic cost.

In Chapter 3, we introduce the language of symplectic geometry. The presentation of the symplectic structure, or equivalently the Poisson bracket, is not classical, but it is naturally introduced to give a geometric description of extremals characterized in the previous chapter. We define the sub-Riemannian Hamiltonian flow, and we specify it for an interesting class of three-dimensional problems. Finally we prove that small pieces of normal trajectories are length-minimizer.

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tangent!vector
tangent!space
vector field

Chapter 1

Vector fields and vector bundles

c:geodiff

In this chapter we collect some basic definitions of differential geometry, in order to recall some useful results and to fix the notation. We assume the reader to be familiar with the definitions of smooth manifold and smooth map between manifolds.

1.1 Differential equations on smooth manifolds

1.1.1 Tangent vectors and vector fields

Let M be a smooth n-dimensional manifold and $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to M$ two smooth curves based at $q = \gamma_1(0) = \gamma_2(0) \in M$. We say that γ_1 and γ_2 are *equivalent* if, in some coordinate chart, they have the same 1-st order Taylor polynomial in some (or, equivalently, in any) coordinate chart. This defines an equivalence relation on the space of smooth curves based at a fixed point.

Definition 1.1. Let M be a smooth n-dimensional manifold and let $\gamma : I \to M$ is a smooth curve such that $\gamma(0) = q \in M$. Its *tangent vector* at $q = \gamma(0)$, denoted by

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(t), \qquad \text{or} \qquad \dot\gamma(0), \tag{1.1}$$

is the equivalence class in the space of all smooth curves in M such that $\gamma(0) = q$.

It is easy to check, using the chain rule, that this is a well-defined object (i.e., it does not depend on the representative curve).

Definition 1.2. Let M be a smooth n-dimensional manifold. The *tangent space* at a point $q \in M$ is the set

$$T_qM := \left\{ \left.\frac{d}{dt}\right|_{t=0} \gamma(t),\ \gamma : (-\varepsilon, \varepsilon) \to M \text{ smooth},\ \gamma(0) = q \right\}.$$

It is a standard fact that T_qM has a natural structure of n-dimensional vector space.

Definition 1.3. A *vector field* on a smooth manifold M is a smooth map

$$X : q \mapsto X(q) \in T_qM,$$

that associates to every point q in M a tangent vector at q. We denote by $\mathrm{Vec}(M)$ the set of smooth vector fields on M.

ve
complete

In coordinates we can write $X = \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x_i}$, and the vector field is smooth if and only if its components $X^i(x)$ are smooth functions. The value of a vector field X at a point q is denoted both with $X(q)$ and $X\big|_q$.

Definition 1.4. Let M be a smooth manifold and $X \in \mathrm{Vec}(M)$. The equation

$$\dot{q} = X(q), \qquad q \in M, \tag{1.2}$$

eq:odeb

is called an *ordinary differential equation* (or *ODE*) on M. A *solution* of (1.2) is a smooth curve $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval, such that

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \forall\, t \in I. \tag{1.3}$$

We also say that γ is an *integral curve* of the vector field X.

A standard theorem on ODE ensures that, for every initial condition, there exists a unique integral curve of a smooth vector field, defined on some interval.

t:ode **Theorem 1.5.** *Let $X \in \mathrm{Vec}(M)$ and consider the Cauchy problem*

$$\begin{cases} \dot{q}(t) = X(q(t)) \\ q(0) = q_0 \end{cases} \tag{1.4}$$

eq:odeb2

For any point $q_0 \in M$ there exists $\delta > 0$ and $\gamma : (-\delta, \delta) \to M$ a unique solution of (1.4), denoted by $\gamma(t; q_0)$. Moreover the map $(t, q) \mapsto \gamma(t; q)$ is smooth on a neighborhood of $(0, q_0)$.

A vector field $X \in \mathrm{Vec}(M)$ is called *complete* if, for every $q_0 \in M$, the solution $\gamma(t; q_0)$ of the equation (1.2) can be extended for all $t \in \mathbb{R}$.

Remark 1.6. Standard results from ODE ensure completeness of the vector field $X \in \mathrm{Vec}(M)$ in the following cases:

(*i*) M is a compact manifold (or more generally X has compact support in M),

(*ii*) $M = \mathbb{R}^n$ and X is sub-linear, i.e. there exists $C_1, C_2 > 0$ such that

$$|X(x)| \leq C_1 |x| + C_2, \qquad \forall\, x \in \mathbb{R}^n.$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.

When we are interested in the behavior of the trajectories of a vector field $X \in \mathrm{Vec}(M)$ in a compact subset K of M, the assumption of completeness is not restrictive.

Indeed consider an open neighborhood O_K with compact closure of a compact K in M. There exists a smooth cut-off function $a : M \to \mathbb{R}$ that is identically 1 on K, and that vanishes out of O_K. Then the vector field aX is complete, since it has compact support in M. Moreover, inside K, the vector fields X and aX coincide, hence the integral curves of the two vector fields coincide too.

1.1.2 Flow of a vector field

Given a complete vector field $X \in \mathrm{Vec}(M)$ we can consider the family of maps

$$\phi_t : M \to M, \qquad \phi_t(q) = \gamma(t;q), \qquad t \in \mathbb{R}. \tag{1.5}$$

In other words, $\phi_t(q)$ is the shift for time t along the integral curve of X that starts from q. By Theorem 1.5 it follows that the map

$$\phi : \mathbb{R} \times M \to M, \qquad \phi(t,q) = \phi_t(q),$$

is smooth in both variables and the family $\{\phi_t,\, t \in \mathbb{R}\}$ is a one parametric subgroup of $\mathrm{Diff}(M)$, namely, it satisfies the following identities:

$$\begin{aligned}
&\phi_0 = \mathrm{Id},\\
&\phi_t \circ \phi_s = \phi_s \circ \phi_t = \phi_{t+s}, \qquad \forall\, t,s \in \mathbb{R},\\
&(\phi_t)^{-1} = \phi_{-t}, \qquad\qquad\qquad \forall\, t \in \mathbb{R},
\end{aligned} \tag{1.6}$$

Moreover, by construction, we have

$$\frac{\partial \phi_t(q)}{\partial t} = X(\phi_t(q)), \qquad \phi_0(q) = q, \quad \forall\, q \in M. \tag{1.7}$$

The family of maps ϕ_t defined by (1.5) is called the *flow* generated by X. For the flow ϕ_t of a vector field X it is convenient to use the exponential notation $\phi_t := e^{tX}$, for every $t \in \mathbb{R}$. Following the exponential notation, the group properties (1.6) take the form:

$$e^{0X} = \mathrm{Id}, \qquad e^{tX} \circ e^{sX} = e^{sX} \circ e^{tX} = e^{(t+s)X}, \qquad (e^{tX})^{-1} = e^{-tX}, \tag{1.8}$$

$$\frac{d}{dt} e^{tX} = X e^{tX}. \tag{1.9}$$

Remark 1.7. When X is a linear vector field on $\mathbb{R}^n$, then $X(x) = Ax$ for some $n \times n$ matrix A. In this case the corresponding flow ϕ_t is the matrix exponential $\phi_t(x) = e^{tA}(x)$.

1.1.3 Nonautonomous vector fields

A family of smooth vector fields $\{X_t\}_{t\in\mathbb{R}}$, where $X_t \in \mathrm{Vec}(M)$ for every $t \in \mathbb{R}$, is said to be *measurable and locally bounded* with respect to t if for every smooth function $a \in \mathcal{C}^\infty(M)$ the function $\varphi_X : \mathbb{R} \to \mathbb{R}$ defined by $\varphi_X(t) = X_t a$ is measurable and locally bounded.

Definition 1.8. A *nonautonomous vector field* is family of smooth vector fields $\{X_t\}_{t\in\mathbb{R}}$ that is measurable and locally bounded with respect to t.

Now we consider a *nonautonomous ODE*, i.e. an equation of the form

$$\dot q = X_t(q), \qquad q \in M, \tag{1.10}$$

where X_t is a nonautonomous vector field. If we consider local coordinates $x = (x_1, \dots, x_n)$ in an open set O on the manifold M, the equation (1.10) is written in coordinates as

$$\dot x = f(t,x), \qquad x \in \mathbb{R}^n,$$

where the map $(t,x) \mapsto f(t,x)$ is defined on a subset of $\mathbb{R} \times \mathbb{R}^n$ and satisfies ratheodory (*i*) f is measurable and locally bounded with respect to t, for any fixed $x \in O$,

(*ii*) f is smooth in x for every fixed $t \in \mathbb{R}$,

(*iii*) f has locally bounded derivatives, i.e.,

$$\left|\frac{\partial f_i}{\partial x}(t,x)\right| \leq C_{I,K}, \qquad I \subset \mathbb{R},\ K \subset O \text{ compact}, \quad i = 1,\ldots,n.$$

where we denote with $f = (f_1, \ldots, f_n)$ the components of the vector function f.

The existence and uniqueness of the solution in the nonautonomous case is guaranteed by the following theorem (see [?]).

t:cara **Theorem 1.9** (Carathéodory theorem)**.** *Assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (i)-(iii). Then the Cauchy problem*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \tag{1.11}$$ eq:nncauchy

has locally a unique solution $x(t; t_0, x_0)$ such that (1.11) is satisfied for almost every t and $x(t_0; t_0, x_0) = x_0$. Moreover the map $(t, x_0) \mapsto x(t; t_0, x_0)$ is Lipschitz with respect to t and smooth with respect to x_0.

Let us assume now that the equation (1.9) is *complete*, i.e. for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ the solution $x(t; t_0, x_0)$ is defined for all $t \in \mathbb{R}$. Let us denote by $P_{t_0,t}(x_0) = x(t; t_0, x_0)$. The family of maps $P_{t_0,t}$ is the *nonautonomous flow* generated by X_t. It satisfies

$$\frac{\partial}{\partial t}\frac{\partial P_{t_0,t}}{\partial x}(x) = \frac{\partial f}{\partial x}(t, P_{t_0,t}(x_0))P_{t_0,t}(x)$$

Moreover the following algebraic identities are satisfied

$$\begin{aligned}
&P_{t,t} = \mathrm{Id}, \\
&P_{t_2,t_3} \circ P_{t_1,t_2} = P_{t_1,t_3}, \qquad \forall\, t_1, t_2, t_3 \in \mathbb{R}, \\
&(P_{t_1,t_2})^{-1} = P_{t_2,t_1}, \qquad\qquad \forall\, t_1, t_2 \in \mathbb{R},
\end{aligned} \tag{1.12}$$ eq:flow4

Conversely, to every family of smooth diffeomorphism $P_{t,s} : M \to M$ satisfying the relations (1.12) one can define its *infinitesimal generator* X_t as follows:

$$X_t(q) = \left.\frac{d}{ds}\right|_{s=0} P_{t,t+s}(q), \qquad \forall\, q \in M. \tag{1.13}$$

The following lemma characterizes the flows whose generator is autonomous.

l:nonautaut **Lemma 1.10.** *Let $\{P_{t,s}\}_{t,s\in\mathbb{R}}$ be a family of smooth diffeomorphisms satisfying (1.12). Its infinitesimal generator is an autonomous vector field if and only if*

$$P_{0,t} \circ P_{0,s} = P_{0,t+s}, \qquad \forall\, t, s \in \mathbb{R}.$$

1.1.4 Vector fields as operators on functions

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A vector field $X \in \mathrm{Vec}(M)$ induces an action on the algebra $\mathcal{C}^\infty(M)$ of the smooth functions on M, defined as follows

$$X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \qquad a \mapsto Xa, \qquad a \in \mathcal{C}^\infty(M), \tag{1.14}$$

where

$$(Xa)(q) = \frac{d}{dt}\bigg|_{t=0} a(e^{tX}(q)), \qquad q \in M. \tag{1.15}$$

eq:Xa

In other words it computes the derivative of the function a restricted on integral curves of the vector field X.

r:at *Remark* 1.11. Let us denote $a_t := a \circ e^{tX}$. Clearly the map $t \mapsto a_t$ is smooth and from (1.15) it immediately follows that Xa represents the first order term in the expansion of a_t:

$$a_t = a + t\, Xa + O(t^2).$$

Exercise 1.12. Let $a \in \mathcal{C}^\infty(M)$ and $X \in \mathrm{Vec}(M)$, and denote $a_t = a \circ e^{tX}$. Prove the following formulas

$$\frac{d}{dt} a_t = X a_t, \tag{1.16}$$

$$a_t = a + t\, Xa + \frac{t^2}{2!}X^2 a + \frac{t^3}{3!}X^3 a + \ldots + \frac{t^k}{k!}X^k a + O(t^{k+1}). \tag{1.17}$$

It is easy to see also that the following Leibnitz rule is satisfied

$$X(ab) = (Xa)b + a(Xb), \qquad \forall\, a, b \in \mathcal{C}^\infty(M), \tag{1.18}$$

eq:leibvf

that means that X, as an operator on functions, is a *derivation* of the algebra $\mathcal{C}^\infty(M)$.

Remark 1.13. Notice that, in coordinates, if $a \in \mathcal{C}^\infty(M)$ and $X = \sum_i X_i(x)\frac{\partial}{\partial x_i}$ then $Xa = \sum_i X_i(x)\frac{\partial a}{\partial x_i}$. In particular, when X is applied to the coordinate functions $a_i(x) = x_i$ then $Xa_i = X_i$, which shows that a vector field is completely charactherized by its action on functions.

ex:tangent **Exercise 1.14.** Let $f_1, \ldots, f_k \in \mathcal{C}^\infty(M)$ and assume that $N = \{f_1 = \ldots = f_k = 0\} \subset M$ where $df_1 \wedge \ldots \wedge df_k \neq 0$ on N. Show that $X \in \mathrm{Vec}(M)$ is tangent to the smooth submanifold N if and only if $Xf_i = 0$ for every $i = 1, \ldots, k$.

1.2 Differential of a smooth map

A smooth map between manifolds induces a map between their tangent spaces, simply by transforming the smooth curves.

Definition 1.15. Let $\varphi : M \to N$ a smooth map between smooth manifolds and $q \in M$. The *differential* of φ at the point q is the linear map

$$\varphi_{*,q} : T_qM \to T_{\varphi(q)}N, \tag{1.19}$$

defined as follows:

$$\varphi_{*,q}(v) = \left.\frac{d}{dt}\right|_{t=0} \varphi(\gamma(t)), \qquad \text{if} \qquad v = \left.\frac{d}{dt}\right|_{t=0} \gamma(t), \quad q = \gamma(0).$$

It is easily checked that this definition depends only on the equivalence class of γ.

Remark 1.16. Applying the definition, one immediately verifies that, if $\varphi : M \to N$, $\psi : N \to Q$ are two smooth maps between manifolds, then the differential of the composition $\psi \circ \varphi : M \to Q$ satisfies $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

The differential $\varphi_{*,q}$ of a smooth map $\varphi : M \to N$, also called its *pushforward*, is sometimes denoted by the symbols $D_q\varphi$ or $d_q\varphi$,

As we said, a smooth map induces a transformation of tangent vectors. If we deal with diffeomorphisms, we can also pushforward a vector field.

Definition 1.17. Let $X \in \mathrm{Vec}(M)$ and $\varphi : M \to N$ be a diffeomorphism. The *pushforward* $\varphi_* X \in \mathrm{Vec}(N)$ is the vector field on N defined by

$$(\varphi_* X)(\varphi(q)) := \varphi_*(X(q)), \qquad \forall\, q \in M. \tag{1.20}$$

eq:star

If $P \in \mathrm{Diff}(M)$ is a diffeomorphism of M, we can rewrite the previous identity as

$$(P_* X)(q) = P_*(X(P^{-1}(q))), \qquad \forall\, q \in M. \tag{1.21}$$

eq:diff

Notice that, in general, if φ is a smooth map, the pushforward of a vector field is not defined.

Remark 1.18. From this definition it follows the useful formula for $X, Y \in \mathrm{Vec}(M)$

$$\left.(e^{tX}_* Y)\right|_q = e^{tX}_*\left(\left.Y\right|_{e^{-tX}(q)}\right) = \left.\frac{d}{ds}\right|_{s=0} e^{tX} \circ e^{sY} \circ e^{-tX}(q).$$

The following lemma shows that $P_* X$ is the vector field whose integral curves are the image under P of integral curves of X. Moreover it shows how the pushforward of a vector field acts on functions:

1:1 **Lemma 1.19.** *Let $P \in \mathrm{Diff}(M)$, $X \in \mathrm{Vec}(M)$ and $a \in \mathcal{C}^\infty(M)$ then*

$$e^{tP_*X} = P \circ e^{tX} \circ P^{-1}, \tag{1.22}$$

eq:relfl

$$(P_* X)a = (X(a \circ P)) \circ P^{-1}. \tag{1.23}$$

eq:pstarf

Proof. From the formula

$$\left.\frac{d}{dt}\right|_{t=0} P \circ e^{tX} \circ P^{-1}(q) = P_*(X(P^{-1}(q))) = (P_* X)(q),$$

it follows that $t \mapsto P \circ e^{tX} \circ P^{-1}(q)$ is an integral curve of $P_* X$, from which (1.22) follows. To prove (1.23) let us compute

$$\left.(P_* X)a\right|_q = \left.\frac{d}{dt}\right|_{t=0} a(e^{tP_*X}(q)).$$

Using (1.22) this is equal to

$$\left.\frac{d}{dt}\right|_{t=0} a(P(e^{tX}(P^{-1}(q))) = \left.\frac{d}{dt}\right|_{t=0} (a \circ P)(e^{tX}(P^{-1}(q))) = (X(a \circ P)) \circ P^{-1}.$$

$\square$

Remark 1.20. From this lemma it follows the following formula: for every $X, Y \in \mathrm{Vec}(M)$ Lie bracket

$$(e_*^{tX} Y)a = Y(a \circ e^{tX}) \circ e^{-tX}. \tag{1.24}$$

eq:pstarX

1.3 Lie brackets

Now we introduce a fundamental notion of all our theory, the *Lie bracket* of two vector fields X and Y. Geometrically it is defined as the infinitesimal version of the pushforward of the second vector field along the flow of the first one. As expalined below, it measures how much Y is modified by the flow of X.

Definition 1.21. Let $X, Y \in \mathrm{Vec}(M)$. We define their *Lie bracket* as the vector field

$$[X, Y] := \frac{\partial}{\partial t}\bigg|_{t=0} e_*^{-tX} Y. \tag{1.25}$$

eq:commutatc

Remark 1.22. The geometric meaning of the Lie bracket can be understood by writing explicitly

$$[X, Y]\big|_q = \frac{\partial}{\partial t}\bigg|_{t=0} e_*^{-tX} Y\big|_q = \frac{\partial}{\partial t}\bigg|_{t=0} e_*^{-tX}(Y\big|_{e^{tX}(q)}) = \frac{\partial}{\partial s \partial t}\bigg|_{t=s=0} e^{-tX} \circ e^{sY} \circ e^{tX}(q). \tag{1.26}$$

eq:liebrtutt

We recover its algebraic properties in the following

p:la **Proposition 1.23.** *As derivations on functions we have*

$$[X, Y] = XY - YX. \tag{1.27}$$

eq:comm

Proof. By definition of Lie bracket we have $[X, Y]a = \frac{\partial}{\partial t}\big|_{t=0}(e_*^{-tX} Y)a$. Hence we have to compute the first order term in the expansion, with respect to t, of the map

$$t \mapsto (e_*^{-tX} Y)a.$$

Using formula (1.24) we have

$$(e_*^{-tX} Y)a = Y(a \circ e^{-tX}) \circ e^{tX}.$$

By Remark 1.11 we have $a \circ e^{-tX} = a_{-t} = a - t\,Xa + O(t^2)$, hence

$$\begin{aligned}(e_*^{-tX} Y)a &= Y(a - t\,Xa + O(t^2)) \circ e^{tX} \\ &= (Ya - t\,YXa + O(t^2)) \circ e^{tX}.\end{aligned}$$

Denoting $b = Ya - t\,YXa + O(t^2)$, $b_t = b \circ e^{tX}$, and using again the expansion above we get

$$\begin{aligned}(e_*^{-tX} Y)a &= (Ya - t\,YXa + O(t^2)) + tX(Ya - t\,YXa + O(t^2)) + O(t^2) \\ &= Ya + t(XY - YX)a + O(t^2).\end{aligned}$$

Hence the first order term is $(XY - YX)a$. $\square$

From this proposition it easily follows also the coordinate expression of the Lie bracket. Indeed if

$$X=\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}, \qquad Y=\sum_{j=1}^{n} Y_j \frac{\partial}{\partial x_j},$$

we have

$$[X,Y]=\sum_{i,j=1}^{n}\left(X_i\frac{\partial Y_j}{\partial x_i}-Y_i\frac{\partial X_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

Proposition p:la 1.23 shows that $\mathrm{Vec}(M)$, being an associative algebra with commutator as multiplication, is a Lie algebra with the Lie bracket.

Now we prove that every diffeomorphism induces a Lie algebra homomorphism on $\mathrm{Vec}(M)$.

Proposition 1.24. *Let $P\in \mathrm{Diff}(M)$. Then P_* is a Lie algebra homomorphism of* $\mathrm{Vec}(M)$*, i.e.*

$$P_*[X,Y]=[P_*X,P_*Y], \qquad \forall\, X,Y\in \mathrm{Vec}(M).$$

Proof. We show that the two terms are equal as derivations on functions. Let $a\in \mathcal{C}^\infty(M)$, preliminarly we see, using (eq:pstarf 1.23), that

$$\begin{aligned}P_*X(P_*Ya)&=P_*X(Y(a\circ P)\circ P^{-1})\\&=X(Y(a\circ P)\circ P^{-1}\circ P)\circ P^{-1}\\&=X(Y(a\circ P))\circ P^{-1},\end{aligned}$$

and using twice this property and (eq:comm 1.27)

$$\begin{aligned}[P_*X,P_*Y]a&=P_*X(P_*Ya)-P_*Y(P_*Xa)\\&=XY(a\circ P)\circ P^{-1}-YX(a\circ P)\circ P^{-1}\\&=(XY-YX)(a\circ P)\circ P^{-1}\\&=P_*[X,Y]a.\end{aligned}$$

□

To end this section, we want to show that the Lie bracket of two vector fields is zero, that means that they commute as operators, if and only if the same holds for their flows.

p:liebflow **Proposition 1.25.** *Let $X,Y\in \mathrm{Vec}(M)$. The following properties are equivalent:*

(i) $[X,Y]=0$,

(ii) $e^{tX}\circ e^{sY}=e^{sY}\circ e^{tX}, \quad \forall\, t,s\in\mathbb{R}$.

Proof. We start the proof with the following

Claim. $[X,Y]=0 \quad\Longrightarrow\quad e_*^{-tX}Y=Y$.

Proof of the Claim. Let us show that $[X,Y]=\left.\frac{d}{dt}\right|_{t=0} e_*^{-tX}Y=0$ implies that $\frac{d}{dt}e_*^{-tX}Y=0$ for all $t\in\mathbb{R}$. Indeed we have

$$\begin{aligned}\frac{d}{dt}e_*^{-tX}Y&=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}e_*^{-(t+\varepsilon)X}Y=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}e_*^{-tX}e_*^{-\varepsilon X}Y\\&=e_*^{-tX}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}e_*^{-\varepsilon X}Y=e_*^{-tX}[X,Y]=0,\end{aligned}$$

and the Claim is proved.
$(i) \Rightarrow (ii)$. Let us show that $P_s := e^{-tX} \circ e^{sY} \circ e^{tX}$ is the flow generated by Y. Indeed we have

$$\begin{aligned}
\frac{\partial}{\partial s} P_s &= \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} e^{-tX} \circ e^{(s+\varepsilon)Y} \circ e^{tX} \\
&= \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} e^{-tX} \circ e^{\varepsilon Y} \circ e^{tX} \circ \underbrace{e^{-tX} \circ e^{sY} \circ e^{tX}}_{P_s} \\
&= e_*^{-tX} Y \circ P_s = Y \circ P_s.
\end{aligned}$$

where in the last equality we used the Claim. Using uniqueness of the flow generated by a vector field we get

$$e^{-tX} \circ e^{sY} \circ e^{tX} = e^{sY}, \quad \forall\, t, s \in \mathbb{R},$$

which is equivalent to (ii).
$(ii) \Rightarrow (i)$. For every function $a \in \mathcal{C}^\infty$ we have

$$XYa = \frac{d^2}{dtds}\Big|_{t=s=0} a \circ e^{sY} \circ e^{tX} = \frac{d^2}{dsdt}\Big|_{t=s=0} a \circ e^{tX} \circ e^{sY} = YXa.$$

Then (i) follows from (1.27). eq:comm $\square$

Exercise 1.26. Let $X, Y \in \mathrm{Vec}(M)$ and $q \in M$. Consider the curve on M

$$\gamma(t) = e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(q).$$

Prove that tangent vector to the curve $\gamma(\sqrt{t})$ is exactly $[X, Y](q)$.

Exercise 1.27. Let $X, Y \in \mathrm{Vec}(M)$. Using the semigroup property of the flow, prove the following expansion

$$e_*^{-tX} Y = Y + t[X, Y] + \frac{t^2}{2}[X, [X, Y]] + \frac{t^3}{6}[X, [X, [X, Y]]] + \ldots \tag{1.28}$$ eq:espcar

Exercise 1.28. Let $X, Y \in \mathrm{Vec}(M)$ and $a \in \mathcal{C}^\infty(M)$. Prove the following Leibnitz rule for the Lie bracket:

$$[X, aY] = a[X, Y] + (Xa)Y.$$

Exercise 1.29. Let $X, Y, Z \in \mathrm{Vec}(M)$. Prove that the Lie bracket satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{1.29}$$ eq:liejacobi

Hint: Differentiate the identity $e_*^{tX}[Y, Z] = [e_*^{tX} Y, e_*^{tX} Z]$.

1.4 Cotangent space

In this section we introduce tangent covectors, that are linear functionals on the tangent space. The space of all covectors at a point $q \in M$, called cotangent space is, in algebraic terms, simply the dual space to the tangent space.

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form

Definition 1.30. Let M be a n-dimensional smooth manifold. The *cotangent space* at a point $q \in M$ is the set

$$T_q^*M := (T_qM)^* = \{\lambda : T_qM \to \mathbb{R}, \lambda \text{ linear}\}.$$

If $\lambda \in T_q^*M$ and $v \in T_qM$, we will denote by $\langle \lambda, v \rangle := \lambda(v)$ the action of the covector λ on the vector v.

As we have seen, a smooth map yields a linear map between tangent spaces. Dualizing this map, we get a linear map on cotangent spaces going in the opposite direction.

Definition 1.31. Let $\varphi : M \to N$ be a smooth map and $q \in M$. The *pullback* of φ at point $\varphi(q)$, where $q \in M$, is the map

$$\varphi^* : T^*_{\varphi(q)}N \to T_q^*M, \qquad \lambda \mapsto \varphi^*\lambda,$$

defined by duality in the following way

$$\langle \varphi^*\lambda, v \rangle := \langle \lambda, \varphi_* v \rangle, \qquad \forall\, v \in T_qM,\ \forall\, \lambda \in T^*_{\varphi(q)}M.$$

Example 1.32. Let $a : M \to \mathbb{R}$ be a smooth function and $q \in M$. The differential d_qa of the function a at the point $q \in M$ is an element of T_q^*M since we have a well defined linear action

$$\langle d_qa, v \rangle := \left.\frac{d}{dt}\right|_{t=0} a(\gamma(t)), \qquad v \in T_qM.$$

where $\gamma(t)$ is any smooth curve such that $\gamma(0) = q$ and $\dot\gamma(0) = v$.

Definition 1.33. A *differential 1-form* on a smooth manifold M is a smooth map

$$\omega : q \mapsto \omega(q) \in T_q^*M,$$

that associates to every point q in M a cotangent vector at q. We denote by $\Lambda^1(M)$ the set of differential forms on M.

Since differential forms are dual objects to vector fields, it is well defined the action of $\omega \in \Lambda^1 M$ on $X \in \mathrm{Vec}(M)$ pointwise, defining a function on M.

$$\langle \omega, X \rangle : q \mapsto \langle \omega(q), X(q) \rangle . \tag{1.30}$$

The differential form ω is *smooth* if and only if, for every smooth vector field $X \in \mathrm{Vec}(M)$, the function $\langle \omega, X \rangle \in \mathcal{C}^\infty(M)$

Definition 1.34. Let $\varphi : M \to N$ be a smooth map and $a : N \to \mathbb{R}$ be a smooth function. The *pullback* φ^*a is the smooth function on M defined by

$$\varphi^*a(q) = a(\varphi(q)), \qquad q \in M.$$

In particular, if $\pi : T^*M \to M$ is the canonical projection and $a \in \mathcal{C}^\infty(M)$, then

$$\pi^*a(\lambda) = a(\pi(\lambda)), \qquad \lambda \in T^*M,$$

which is constant on fibers.

1.5 Vector bundles

vector bundle
vector bundle!l trivialization
vector bundle!c projection
vector bundle!r
trivializable!vec
tangent!bundle
cotangent!bund

Heuristically, a smooth vector bundle on a manifold M, is a smooth family of vector spaces parametrized by points in M.

Definition 1.35. Let M be a n-dimensional manifold. A *smooth vector bundle of rank* k over M is a smooth manifold E with a surjective smooth map $\pi : E \to M$ such that

(*i*) the set $E_q := \pi^{-1}(q)$, the *fiber* of E at q, is a k-dimensional vector space

(*ii*) for every $q \in M$ there exist a neighborhood O_q of q and a linear-on-fiber diffeomorphism (also called *local trivialization*) $\psi : \pi^{-1}(O_q) \to O_q \times \mathbb{R}^k$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(O_q) \arrow[r, "\psi"] \arrow[dr, "\pi"'] & O_q \times \mathbb{R}^k \arrow[d, "\pi_1"] \\ & O_q \end{tikzcd} \tag{1.31}$$

eq:diagr0

The space E is called *total space* and M is the *base* of the vector bundle. We will refer at π as the *canonical projection* and rank E will denote the rank of the bundle.

Remark 1.36. The existence of local trivialization maps ψ says that E, as smooth manifold, has dimension

$$\dim E = \dim M + \text{rank } E = n + k.$$

In the case when there exists a global trivialization map, i.e. a local trivialization with $O_q = M$, then $E \simeq M \times \mathbb{R}^k$ and we say that E is *trivializable*.

Example 1.37. For any smooth n-dimensional manifold M, the *tangent bundle* TM, defined as the disjoint union of the tangent spaces at all points of M,

$$TM = \bigcup_{q\in M} T_qM,$$

has a natural structure of $2n$-dimensional smooth manifold, equipped with the vector bundle structure (of rank n) induced by the canonical projection map

$$\pi : TM \to M, \qquad \pi(v) = q \quad \text{if} \quad v \in T_qM.$$

In the same way one can consider the *cotangent bundle* T^*M, defined as

$$T^*M = \bigcup_{q\in M} T_q^*M.$$

Again, it is a $2n$-dimensional manifold, and the canonical projection map

$$\pi : T^*M \to M, \qquad \pi(\lambda) = q \quad \text{if} \quad \lambda \in T_q^*M,$$

endows T^*M with a structure of rank n vector bundle.

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Let $O \subset M$ be a coordinate neighborhood where

$$\psi : O \to \mathbb{R}^n, \qquad \psi(q) = (x_1, \ldots, x_n),$$

define a local coordinate system. The differentials of the coordinate functions

$$dx_i\big|_q, \qquad i = 1, \ldots, n, \qquad q \in O,$$

form a basis of the cotangent space T_q^*M. The dual basis in the tangent space T_qM is defined by the vectors

$$\left.\frac{\partial}{\partial x_i}\right|_q \in T_qM, \qquad i = 1, \ldots, n, \qquad q \in O, \tag{1.32}$$

$$\left\langle dx_i, \frac{\partial}{\partial x_j} \right\rangle = \delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.33}$$

Thus any tangent vector $v \in T_qM$ and any covector $\lambda \in T_q^*M$ can be decomposed in these basis

$$v = \sum_{i=1}^n v_i \left.\frac{\partial}{\partial x_i}\right|_q, \qquad\qquad \lambda = \sum_{i=1}^n p_i dx_i\big|_q,$$

and the maps

$$\psi_v : v \mapsto (x_1, \ldots, x_n, v_1, \ldots, v_n), \qquad \psi_\lambda : \lambda \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n), \tag{1.34}$$

define local coordinates on TM and T^*M respectively, which we call *canonical coordinates* induced by the coordinates ψ on M.

Definition 1.38. A *morphism* $f : E \to E'$ between two vector bundles E, E' on the base M (also called a *bundle map*) is a smooth map such that the following diagram is commutative

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "\pi"'] & E' \arrow[d, "\pi'"] \\ & M \end{tikzcd} \tag{1.35}$$ eq:diagr1

where f is linear on fibers. Here π and π' denote the canonical projections.

Definition 1.39. Let $\pi : E \to M$ be a smooth vector bundle over M. A *section* of E is a smooth map[1] $\sigma : A \subset M \to E$ satisfying $\pi \circ \sigma = \mathrm{Id}_A$. In other words $\sigma(q)$ belongs to E_q for each $q \in A$, smoothly with respect to q. If σ is defined on all M it is said to be a *global section*.

ex:zerosec **Example 1.40.** Let $\pi : E \to M$ be a smooth vector bundle over M. The *zero section* of E is the global section

$$\zeta : M \to E, \qquad \zeta(q) = 0 \in E_q, \qquad \forall\, q \in M.$$

We will denote by $M_0 := \zeta(M) \subset E$.

[1] as a map between manifolds.

Remark 1.41. Notice that vector fields and differential forms are, by definition, sections of the vector bundles TM and T^*M respectively. induced bundle

nducedbundle **Definition 1.42.** Let $\varphi : M \to N$ be a smooth map between smooth manifolds and E be a vector bundle on N, with fibers $\{E_{q'}, q' \in N\}$. The *induced bundle* φ^*E is a vector bundle on the base M defined by

$$\varphi^*E := \{(q, v) \mid q \in M, v \in E_{\varphi(q)}\} \subset M \times E.$$

Notice that $\operatorname{rank} \varphi^*E = \operatorname{rank} E$, hence $\dim \varphi^*E = \dim M + \operatorname{rank} E$.

ex:sphb **Example 1.43.** (i). Let M be a smooth manifold and TM its tangent bundle, endowed with an Euclidean structure. The *spherical bundle* SM is the vector subbundle of TM defined as follows

$$SM = \bigcup_{q \in M} S_qM, \qquad S_qM = \{v \in T_qM \mid |v| = 1\}.$$

(ii). Let E, E' be two vector bundles over a smooth manifold M. The *direct sum* $E \oplus E'$ is the vector bundle over M defined by

$$(E \oplus E')_q := E_q \oplus E'_q.$$

1.6 Submersions and level sets of smooth maps

If $\varphi : M \to N$ is a smooth map, we define the *rank of* φ at $q \in M$ to be the rank of the linear map $\varphi_{*,q} : T_qM \to T_{\varphi(q)}N$. It is of course just the rank of the matrix of partial derivatives of φ in any coordinate chart, or the dimension of $\operatorname{Im}(\varphi_{*,q}) \subset T_{\varphi(q)}N$. If φ has the same rank k at every point, we say φ has *constant rank*, and write $\operatorname{rank} \varphi = k$.

An *immersion* is a smooth map $\varphi : M \to N$ with the property that φ_* is injective at each point (or equivalently $\operatorname{rank} \varphi = \dim M$). Similarly, a *submersion* is a smooth map $\varphi : M \to N$ such that φ_* is surjective at each point (equivalently, $\operatorname{rank} \varphi = \dim N$).

t:constrank **Theorem 1.44** (Rank Theorem)**.** . *Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $\varphi : M \to N$ is a smooth map with constant rank k in a neighborhood of $q \in M$. Then there exist coordinates $(x_1, \ldots, x_m)$ centered at q and $(y_1, \ldots, y_n)$ centered at $\varphi(q)$ in which φ has the following coordinate representation:*

$$\varphi(x_1, \ldots, x_m) = (x_1, \ldots, x_k, 0, \ldots, 0). \tag{1.36}$$

Remark 1.45. The previous theorem can be rephrased in the following more invariant way. Let $\varphi : M \to N$ be a smooth map between two smooth manifolds. Then the following are equivalent:

(i) φ has constant rank in a neighborhood of $q \in M$.

(ii) There exist coordinates near $q \in M$ and $\varphi(q) \in N$ in which the coordinate representation of φ is linear.

In the case of a submersion, from Theorem t:constrank 1.44 on can deduce the following result

:submersionc **Corollary 1.46.** *Assume $\varphi : M \to N$ is a smooth submersion at q. Then φ admits a local right inverse at $\varphi(q)$. Moreover φ is open at q. More precisely it exist $\varepsilon > 0$ and $C > 0$ such that*

$$B_{\varphi(q)}(C^{-1}r) \subset \varphi(B_q(r)), \qquad \forall\, r \in [0, \varepsilon[. \tag{1.37}$$ eq:submersio

Remark 1.47. The constant C appearing in (eq:submersionc 1.37) is the norm of the differential of the local right inverse. In the case when φ is a diffeomorphism it can be taken as the norm of the differential of the inverse of φ and we recover the well known statement of the inverse function theorem.

Using these results, one can give some very general criteria for level sets of smooth maps (or smooth functions) to be submanifolds.

t:crlst **Theorem 1.48** (Constant Rank Level Set Theorem)**.** *Let M and N be smooth manifolds, and let $\varphi : M \to N$ be a smooth map with constant rank k. Each level set $\varphi^{-1}(y)$, for $y \in N$ is a closed embedded submanifold of codimension k in M.*

Remark 1.49. It is worth to specify the following two important sub cases of Theorem t:crlst 1.48:

(a) If $\varphi : M \to N$ is a submersion at every $q \in \varphi^{-1}(y)$ for some $y \in N$, then $\varphi^{-1}(y)$ is a closed embedded submanifold whose codimension is equal to the dimension of N.

(b) If $a : M \to \mathbb{R}$ is a smooth function such that $d_q a \neq 0$ for every $q \in a^{-1}(c)$, where $c \in \mathbb{R}$, then the level set $a^{-1}(c)$ is a smooth hypersurface of M

:isamanifold **Exercise 1.50.** Let $a : M \to \mathbb{R}$ be a smooth function. Assume that $c \in \mathbb{R}$ is a regular value of a, i.e., $d_q a \neq 0$ for every $q \in a^{-1}(c)$. Then $N_c = a^{-1}(c) = \{q \in M \,|\, a(q) = c\} \subset M$ is a smooth submanifold. Prove that for every $q \in N_c$

$$T_q N_c = \ker d_q a = \{v \in T_q M \mid \langle d_q a, v\rangle = 0\}.$$

Bibliographical notes

The material presented in this chapter is classical and covered by many textbook in differential geometry, as for instance boothby,lee,docarmo,chavel,spivak [?, ?, ?, ?, ?]

bracket-generat
vector field!bracket-g family
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Chapter 2

Sub-Riemannian structures

c:srbasic

2.1 Basic definitions

In this section we introduce a definition of sub-Riemannian structure which is quite general. Indeed, this definition includes all the classical notions of Riemannian structure, constant-rank sub-Riemannian structure, rank-varying sub-Riemannian structure, almost-Riemannian structure etc.

Definition 2.1. Let M be a smooth manifold and let $\mathcal{F} \subset \mathrm{Vec}(M)$ be a family of smooth vector fields. The *Lie algebra generated* by $\mathcal{F}$ is the smallest sub-algebra of $\mathrm{Vec}(M)$ containing $\mathcal{F}$, namely

$$\mathrm{Lie}\,\mathcal{F} := \mathrm{span}\{[X_1, \dots, [X_{j-1}, X_j]], X_i \in \mathcal{F}, j \in \mathbb{N}\}. \tag{2.1}$$ eq:brgen

We will say that $\mathcal{F}$ is *bracket-generating* (or that satisfies the *Hörmander condition*) if

$$\mathrm{Lie}_q\mathcal{F} := \{X(q), X \in \mathrm{Lie}\,\mathcal{F}\} = T_qM, \quad \forall\, q \in M.$$

d:srm **Definition 2.2. (sub-Riemannian manifold)** Let M be a connected smooth manifold. A *sub-Riemannian structure* on M is a pair $(\mathbf{U}, f)$ where:

(i) $\mathbf{U}$ is an Euclidean bundle with base M and Euclidean fiber U_q, i.e. for every $q \in M$, U_q is a vector space equipped with a scalar product g_q, smooth with respect to q. For $u \in U_q$ we denote the norm of u as $|u| = \sqrt{(u|u)_q}$.

(ii) $f : \mathbf{U} \to TM$ is a smooth map that is a morphism of vector bundles, i.e. the following diagram is commutative (here $\pi_{\mathbf{U}} : \mathbf{U} \to M$ and $\pi : TM \to M$ are the canonical projections)

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[dr, "\pi_{\mathbf{U}}"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd} \tag{2.2}$$ eq:diagr2

and f is *linear* on fibers.

(iii) The *set of horizontal vector fields* $\mathcal{D} := \{f(\sigma), \sigma \text{ smooth section of } \mathbf{U}\}$, is a bracket-generating family of vector fields.

When the vector bundle $\mathbf{U}$ admits a global trivialization we say that $(M, \mathbf{U}, f)$ is a *free sub-Riemannian structure.*

A smooth manifold endowed with a sub-Riemannian structure (i.e. the triple $(M, \mathbf{U}, f)$) is called a *sub-Riemannian manifold.* When the map $f : \mathbf{U} \to TM$ is fiberwise surjective, $(M, \mathbf{U}, f)$) is called a *Riemannian manifold* (cf. Exercise es:riemannian 2.23).

def:iso0 **Definition 2.3.** Let $(M, \mathbf{U}, f)$ be a sub-Riemannian manifold. The *distribution* is the family of subspaces

$$\{\mathcal{D}_q\}_{q \in M}, \qquad \text{where} \qquad \mathcal{D}_q := f(U_q) \subset T_qM.$$

We call $k(q) := \dim \mathcal{D}_q$ the *rank* of the sub-Riemannian structure at $q \in M$. We say that the sub-Riemannian structure $(\mathbf{U}, f)$ on M has *constant rank* if $k(q)$ is constant.

The set of horizontal vector fields $\mathcal{D} \subset \mathrm{Vec}(M)$ has the structure of a finitely generated $\mathcal{C}^\infty(M)$-module, whose elements are vector fields tangent to the distribution at each point, i.e.

$$\mathcal{D}_q = \{X(q) | \, X \in \mathcal{D}\}.$$

The rank of a sub-Riemannian structure $(M, \mathbf{U}, f)$ satisfies

$$k(q) \leq m, \qquad \text{where } m = \operatorname{rank} \mathbf{U}, \tag{2.3}$$

$$k(q) \leq n, \qquad \text{where } n = \dim M. \tag{2.4}$$

In what follows we denote points in $\mathbf{U}$ as pairs (q, u), where $q \in M$ is an element of the base and $u \in U_q$ is an element of the fiber. Following this notation we can write the value of f at this point as

$$f(q, u) \qquad \text{or} \qquad f_u(q).$$

We prefer the second notation to stress that, for each $q \in M$, $f_u(q)$ is a vector in T_qM.

Definition 2.4. (Admissible Curves) A Lipschitz curve $\gamma : [0, T] \to M$ is said to be *admissible* (or *horizontal*) for a sub-Riemannian structure if there exists a measurable essentially bounded function

$$u : t \in [0, T] \mapsto u(t) \in U_{\gamma(t)}, \tag{2.5}$$

called the *control function*, such that

$$\dot{\gamma}(t) = f(\gamma(t), u(t)), \qquad \text{for a.e. } t \in [0, T]. \tag{2.6}$$

eq:intcurve

In this case we say that $u(\cdot)$ is a *control corresponding* to γ. Notice that different controls could correspond to the same trajectory.

Remark 2.5. Once we have chosen a local trivialization $O_q \times \mathbb{R}^m$ for the vector bundle $\mathbf{U}$, where O_q is a neighborhood of a point $q \in M$, we can choose a basis in the fibers and the map f is written $f(q, u) = \sum_{i=1}^m u_i f_i(q)$, where m is the rank of $\mathbf{U}$. In this trivialization, a Lipschitz curve $\gamma : [0, T] \to M$ is *admissible* if there exists $u = (u_1, \ldots, u_m) \in L^\infty([0, T], \mathbb{R}^m)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) f_i(\gamma(t)), \qquad \text{for a.e. } t \in [0, T]. \tag{2.7}$$

eq:cs2

Thanks to this local characterization and Theorem t:cara 1.9, for each initial condition $q \in M$ and $u \in L^\infty([0, T], \mathbb{R}^m)$ there exists an admissible curve γ, defined on a sufficiently small interval, such that u is the control associated with γ and $\gamma(0) = q$.

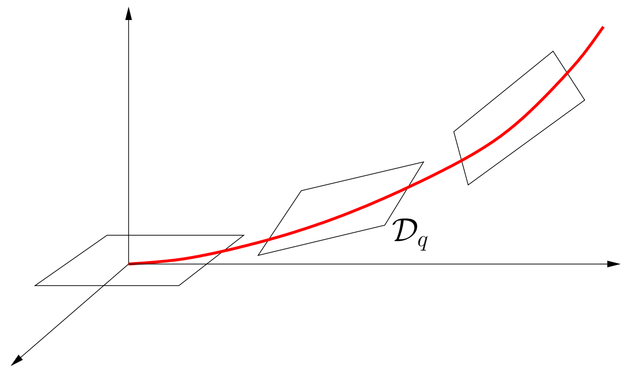


Figure 2.1: An horizontal curve f-1-distr

r:noequiv *Remark* 2.6. Notice that, for a curve to be admissible, it is not sufficient to satisfy $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for almost every $t \in [0, T]$. Take for instance the two free sub-Riemannian structures on $\mathbb{R}^2$ having rank two and defined by

$$f(x, y, u_1, u_2) = (x, y, u_1, u_2 x), \qquad f'(x, y, u_1, u_2) = (x, y, u_1, u_2 x^2). \tag{2.8}$$

and let $\mathcal{D}$ and $\mathcal{D}'$ the corresponding moduli of horizontal vector fields. It is easily seen that the curve $\gamma : [-1, 1] \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$ satisfies $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ and $\dot\gamma(t) \in \mathcal{D}'_{\gamma(t)}$ for every $t \in [-1, 1]$.

Moreover, γ is admissible for f, since its corresponding control is $(u_1, u_2) = (1, 2)$ for a.e. $t \in [-1, 1]$, but it is not admissible for f', since its corresponding control is uniquely determined as $(u_1(t), u_2(t)) = (1, 2/t)$ for a.e. $t \in [-1, 1]$, which is not essentially bounded.

This example shows that, for two different sub-Riemannian structures $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ on the same manifold M, one can have $\mathcal{D}_q = \mathcal{D}'_q$ for every $q \in M$, but $\mathcal{D} \neq \mathcal{D}'$. Notice however that, in the case of constant rank distribution, we have that $\mathcal{D}_q = \mathcal{D}'_q$ for every $q \in M$ if and only if $\mathcal{D} = \mathcal{D}'$.

2.1.1 The minimal control and the length of an admissible curve

We start by defining a norm for vectors that belong to the distribution.

Definition 2.7. Let $v \in \mathcal{D}_q$. We define the sub-Riemannian *norm* of v as follows

$$\|v\| := \min\{|u|,\ u \in U_q \ \text{ s.t. } \ v = f(q, u)\}. \tag{2.9}$$ eq:mincontr

Notice that since f is linear with respect to u, the minimum in (2.9) is always attained at a unique point. Indeed the condition $f(q, \cdot) = v$ defines an affine subspace of U_q (which is nonempty since $v \in \mathcal{D}_q$) and the minimum (2.9) is uniquely attained at the orthogonal projection of the origin onto this subspace (see Figure 2.2).

es:inner **Exercise 2.8.** Show that $\|\cdot\|$ is a norm in $\mathcal{D}_q$. Moreover prove that it satisfies the parallelogram law, i.e. it is induced by a scalar product $\langle \cdot | \cdot \rangle_q$ on $\mathcal{D}_q$, that can be recovered by the polarization identity

$$\langle v | w \rangle_q = \frac{1}{4}\|v + w\|^2 - \frac{1}{4}\|v - w\|^2, \qquad v, w \in \mathcal{D}_q. \tag{2.10}$$

nian!length

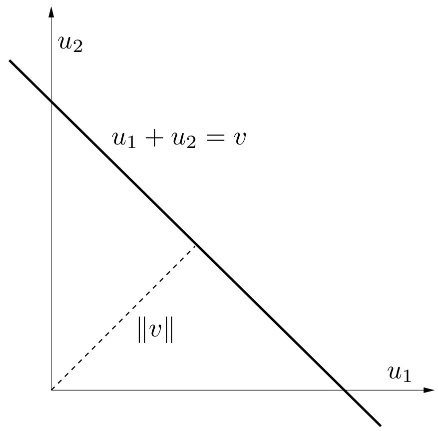


Figure 2.2: The norm of a vector v for $f(x, u_1, u_2) = u_1 + u_2$ f-3-affine

Exercise 2.9. Let $u_1, \ldots, u_m \in U_q$ be an orthonormal basis for U_q. Define $v_i = f(q, u_i)$. Show that if $f(q, \cdot)$ is injective then $v_1, \ldots, v_m$ is an orthonormal basis for $\mathcal{D}_q$.

An admissible curve $\gamma : [0, T] \to M$ is Lipschitz, hence differentiable at almost every point. Hence it is well defined the unique control $t \mapsto u^*(t)$ associated with γ and realizing the minimum.

d:ustar **Definition 2.10.** Given an admissible curve $\gamma : [0, T] \to M$, we say that the control $t \mapsto u^*(t)$ is the *minimal control* associated with γ.

The proof of the following crucial Lemma is postponed to the Section 2.A.

l:measlemma **Lemma 2.11.** *Let $\gamma : [0, T] \to M$ be an admissible curve. Then its minimal control $u^*(\cdot)$ is measurable and essentially bounded.*

We stress that $u^*(t)$ is pointwise defined for a.e. $t \in [0, T]$. In particular, if the admissible curve $\gamma : [0, T] \to M$ is C^1, the minimal control is defined everywhere on $[0, T]$.

r:nocont *Remark* 2.12. Notice that, even if an admissible curve is smooth, its minimal control could be not continuous. Consider, as in Remark 2.6 the free sub-Riemannian structure on $\mathbb{R}^2$

$$f(x, y, u_1, u_2) = (x, y, u_1, u_2 x), \tag{2.11}$$

and let $\gamma : [-1, 1] \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$. Its minimal control $u^*(t)$ satisfies $(u_1^*(t), u_2^*(t)) = (1, 2)$ when $t \neq 0$, while $(u_1^*(0), u_2^*(0)) = (1, 0)$, hence is not continuous.

Thanks to Lemma 2.11 we are allowed to introduce the following definition.

def:mincontr **Definition 2.13.** Let $\gamma : [0, T] \to M$ be an admissible curve. We define the *sub-Riemannian length* of γ as

$$\ell(\gamma) := \int_0^T \|\dot\gamma(t)\| dt. \tag{2.12}$$ eq:deflength

We say that γ is *length-parametrized* if $\|\dot\gamma(t)\| = 1$ for a.e. $t \in [0, T]$. For a length-parametrized curve we have that $\ell(\gamma) = T$.

Notice that (eq:deflength 2.12) says that the length of an admissible curve is the integral of the norm of its minimal control.

$$\ell(\gamma) = \int_0^T |u^*(t)|dt. \tag{2.13}$$ eq:deflength

In particular any admissible curve has finite length.

l:linvariant **Lemma 2.14.** *The length of an admissible curve is invariant by Lipschitz reparametrization.*

Proof. Let $\gamma : [0, T] \to M$ be an admissible curve and $\varphi : [0, T'] \to [0, T]$ a Lipschitz reparametrization, i.e. a Lipschitz and monotone surjective map. Consider the reparametrized curve

$$\gamma_\varphi : [0, T'] \to M, \qquad \gamma_\varphi := \gamma \circ \varphi.$$

First observe that γ_φ is a composition of Lipschitz functions, hence Lipschitz. Moreover γ_φ is admissible since, by the linearity of f, it has minimal control $(u^* \circ \varphi)\dot{\varphi} \in L^\infty$, where u^* is the minimal control of γ. Using the change of variables $t = \varphi(s)$, one gets

$$\ell(\gamma_\varphi) = \int_0^{T'} \|\dot{\gamma}_\varphi(s)\| ds = \int_0^{T'} |u^*(\varphi(s))||\dot{\varphi}(s)| ds = \int_0^T |u^*(t)| dt = \int_0^T \|\dot{\gamma}(t)\| dt = \ell(\gamma). \tag{2.14}$$ eq:chvar

$\square$

l:reparam **Lemma 2.15.** *Every admissible curve of positive length is a Lipschitz reparametrization of a length-parametrized admissible one.*

Proof. Let $\psi : [0, T] \to M$ be an admissible curve with minimal control u^*. Consider the Lipschitz monotone function $\varphi : [0, T] \to [0, \ell(\psi)]$ defined by

$$\varphi(t) := \int_0^t |u^*(\tau)| d\tau.$$

Notice that if $\varphi(t_1) = \varphi(t_2)$, the monotonicity of φ ensures $\psi(t_1) = \psi(t_2)$. Hence we are allowed to define $\gamma : [0, \ell(\psi)] \to M$ by

$$\gamma(s) := \psi(t), \qquad \text{if } s = \varphi(t) \text{ for some } t \in [0, T].$$

In other words, it holds $\psi = \gamma \circ \varphi$. To show that γ is Lipschitz let us first show that there exists a constant $C > 0$ such that, for every $t_0, t_1 \in [0, T]$ one has, in some local coordinates (where $|\cdot|$ denotes the Euclidean norm in coordinates)

$$|\psi(t_1) - \psi(t_0)| \leq C \int_{t_0}^{t_1} |u^*(\tau)| d\tau$$

Indeed

$$\begin{aligned}
|\psi(t_1) - \psi(t_0)| &\leq \int_{t_0}^{t_1} \sum_{i=1}^m |u_i^*(t) f_i(\psi(t))| \, dt \\
&\leq \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^m |u_i^*(t)|^2} \sqrt{\sum_{i=1}^m |f_i(\psi(t))|^2 dt} \\
&\leq C \int_{t_0}^{t_1} |u^*(t)| dt
\end{aligned}$$

an!structure!equivalent where K is a compact set such that $\psi([0,T]) \subset K$ and $C = \max_{x \in K} \left(\sum_{i=1}^{m} |f_i(x)|^2 \right)^{1/2}$. Then if $s_1 = \varphi(t_1)$ and $s_0 = \varphi(t_0)$ one has

$$|\gamma(s_1) - \gamma(s_0)| = |\psi(t_1) - \psi(t_0)| \leq C \int_{t_0}^{t_1} |u^*(\tau)| d\tau = C|s_1 - s_0|,$$

hence γ is Lipschitz. It follows that $\dot{\gamma}(s)$ exists for a.e. $s \in [0, \ell(\psi)]$.

We are going to prove that γ is admissible and its minimal control has norm one. Define for every s such that $s = \varphi(t)$, $\dot{\varphi}(t)$ exists and $\dot{\varphi}(t) \neq 0$, the control

$$v(s) := \frac{u^*(t)}{\dot{\varphi}(t)} = \frac{u^*(t)}{|u^*(t)|}.$$

By Exercise es:mis0 2.16 the control v is defined for a.e. s. Moreover, by construction, $|v(s)| = 1$ for a.e. s and v is the minimal control associated with γ. □

es:mis0 **Exercise 2.16.** Show that for a Lipschitz and monotone function $\varphi : [0,T] \to \mathbb{R}$, the Lebesgue measure of the set $\{s \in \mathbb{R} \,|\, s = \varphi(t), \dot{\varphi}(t) \text{ exists}, \dot{\varphi}(t) = 0\}$ is zero.

By the previuos discussion, in what follows, it will be often convenient to assume that admissible curves are length-parametrized (or parametrized such that $\|\dot{\gamma}(t)\| = \text{const}$).

2.1.2 Equivalence of sub-Riemannian structures

In this section we discuss the notion of equivalence for sub-Riemannian structures on the same base manifold M and the notion of isometry between sub-Riemannian manifolds.

def:iso **Definition 2.17.** Let $(\mathbf{U}, f), (\mathbf{U}', f')$ be two sub-Riemannian structures on a smooth manifold M. They are said to be *equivalent* if the following conditions are satisfied

(*i*) there exist an Euclidean bundle $\mathbf{V}$ and two surjective vector bundle morphisms $p : \mathbf{V} \to \mathbf{U}$ and $p' : \mathbf{V} \to \mathbf{U}'$ such that the following diagram is commutative

$$\begin{tikzcd} & \mathbf{U} \arrow[dr, "f"] & \\ \mathbf{V} \arrow[ur, "p"] \arrow[dr, "p'"'] & & TM \\ & \mathbf{U}' \arrow[ur, "f'"'] & \end{tikzcd} \tag{2.15}$$ eq:diagr3

(*ii*) the projections p, p' are compatible with the scalar product, i.e. it holds

$$|u| = \min\{|v|, p(v) = u\}, \qquad \forall\, u \in \mathbf{U},$$
$$|u'| = \min\{|v|, p'(v) = u'\}, \qquad \forall\, u' \in \mathbf{U}',$$

Remark 2.18. Notice that if $(\mathbf{U}, f), (\mathbf{U}', f')$ are equivalent sub-Riemannian structures on M, then:

(a) the distributions $\mathcal{D}_q$ and $\mathcal{D}'_q$ defined by f and f' coincide, since $f(U_q) = f'(U'_q)$ for all $q \in M$. sub-Riemannia sub-Riemannia rank sub-Riemannia

(b) for each $w \in \mathcal{D}_q$ we have $\|w\| = \|w\|'$, where the norms are induced by $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ respectively.

In particular the length of an admissible curve for two equivalent sub-Riemannian structures is the same.

Remark 2.19. Notice that (i) is satisfied, with the vector bundle $\mathbf{V}$ possibly non Euclidean, if and only if the two moduli of horizontal vector fields $\mathcal{D}$ and $\mathcal{D}'$ defined by $\mathbf{U}$ and $\mathbf{U}'$ (cf. Definition 2.2) are equal.

Definition 2.20. Let M be a sub-Riemannian manifold. We define the *minimal bundle rank* of M as the infimum of rank of bundles that induce equivalent structures on M. Given $q \in M$ the *local minimal bundle rank* of M at q is the minimal bundle rank of the structure restricted on a sufficiently small neighborhood O_q of q.

Exercise 2.21. Prove that the free sub-Riemannian structure on $\mathbb{R}^2$ defined by $f : \mathbb{R}^2 \times \mathbb{R}^3 \to T\mathbb{R}^2$ defined by

$$f(x, y, u_1, u_2, u_3) = (x, y, u_1, u_2 x + u_3 y)$$

has non constant local minimal bundle rank.

For equivalence classes of sub-Riemannian structures we introduce the following definition.

Definition 2.22. Two equivalent classes of sub-Riemannian manifolds are said to be *isometric* if there exist two representatives $(M, \mathbf{U}, f), (M', \mathbf{U}', f')$, a diffeomorphism $\phi : M \to M'$ and an isomorphism[1] of Euclidean bundles $\psi : \mathbf{U} \to \mathbf{U}'$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbf{U} \arrow[r, "f"] \arrow[d, "\psi"'] & TM \arrow[d, "\phi_*"] \\ \mathbf{U}' \arrow[r, "f'"'] & TM' \end{tikzcd} \tag{2.16}$$

eq:diagr4

2.1.3 Examples

s:esempi

Our definition of sub-Riemannian manifold is quite general. In the following we list some classical geometric structures which are included in our setting.

1. **Riemannian structures**.
   Classically a Riemannian manifold is defined as a pair $(M, \langle\cdot|\cdot\rangle)$, where M is a smooth manifold and $\langle\cdot|\cdot\rangle_q$ is a family of scalar product on T_qM, smoothly depending on $q \in M$. This definition is included in Definition 2.2 by taking $\mathbf{U} = TM$ endowed with the Euclidean structure induced by $\langle\cdot|\cdot\rangle$ and $f : TM \to TM$ the identity map.

   s:riemannian

   Exercise 2.23. Show that every Riemannian manifold in the sense of Definition 2.2 is indeed equivalent to a Riemannian structure in the classical sense above (cf. Exercise 2.8).

[1] isomorphism of bundles in the broad sense, it is fiberwise but is not obliged to send fiber in the same fiber.

emannian

2. **Constant rank sub-Riemannian structures.**
   an!structure!free
   Classically a constant rank sub-Riemannian manifold is a triple $(M, D, \langle\cdot|\cdot\rangle)$, where D is a vector subbundle of TM and $\langle\cdot|\cdot\rangle_q$ is a family of scalar product on D_q, smoothly depending on $q \in M$. This definition is included in Definition 2.2 by taking $\mathbf{U} = D$, endowed with its Euclidean structure, and $f : D \hookrightarrow TM$ the canonical inclusion.

3. **Almost-Riemannian structures.**
   An *almost-Riemannian structure* on M is a sub-Riemannian structure $(\mathbf{U}, f)$ on M such that its local minimal bundle rank is equal to the dimension of the manifold, at every point.

ex:tr

4. **Free sub-Riemannian structures.**
   Let $\mathbf{U} = M \times \mathbb{R}^m$ be the trivial Euclidean bundle of rank m on M. A point in $\mathbf{U}$ can be written as (q, u), where $q \in M$ and $u = (u_1, \dots, u_m) \in \mathbb{R}^m$.

   If we denote by $\{e_1, \dots, e_m\}$ an orthonormal basis of $\mathbb{R}^m$, then we can define globally m smooth vector fields on M by $f_i(q) := f(q, e_i)$ for $i = 1, \dots, m$. Then we have

   $$f(q,u) = f\left(q, \sum_{i=1}^{m} u_i e_i\right) = \sum_{i=1}^{m} u_i f_i(q), \qquad q \in M. \tag{2.17}$$

   In this case, the problem of finding an admissible curve joining two fixed points $q_0, q_1 \in M$ and with minimal length is rewritten as the optimal control problem

   $$\begin{cases} \dot\gamma(t) = \displaystyle\sum_{i=1}^{m} u_i(t) f_i(\gamma(t)) \\ \displaystyle\int_0^T |u(t)| dt \to \min \\ \gamma(0) = q_0, \quad \gamma(T) = q_1 \end{cases} \tag{2.18}$$ c-GSR-oc

   For a free sub-Riemannian structure, the set of vector fields $f_1, \dots, f_m$ build as above is called a *generating family*. Notice that, in general, a generating family is not orthonormal when f is not injective.

5. **Surfaces in $\mathbb{R}^3$ as free sub-Riemannian structures**
   Due to topological constraints, in general it not possible to regard a surface as a free sub-Riemannian structure of rank 2, i.e. defined by a pair of globally defined orthonormal vector fields. However, it is always possible to regard it as a free sub-Riemannian structure of rank 3.

   Indeed, for an embedded surface M in $\mathbb{R}^3$, consider the trivial Euclidean bundle $\mathbf{U} = M \times \mathbb{R}^3$, where points are denoted as usual (q, u), with $u \in \mathbb{R}^3, q \in M$, and the map

   $$f : \mathbf{U} \to TM, \qquad f(q,u) = \pi_q^\perp(u) \in T_qM. \tag{2.19}$$ eq:M3d

   where $\pi_q^\perp : \mathbb{R}^3 \to T_qM \subset \mathbb{R}^3$ is the orthogonal projection.

   Notice that f is a surjective bundle map and the set of vector fields $\{\pi_q^\perp(\partial_x), \pi_q^\perp(\partial_y), \pi_q^\perp(\partial_z)\}$ is a generating family for this structure.

Exercise 2.24. Show that $(\mathbf{U}, f)$ defined in (2.19) is equivalent to the Riemannian structure on M induced by the embedding in $\mathbb{R}^3$.

s:trivial

2.1.4 Every sub-Riemannian structure is equivalent to a free one

The purpose of this section is to show that every sub-Riemannian structure $(\mathbf{U}, f)$ on M is equivalent to a sub-Riemannian structure $(\mathbf{U}', f')$ where $\mathbf{U}'$ is a trivial bundle with sufficiently big rank.

l:trivial **Lemma 2.25.** *Let M be a n-dimensional smooth manifold and $\pi : E \to M$ a smooth vector bundle of rank m. Then, there exists a vector bundle $\pi_0 : E_0 \to M$ with $\operatorname{rank} E_0 \leq 2n + m$ such that $E \oplus E_0$ is a trivial vector bundle.*

Proof. Remember that E, as a smooth manifold, has dimension

$$\dim E = \dim M + \operatorname{rank} E = n + m.$$

Consider the map $i : M \hookrightarrow E$ which embeds M into the vector bundle E as the zero section M_0. If we denote with $T_M E$ the vector bundle $i^*(TE)$, i.e. the restriction of TE to the section M_0, we have the isomorphism (as vector bundles on M)

$$T_M E \simeq E \oplus TM. \tag{2.20}$$ eq:isovb

Eq. (2.20) is a consequence of the fact that the tangent to every fibre E_q, being a vector space, is canonically isomorphic to its tangent space $T_q E_q$ so that

$$T_q E = T_q E_q \oplus T_q M \simeq E_q \oplus T_q M, \qquad \forall\, q \in M.$$

By Whitney theorem we have a (nonlinear on fibers, in general) immersion

$$\Psi : E \to \mathbb{R}^N, \qquad \Psi_* : T_M E \subset TE \hookrightarrow T\mathbb{R}^N,$$

for $N = 2(n+m)$, and Ψ_* is injective as bundle map, i.e. $T_M E$ is a sub-bundle of $T\mathbb{R}^N \simeq \mathbb{R}^N \times \mathbb{R}^N$. Thus we can choose as a complement E', the orthogonal bundle (on the base M) with respect to the Euclidean metric in $\mathbb{R}^N$, i.e.

$$E' = \bigcup_{q \in M} E'_q, \qquad E'_q = (T_q E_q \oplus T_q M)^\perp,$$

and considering $E_0 := T_M E \oplus E'$ we have that E_0 is trivial since its fibers are sum of orthogonal complements and by (2.20) we are done.

□

c:tr **Corollary 2.26.** *Every sub-Riemannian structure $(\mathbf{U}, f)$ on M is equivalent to a sub-Riemannian structure $(\mathbf{U}', f')$ where $\mathbf{U}'$ is a trivial bundle.*

Proof. By Lemma 2.25 there exists a vector bundle $\mathbf{U}'$ such that the direct sum $\widetilde{\mathbf{U}} := \mathbf{U} \oplus \mathbf{U}'$ is a trivial bundle. Endow $\mathbf{U}'$ with any metric structure g'. Define a metric on $\widetilde{\mathbf{U}}$ in such a way that $\widetilde{g}(u + u', v + v') = g(u, v) + g'(u', v')$ on each fiber $\widetilde{U}_q = U_q \oplus U'_q$. Notice that U_q and U'_q are orthogonal.

Let us define the sub-Riemannian structure $(\widetilde{\mathbf{U}}, \widetilde{f})$ on M by

$$\widetilde{f} : \widetilde{\mathbf{U}} \to TM, \qquad \widetilde{f} := f \circ p_1,$$

nian!distance
atheodory

where $p_1 : \mathbf{U} \oplus \mathbf{U}' \to \mathbf{U}$ denotes the projection on the first factor. By construction, the diagram

$$
\begin{tikzcd}
 & \widetilde{\mathbf{U}} \arrow[dr, "\widetilde{f}"] & \\
\mathbf{U} \oplus \mathbf{U}' \arrow[ur, "\mathrm{Id}"] \arrow[dr, "p_1"'] & & TM \\
 & \mathbf{U} \arrow[ur, "f"'] &
\end{tikzcd}
\tag{2.21}
$$

eq:diagr5

is commutative. Moreover condition (ii) of Definition 2.17 is satisfied since for every $\widetilde{u} = u + u'$, with $u \in U_q$ and $u' \in U'_q$, we have $|\widetilde{u}|^2 = |u|^2 + |u'|^2$, hence $|u| = \min\{|\widetilde{u}|, p_1(\widetilde{u}) = u\}$. $\square$

Since every sub-Riemannian structure is equivalent to a free one, in what follows we can assume that there exists a global *generating family*, i.e., a family of $f_1, \dots, f_m$ of vector fields globally defined on M such that every admissible curve of the sub-Riemannian structure satisfies

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t) f_i(\gamma(t)), \tag{2.22}$$

eq:global

Moreover, by the classical Gram-Schmidt procedure, we can assume that f_i are the image of an orthonormal frame defined on the fiber. (cf. Example 4 of Section 2.1.3)

Hence the length of an admissible curve γ is given by

$$\ell(\gamma) = \int_0^T |u^*(t)| dt = \int_0^T \sqrt{\sum_{i=1}^{m} u_i^*(t)^2} dt,$$

where $u^*(t)$ is the minimal control.

Notice that Corollary 2.26 implies that the modulus of horizontal vector fields $\mathcal{D}$ is globally generated by $f_1, \dots, f_m$.

r:menodiT *Remark* 2.27. Notice that the integral curve $\gamma(t) = e^{tf_i}$, defined on $[0, T]$, of an element f_i of a generating family $\mathcal{F} = \{f_1, \dots, f_m\}$ is admissible and $\ell(\gamma) \leq T$. If $\mathcal{F} = \{f_1, \dots, f_m\}$ are linearly independent then they are an orthonormal frame and $\ell(\gamma) = T$.

2.2 Sub-Riemannian distance and Chow-Rashevskii Theorem

sec:chow

In this section we introduce the sub-Riemannian distance between two points as the infimum of the length of admissible curves joining them.

Recall that, in the definition of sub-Riemannian manifold, M is assumed to be connected. Moreover, thanks to the construction of Section 2.1.4, in what follows we can assume that the sub-Riemannian structure is free, with generating family $\mathcal{F} = \{f_1, \dots, f_m\}$. Notice that, by definition, $\mathcal{F}$ is assumed to be bracket generating.

def:dist **Definition 2.28.** Let M be a sub-Riemannian manifold and $q_0, q_1 \in M$. The *sub-Riemannian distance* (or *Carnot-Caratheodory distance*) between q_0 and q_1 is

$$\mathsf{d}(q_0, q_1) = \inf\{\ell(\gamma),\ \gamma \text{ admissible},\ \gamma(0) = q_0,\ \gamma(T) = q_1\}, \tag{2.23}$$

eq:dist

One of the purpose of this section is to show that, thanks to the bracket generating condition, eq:dist (2.23) is well-defined since, for every $q_0, q_1 \in M$, there exists an admissible curve that joins q_0 to q_1 and $\mathsf{d}(q_0, q_1) < +\infty$. *theorem!Chow-*

th:chow **Theorem 2.29** (Chow-Raschevskii)**.** *Let M be a sub-Riemannian manifold. Then*

(i) (M, d) is a metric space,

(ii) the topology induced by (M, d) is equivalent to the manifold topology.

In particular, $\mathsf{d} : M \times M \to \mathbb{R}$ is continuous.

In what follows $B(q, r)$ denotes the (open) sub-Riemannian ball of radius r and center q

$$B(q, r) := \{q' \in M \,|\, \mathsf{d}(q, q') < r\}.$$

The rest of this section is devoted to the proof of Theorem th:chow 2.29. To prove Theorem th:chow 2.29 we have to show that d is actually a distance, i.e.,

(a) $0 \le \mathsf{d}(q_0, q_1) < +\infty$ for all $q_0, q_1 \in M$,

(b) $\mathsf{d}(q_0, q_1) = 0$ if and only if $q_0 = q_1$,

(c) $\mathsf{d}(q_0, q_1) = \mathsf{d}(q_1, q_0)$ and $\mathsf{d}(q_0, q_2) \le \mathsf{d}(q_0, q_1) + \mathsf{d}(q_1, q_2)$ for all $q_0, q_1, q_2 \in M$,

and the equivalence between the metric and the manifold topology: for every $q_0 \in M$ we have

(d) for every $\varepsilon > 0$ there exists a neighborhood O_{q_0} of q_0 such that $O_{q_0} \subset B(q_0, \varepsilon)$,

(e) for every neighborhood O_{q_0} of q_0 there exists $\delta > 0$ such that $B(q_0, \delta) \subset O_{q_0}$.

2.2.1 Proof of Chow-Raschevskii Theorem

The symmetry of d is a direct consequence of the fact that if $\gamma : [0, T] \to M$ is admissible, then the curve $\widetilde{\gamma} : [0, T] \to M$ defined by $\widetilde{\gamma}(t) = \gamma(T - t)$ is admissible and $\ell(\widetilde{\gamma}) = \ell(\gamma)$. The triangular inequality follows from the fact that the concatenation of two admissible curves is still admissible. This proves (c).

We divide the rest of the proof of the Theorem in the following steps.

S1. We prove that, for every $q_0 \in M$, there exists a neighborhood O_{q_0} of q_0 such that $\mathsf{d}(q_0, \cdot)$ is finite and continuous in O_{q_0}. This proves (d).

S2. We prove that d is finite on $M \times M$. This proves (a).

S3. We prove (b) and (e).

To prove Step 1 we first need the following lemmas:

l:ugo **Lemma 2.30.** *Let $N \subset M$ be a submanifold and $\mathcal{F} \subset \mathrm{Vec}(M)$ be a family of vector fields tangent to N, i.e. $X(q) \in T_qN, \forall\, q \in N, X \in \mathcal{F}$. Then for all $q \in N$ we have $\mathrm{Lie}_q\mathcal{F} \subset T_qN$. In particular $\dim \mathrm{Lie}_q\mathcal{F} \le \dim N$.*

Proof. Let $X \in \mathcal{F}$. As a consequence of the local existence and uniqueness of the two Cauchy problems

$$\begin{cases} \dot{q} = X(q), & q \in M, \\ q(0) = q_0, & q_0 \in N. \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{q} = X\big|_N(q), & q \in N, \\ q(0) = q_0, & q_0 \in N. \end{cases}$$

it follows that $e^{tX}(q) \in N$ for every $q \in N$ and t small enough.

This property, together with the definition of Lie bracket (see formula (1.26)) implies that, if X, Y are tangent to N, the vector field $[X, Y]$ is tangent to N as well.

Iterating this argument we get that $\mathrm{Lie}_q \mathcal{F} \subset T_q N$ for every $q \in N$, from which the conclusion follows. $\square$

Lemma 2.31. *Let M be an n-dimensional sub-Riemannian manifold with generating family $\mathcal{F} = \{f_1, \ldots, f_m\}$. Then, for every $q_0 \in M$ and every neighborhood V of the origin in $\mathbb{R}^n$ there exist $\widehat{s} = (\widehat{s}_1, \ldots, \widehat{s}_n) \in V$, and a choice of n vector fields $f_{i_1}, \ldots, f_{i_n} \in \mathcal{F}$, such that $\widehat{s}$ is a regular point of the map*

$$\psi : \mathbb{R}^n \to M, \qquad \psi(s_1, \ldots, s_n) = e^{s_n f_{i_n}} \circ \cdots \circ e^{s_1 f_{i_1}}(q_0).$$

Remark 2.32. Notice that, if $\mathcal{D}_{q_0} \neq T_{q_0} M$, then $\widehat{s} = 0$ cannot be a regular point of the map ψ. Indeed in this case, for each choice of the vector fields $f_{i_1}, \ldots, f_{i_n} \in \mathcal{F}$, the image of the differential of ψ at $s = 0$ is $\mathrm{span}_{q_0}\{f_{i_j}, j = 1, \ldots, n\} \subset \mathcal{D}_{q_0}$ and the differential of ψ is not surjective.

We stress that, in the choice of $f_{i_1}, \ldots, f_{i_n} \in \mathcal{F}$, a vector field can appear more than once, as for instance in the case $m < n$.

Proof of Lemma 2.31. We prove the lemma by steps.

1. There exists a vector field $f_{i_1} \in \mathcal{F}$ such that $f_{i_1}(q_0) \neq 0$, otherwise all vector fields in $\mathcal{F}$ vanish at q_0 and $\dim \mathrm{Lie}_{q_0} \mathcal{F} = 0$, which contradicts the bracket generating condition. Then, for $|s|$ small enough, the map

   $$\phi_1 : s_1 \mapsto e^{s_1 f_{i_1}}(q_0),$$

   is a local diffeomorphism onto its image Σ_1. If $\dim M = 1$ the Lemma is proved.

2. Assume $\dim M \geq 2$. Then there exist $t_1^1 \in \mathbb{R}$, with $|t_1^1|$ small enough, and $f_{i_2} \in \mathcal{F}$ such that, if we denote by $q_1 = e^{t_1^1 f_{i_1}}(q_0)$, the vector $f_{i_2}(q_1)$ is not tangent to Σ_1. Otherwise, by Lemma 2.30, $\dim \mathrm{Lie}_q \mathcal{F} = 1$, which contradicts the bracket generating condition. Then the map

   $$\phi_2 : (s_1, s_2) \mapsto e^{s_2 f_{i_2}} \circ e^{s_1 f_{i_1}}(q_0),$$

   is a local diffeomorphism near $(t_1^1, 0)$ onto its image Σ_2. Indeed the vectors

   $$\left.\frac{\partial \phi_2}{\partial s_1}\right|_{(t_1^1, 0)} \in T_{q_1}\Sigma_1, \qquad \left.\frac{\partial \phi_2}{\partial s_2}\right|_{(t_1^1, 0)} = f_{i_2}(q_1),$$

   are linearly independent by construction. If $\dim M = 2$ the Lemma is proved.

3. Assume $\dim M \geq 3$. Then there exist t_2^1, t_2^2, with $|t_2^1 - t_1^1|$ and $|t_2^2|$ small enough, and $f_{i_3} \in \mathcal{F}$ such that, if $q_2 = e^{t_2^2 f_{i_2}} \circ e^{t_2^1 f_{i_1}}(q_0)$ we have that $f_{i_3}(q_2)$ is not tangent to Σ_2. Otherwise, by Lemma 2.30, $\dim \mathrm{Lie}_{q_1} \mathcal{D} = 2$, which contradicts the bracket generating condition. Then the map

   $$\phi_3 : (s_1, s_2, s_3) \mapsto e^{s_3 f_{i_3}} \circ e^{s_2 f_{i_2}} \circ e^{s_1 f_{i_1}}(q_0),$$

is a local diffeomorphism near $(t_2^1, t_2^2, 0)$. Indeed the vectors

$$\left.\frac{\partial \phi_3}{\partial s_1}\right|_{(t_2^1,t_2^2,0)}, \left.\frac{\partial \phi_3}{\partial s_2}\right|_{(t_2^1,t_2^2,0)} \in T_{q_2}\Sigma_2, \qquad \left.\frac{\partial \phi_3}{\partial s_3}\right|_{(t_2^1,t_2^2,0)} = f_{i_3}(q_2),$$

are linearly independent since the last one is transversal to $T_{q_2}\Sigma_2$ by construction, while the first two are linearly independent since $\phi_3(s_1, s_2, 0) = \phi_2(s_1, s_2)$ and ϕ_2 is a local diffeomorphisms at (t_2^1, t_2^2) which is close to $(t_1^1, 0)$.

Repeating the same argument n times (with $n = \dim M$), the lemma is proved. □

Proof of Step 1. Thanks to Lemma 2.31 there exists a neighborhood $\widehat{V} \subset V$ of $\widehat{s}$ such that ψ is a diffeomorphism from $\widehat{V}$ to $\psi(\widehat{V})$, see Figure 2.3. We stress that in general $q_0 = \psi(0)$ is not contained $\psi(\widehat{V})$, cf. Remark 2.32.

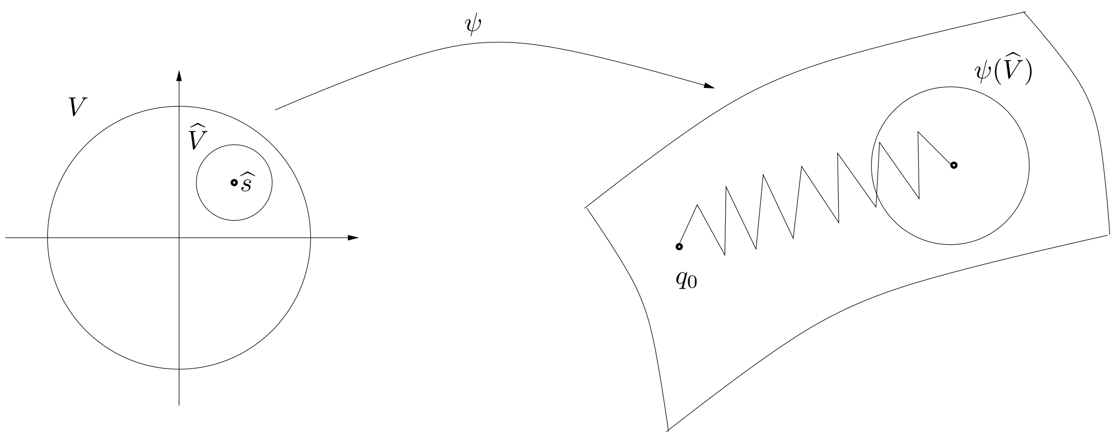


Figure 2.3: Proof of Lemma 2.31

To build a local diffeomorphism whose image contains q_0, we consider the map

$$\widehat{\psi} : \mathbb{R}^n \to M, \qquad \widehat{\psi}(s_1, \dots, s_n) = e^{-\widehat{s}_1 f_{i_1}} \circ \dots \circ e^{-\widehat{s}_n f_{i_n}} \circ \psi(s_1, \dots, s_n),$$

which has the following property: $\widehat{\psi}$ is a diffeomorphism from a neighborhood of $\widehat{s} \in V$, that we still denote $\widehat{V}$, to a neighborhood of $\widehat{\psi}(\widehat{s}) = q_0$.

Fix now $\varepsilon > 0$ and apply the construction above where V is the neighborhood of the origin in $\mathbb{R}^n$ defined by $V = \{s \in \mathbb{R}^n, \sum_{i=1}^n |s_i| < \varepsilon\}$. Let us show that the claim of Step 1 holds with $O_{q_0} = \widehat{\psi}(\widehat{V})$. Indeed, for every $q \in \widehat{\psi}(\widehat{V})$, let $s = (s_1, \dots, s_n)$ such that $q = \widehat{\psi}(s)$, and denote by γ the admissible curve joining q_0 to q, built by $2n$-pieces, as in Figure 2.4.

In other words γ is the concatenation of integral curves of the vector fields f_{i_j}, i.e. admissible curves of the form $t \mapsto e^{tf_{i_j}}(q)$ defined on some interval $[0, T]$, whose length is less or equal than T (cf. Remark 2.27). Since $s, \widehat{s} \in \widehat{V} \subset V$, it follows that:

$$\mathsf{d}(q_0, q) \leq \ell(\gamma) \leq |s_1| + \ldots + |s_n| + |\widehat{s}_1| + \ldots + |\widehat{s}_n| < 2\varepsilon,$$

which ends the proof of Step 1. □

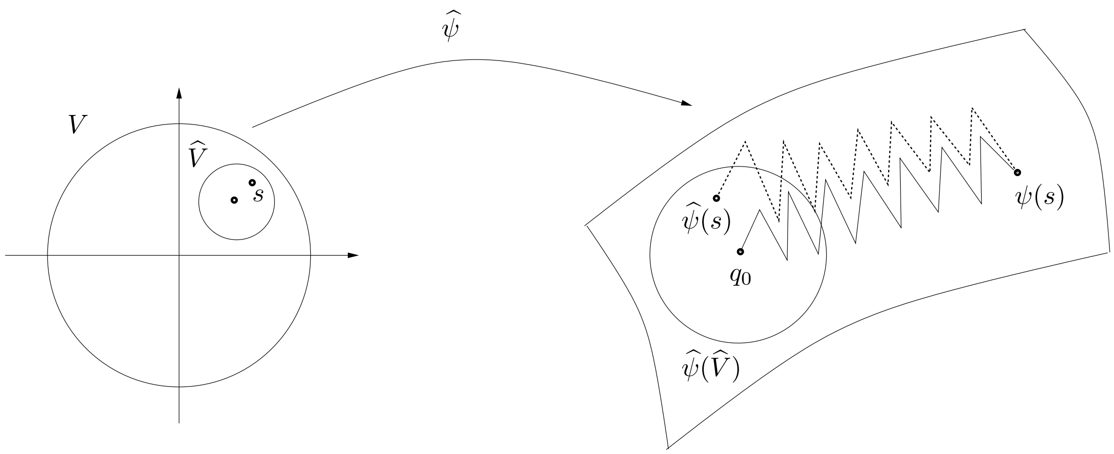


Figure 2.4: The map $\widehat{\psi}$ f-3-lemma-ch

Proof of Step 2. To prove that d is finite on $M \times M$ let us consider the equivalence classes of points in M with respect to the relation

$$q_1 \sim q_2 \qquad \text{if} \qquad \mathsf{d}(q_1, q_2) < +\infty. \tag{2.24}$$

From the triangular inequality and the proof of Step 1, it follows that each equivalence class is open. Moreover, by definition, the equivalence classes are disjoint. Since M is connected, it cannot be the union of open disjoint and nonempty subsets. It follows that there exists only one equivalence class. □

l:deltaK **Lemma 2.33.** *Let $q_0 \in M$ and $K \subset M$ a compact set with $q_0 \in \operatorname{int} K$. Then there exists $\delta_K > 0$ such that every admissible curve γ starting from q_0 and with $\ell(\gamma) \leq \delta_K$ is contained in K.*

Proof. Without loss of generality we can assume that K is contained in a coordinate chart of M, where we denote by $|\cdot|$ the Euclidean norm in the coordinate chart. Let us define

$$C_K := \max_{x \in K} \left(\sum_{i=1}^{m} |f_i(x)|^2 \right)^{1/2} \tag{2.25}$$ eq:CK

and fix $\delta_K > 0$ such that $\operatorname{dist}(q_0, \partial K) > C_K \delta_K$ (here dist is the Euclidean distance in coordinates).

Let us show that for any admissible curve $\gamma : [0, T] \to M$ such that $\gamma(0) = q_0$ and $\ell(\gamma) \leq \delta_K$ we have $\gamma([0, T]) \subset K$. Indeed, if this is not true, there exists an admissible curve $\gamma : [0, T] \to M$

with $\ell(\gamma) \le \delta_K$ and $t^* := \sup\{t \in [0,T], \gamma([0,t]) \subset K\}$, with $t^* < T$. Then

$$\begin{aligned}
|\gamma(t^*) - \gamma(0)| &\le \int_0^{t^*} |\dot\gamma(t)| dt = \int_0^{t^*} \sum_{i=1}^m |u_i^*(t) f_i(\gamma(t))|\, dt && (2.26) \\
&\le \int_0^{t^*} \sqrt{\sum_{i=0}^m |f_i(\gamma(t))|^2}\, \sqrt{\sum_{i=0}^m u_i^*(t)^2}\, dt && (2.27) \\
&\le C_K \int_0^{t^*} \sqrt{\sum_{i=0}^m u_i^*(t)^2}\, dt \le C_K \ell(\gamma) && (2.28) \\
&\le C_K \delta_K < \mathrm{dist}(q_0, \partial K). && (2.29)
\end{aligned}$$

eq:deltaK1 eq:deltaK2

which contradicts the fact that, at t^*, the curve γ leaves the compact K. Thus $t^* = T$. ☐

Proof of Step 3. Let us prove that Lemma 2.33 implies property (b). Indeed the only nontrivial implication is that $\mathsf{d}(q_0, q_1) > 0$ whenever $q_0 \ne q_1$. To prove this, fix a compact neighborhood K of q_0 such that $q_1 \notin K$. By Lemma 2.33, each admissible curve joining q_0 and q_1 has length greater than δ_K, hence $\mathsf{d}(q_0, q_1) \ge \delta_K > 0$.

Let us now prove property (e). Fix $\varepsilon > 0$ and a a compact neighborhood K of q_0. Define C_K and δ_K as in Lemma 2.33, and set $\delta := \min\{\delta_K, \varepsilon/C_K\}$. Let us show that $|q - q_0| < \varepsilon$ whenever $\mathsf{d}(q_0, q) < \delta$, where again $|\cdot|$ is the Euclidean norm in a coordinate chart.

Consider a minimizing sequence $\gamma_n : [0,T] \to M$ of admissible trajectories joining q_0 and q such that $\ell(\gamma_n) \to \mathsf{d}(q_0, q)$ for $n \to \infty$. Without loss of generality, we can assume that $\ell(\gamma_n) \le \delta$ for all n. By Lemma 2.33, $\gamma_n([0,T]) \subset K$ for all n.

We can repeat estimates (2.26)-(2.28) proving that $|q - q_0| = |\gamma_n(T) - \gamma_n(0)| \le C_K \ell(\gamma_n)$ for all n. Passing to the limit for $n \to \infty$, one gets

$$|q - q_0| \le C_K \mathsf{d}(q_0, q) \le C_K \delta < \varepsilon. \tag{2.30}$$

☐

mallballscpt **Corollary 2.34.** *The metric space (M, d) is locally compact, i.e., for any $q \in M$ there exists $\varepsilon > 0$ such that the closed sub-Riemannian ball $\overline{B}(q, r)$ is compact for all $0 \le r \le \varepsilon$.*

Proof. By the continuity of d, the set $\overline{B}(q, r) = \{\mathsf{d}(q, \cdot) \le r\}$ is closed for all $q \in M$ and $r \ge 0$. Moreover the sub-Riemannian metric d induces the manifold topology on M. Hence, for radius small enough, the sub-Riemannian ball is bounded. Thus small sub-Riemannian balls are compact. ☐

2.3 Existence of minimizers

s:filippov

In this section we want to discuss the existence of minimizers of the distance.

Definition 2.35. Let $\gamma : [0,T] \to M$ be an admissible curve. We say that γ is a *length-minimizer* if it minimizes the length among admissible curves with same endpoints, i.e., $\ell(\gamma) = \mathsf{d}(\gamma(0), \gamma(T))$.

Remark 2.36. The example $M = \mathbb{R}^2 \setminus \{0\}$ endowed with the Euclidean distance shows that in general there may be no minimizers between two points. However there may be several minimizers between two fixed points, as it happens for two antipodal points on the sphere S^2.

Before proving the existence of length minimizers we show a general property of the length functional.

Theorem 2.37. *Let γ_n be a sequence of admissible curves on M such that $\gamma_n \to \gamma$ uniformly. Then*

$$\ell(\gamma) \leq \liminf_{n\to\infty} \ell(\gamma_n). \tag{2.31}$$

If moreover $\liminf_{n\to\infty} \ell(\gamma_n) < +\infty$, then γ is also admissible.

Proof. Without loss of generality we assume that γ_n and γ are parametrized with constant speed on the interval $[0,1]$. Moreover, denote $L := \liminf \ell(\gamma_n)$ and choose a subsequence, which we still denote by the same symbol, such that $\ell(\gamma_n) \to L$. If $L = +\infty$ the inequality (2.31) is clearly true, thus assume $L < +\infty$.

Fix $\delta > 0$. By uniform convergence, it is not restrictive to assume that, for n large enough, $\ell(\gamma_n) \leq L + \delta$ and that the image of γ_n are all contained in a common compact set K. Since γ_n is parametrized by constant speed on $[0,1]$ we have that $\dot\gamma_n(t) \in V_{\gamma_n(t)}$ where

$$V_q = \{f_u(q), |u| \leq L + \delta\} \subset T_qM, \qquad f_u(q) = \sum_{i=1}^m u_i f_i(q).$$

Notice that V_q is convex for every $q \in M$, thanks to the linearity of f in u. Let us prove that γ is admissible and satisfies $\ell(\gamma) \leq L + \delta$. Since δ is arbitrary, this implies $\ell(\gamma) \leq L$, that is (2.31).

In local coordinates, we have for every $\varepsilon > 0$

$$\frac{1}{\varepsilon}(\gamma_n(t+\varepsilon) - \gamma_n(t)) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f_{u_n(\tau)}(\gamma_n(\tau))d\tau \in \operatorname{conv}\{V_{\gamma_n(\tau)}, \tau \in [t, t+\varepsilon]\}. \tag{2.32}$$

Moreover, for n sufficiently large, we have for $\tau \in [t, t+\varepsilon]$

$$|\gamma_n(\tau) - \gamma(t)| \leq |\gamma_n(t) - \gamma_n(\tau)| + |\gamma_n(t) - \gamma(t)| \leq C'\varepsilon, \tag{2.33}$$

where C' is independent on n, ε. Indeed $|\gamma_n(t) - \gamma(t)| < \varepsilon$ (by uniform convergence) and an estimate similar to (2.28) gives for $\tau \in [t, t+\varepsilon]$

$$|\gamma_n(t) - \gamma_n(\tau)| \leq \int_t^\tau |\dot\gamma_n(s)| ds \leq C_K(L+\delta)\varepsilon. \tag{2.34}$$

where C_K is the constant (2.25) defined by the compact K. From the estimate (2.33) and the equivalence of the manifold and metric topology we have that, for all $\tau \in [t, t+\varepsilon]$ and n big enough, $\gamma_n(\tau) \in B_{\gamma(t)}(r_\varepsilon)$, where $r_\varepsilon \to 0$ for $\varepsilon \to 0$. In particular

$$\operatorname{conv}\{V_{\gamma_n(\tau)}, \tau \in [t, t+\varepsilon]\} \subset \operatorname{conv}\{V_q, q \in B_{\gamma(t)}(r_\varepsilon)\}, \tag{2.35}$$

Plugging (2.35) in (2.32) and passing to the limit for $n \to \infty$ we get:

$$\frac{1}{\varepsilon}(\gamma(t+\varepsilon) - \gamma(t)) \in \operatorname{conv}\{V_q, q \in B_{\gamma(t)}(r_\varepsilon)\}. \tag{2.36}$$

Assume now that $t \in [0,1]$ is a differentiability point of γ. Then the limit for $\varepsilon \to 0$ in (2.36) gives $\dot\gamma(t) \in \operatorname{conv} V_{\gamma(t)} = V_{\gamma(t)}$. For every such t we can define the unique solution $u^*(t)$ to the problem $\dot\gamma(t) = f(\gamma(t), u^*(t))$ and $|u^*(t)| = \|\dot\gamma(t)\|$. Using the argument contained in Appendix 2.A it follows that $u^*(t)$ is measurable in t. Moreover it is bounded since, by construction, $|u^*(t)| \leq L + \delta$. Hence γ is admissible. Moreover $\ell(\gamma) \leq L + \delta$ since γ is parametrized on $[0,1]$.

□

Corollary 2.38 (Existence of minimizers)**.** *Let M be a sub-Riemannian manifold and $q_0 \in M$. Assume that the ball $\overline{B}_{q_0}(r)$ is compact, for some $r > 0$. Then for all $q_1 \in B_{q_0}(r)$ there exists a length minimizer joining q_0 and q_1, i.e., we have*

$$\mathsf{d}(q_0, q_1) = \min\{\ell(\gamma), \gamma \text{ admissible}, \gamma(0) = q_0, \gamma(T) = q_1\}.$$

Proof. Fix $q_1 \in B_{q_0}(r)$ and consider a minimizing sequence of admissible trajectories $\gamma_n : [0,1] \to M$, parametrized with constant speed, joining q_0 and q_1 and such that $\ell(\gamma_n) \to \mathsf{d}(q_0, q_1)$.

Since $\mathsf{d}(q_0, q_1) < r$, we have $\ell(\gamma_n) \leq r$ for all n large enough, hence we can assume without loss of generality that the image of γ_n is contained in the common compact $K = \overline{B}_{q_0}(r)$ for all n. In particular, the same argument leading to (2.34) shows that for all n

$$|\gamma_n(t) - \gamma_n(\tau)| \leq \int_\tau^t |\dot{\gamma}_n(s)| ds \leq C_K r |t - \tau|, \qquad \forall\, t, \tau \in [0,1]. \tag{2.37}$$

In other words all trajectories in the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ are Lipschitz with the same Lipschitz constant. Thus the sequence is equicontinuous and uniformly bounded.

By the classical Ascoli-Arzelà Theorem there exist a subsequence of γ_n, which we still denote by the same symbol, and a Lipschitz curve $\gamma : [0, T] \to M$ such that we have uniform convergence $\gamma_n \to \gamma$. By Theorem 2.37 the curve γ is admissible and has length $\ell(\gamma) \leq \liminf \ell(\gamma_n) = \mathsf{d}(q_0, q_1)$. □

Corollary 2.39. *Let $q_0 \in M$. Under the hypothesis of Corollary 2.38 there exists $\varepsilon > 0$ such that for all $r \leq \varepsilon$ and $q_1 \in B_{q_0}(r)$ there exists a minimizing curve joining q_0 and q_1.*

Proof. It is a direct consequence of Corollary 2.38 and Corollary 2.34. □

Remark 2.40. It is well known that a length space is complete if and only if all closed balls are compact, see [?, Ch. 2]. In particular, if (M, d) is complete with respect to the sub-Riemannian distance, then for every $q_0, q_1 \in M$ there exists a length minimizer joining q_0 and q_1.

2.4 Pontryagin extremals

In this section we want to give necessary conditions to characterize the length minimizers. To begin with, we would like to motivate our Hamiltonian approach that we develop in the sequel.

In classical Riemannian geometry geodesics are local (in time) length-minimizers, appropriately parametrized. They satisfy a second order differential equation in M, which can be reduced to a first-order differential equation in TM. Hence the set of all geodesics can be parametrized by initial position and velocity.

In our setting (which includes Riemannian and sub-Riemannian geometry) we cannot use the initial velocity to parametrize geodesics. This can be easily understood by a dimensional argument. If the rank of the sub-Riemannian structure is smaller than the dimension of the manifold, the initial velocity $\dot{\gamma}(0)$ of an admissible curve $\gamma(t)$ starting from q_0, belongs to the proper subspace $\mathcal{D}_{q_0}$ of the tangent space $T_{q_0}M$. Hence the set of admissible velocities form a set whose dimension is smaller than the dimension of M, even if, by the Chow and Filippov theorems, geodesics starting from a point q_0 cover a full neighborhood of q_0.

The right approach is to parametrize the geodesics by their initial point and an initial *covector* $\lambda_0 \in T^*_{q_0}M$, which can be thought as the linear form annihilating the "front", i.e. the set $\{\gamma_{q_0}(\varepsilon),$ where γ_{q_0} is a geodesic starting from $q_0\}$ on the corresponding geodesic for $\varepsilon \to 0$.

Next theorem is the first version of Pontryagin maximum principle, whose proof is given in the next section.

Theorem 2.41 (Characterization of Pontryagin extremals)**.** *Let $\gamma : [0,T] \to M$ be an admissible curve which is a length-minimizer, parametrized by constant speed. Let $\widetilde{u}(\cdot)$ be the corresponding minimal control, i.e.,*

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \widetilde{u}_i(t) f_i(\gamma(t)), \qquad \ell(\gamma) = \int_0^T |\widetilde{u}(t)| dt = \mathsf{d}(\gamma(0), \gamma(T)), \qquad |\widetilde{u}(t)| = const.\ a.e.$$

Denote with $P_{0,t}$ the flow[2] *of the nonautonomous vector field $f_{\widetilde{u}(t)} = \sum_{i=1}^k \widetilde{u}_i(t) f_i$. Then there exists $\lambda_0 \in T^*_{\gamma(0)}M$ such that defining*

$$\lambda(t) := (P_{0,t}^{-1})^* \lambda_0, \qquad \lambda(t) \in T^*_{\gamma(t)}M, \tag{2.38}$$

we have that one of the following conditions is satisfied:

(N) $\widetilde{u}_i(t) \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle, \quad \forall\, i = 1, \ldots, m,$

(A) $0 \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle, \quad \forall\, i = 1, \ldots, m.$

Moreover in case (A) one has $\lambda_0 \neq 0$.

Notice that, by definition, the curve $\lambda(t)$ is Lipschitz continuous. Moreover the conditions (N) and (A) are mutually exclusive, unless $\widetilde{u}(t) \equiv 0$ a.e., i.e., γ is the trivial trajectory.

Definition 2.42. Let $\gamma : [0,T] \to M$ be an admissible curve with minimal control $\widetilde{u} \in L^\infty([0,T], \mathbb{R}^m)$. Fix $\lambda_0 \in T^*_{\gamma(0)}M \setminus \{0\}$, and define $\lambda(t)$ by (2.38).

- If $\lambda(t)$ satisfies (N) then it is called *normal extremal* (and $\gamma(t)$ a *normal extremal trajectory*).
- If $\lambda(t)$ satisfies (A) then it is called *abnormal extremal* (and $\gamma(t)$ a *abnormal extremal trajectory*).

Remark 2.43. In the Riemannian case there are no abnormal extremals. Indeed, since the map f is fiberwise surjective, we can always find m vector fields $f_1, \ldots, f_m$ on M such that

$$\mathrm{span}_{q_0}\{f_1, \ldots, f_m\} = T_{q_0}M,$$

and (A) would imply that $\langle \lambda_0, v \rangle = 0$, for all $v \in T_{q_0}M$, that gives the contradiction $\lambda_0 = 0$.

Remark 2.44. If the sub-Riemannian structure is not surjective at q_0, i.e., $\mathrm{span}_{q_0}\{f_1, \ldots, f_m\} \neq T_{q_0}M$, then the trivial trajectory, corresponding to $\widetilde{u}(t) \equiv 0$, is always normal and abnormal.

Notice that even a nontrivial admissible trajectory γ can be both normal and abnormal, since there may exist two different lifts $\lambda(t), \lambda'(t) \in T^*_{\gamma(t)}M$, such that $\lambda(t)$ satisfies (N) and $\lambda'(t)$ satisfies (A).

Exercise 2.45. Prove that condition (N) of Theorem 2.38 implies that the minimal control $\widetilde{u}(t)$ is smooth. In particular normal extremals are smooth.

At this level it seems not obvious how to use Theorem 2.41 to find the explicit expression of extremals for a given problem. In the next chapter we provide another formulation of Theorem 2.41 which gives Pontryagin extremals as solutions of a Hamiltonian system.

The rest of this section is devoted to the proof of Theorem 2.41.

[2] defined for $t \in [0,T]$ and in a neighborhood of $\gamma(0)$

2.4.1 The energy functional

energy functior

Let $\gamma : [0,T] \to M$ be an admissible curve. We define the *energy* functional J as follows

$$J(\gamma) = \frac{1}{2}\int_0^T \|\dot\gamma(t)\|^2 dt.$$

Remark 2.46. Notice that, while ℓ is invariant by reparametrization (see Remark 2.14), J is not. Indeed consider, for every $\alpha > 0$, the reparametrized curve

$$\gamma_\alpha : [0, T/\alpha] \to M, \qquad \gamma_\alpha(t) = \gamma(\alpha t).$$

Using that $\dot\gamma_\alpha(t) = \alpha\,\dot\gamma(\alpha t)$, we have

$$J(\gamma_\alpha) = \frac{1}{2}\int_0^{T/\alpha} \|\dot\gamma_\alpha(t)\|^2 dt = \frac{1}{2}\int_0^{T/\alpha} \alpha^2 \|\dot\gamma(\alpha t)\|^2 dt = \alpha\, J(\gamma).$$

Thus, if the final time is not fixed, the infimum of J, among admissible curves joining two fixed points, is always zero. The following lemma relates minimizers of J with fixed final time with minimizers of ℓ.

l:Jell **Lemma 2.47.** *Fix $T > 0$ and let Ω_{q_0,q_1} be the set of admissible curves joining $q_0, q_1 \in M$. An admissible curve $\gamma : [0,T] \to M$ is a minimizer of J on Ω_{q_0,q_1} if and only if it is a minimizer of ℓ on Ω_{q_0,q_1} and has constant speed.*

Proof. Applying the Cauchy-Schwarz inequality

$$\left(\int_0^T f(t)g(t)dt\right)^2 \leq \int_0^T f(t)^2 dt \int_0^T g(t)^2 dt, \tag{2.39}$$ eq:cs0

with $f(t) = \|\dot\gamma(t)\|$ and $g(t) = 1$ we get

$$\ell(\gamma)^2 \leq 2J(\gamma)T. \tag{2.40}$$ eq:cs

Moreover in (2.39) equality holds if and only if f is proportional to g, i.e. $\|\dot\gamma(t)\| = \text{const.}$ in (2.40). Since, by Lemma 2.15, every curve is a Lipschitz reparametrization of a length-parametrized one, the minima of J are attained at admissible curves with constant speed, and the statement follows. □

2.4.2 Proof of Theorem 2.41

By Lemma 2.47 we can assume that γ is a minimizer of the functional J among admissible curves joining $q_0 = \gamma(0)$ and $q_1 = \gamma(T)$ in fixed time $T > 0$. In particular, if we define the functional

$$\bar J(u(\cdot)) := \frac{1}{2}\int_0^T |u(t)|^2 dt, \tag{2.41}$$ eq:costugo

on the space of controls $u(\cdot) \in L^\infty([0,T], \mathbb{R}^m)$, the minimal control $\widetilde u(\cdot)$ of γ is a minimizer for the energy functional $\bar J$

$$\bar J(\widetilde u(\cdot)) \leq \bar J(u(\cdot)), \quad \forall\, u \in L^\infty([0,T], \mathbb{R}^m),$$

where trajectories corresponding to $u(\cdot)$ join $q_0, q_1 \in M$. In the following we denote the functional $\bar{J}$ by J.

Consider now a variation $u(\cdot) = \widetilde{u}(\cdot) + v(\cdot)$ of the control $\widetilde{u}(\cdot)$, and its associated trajectory $q(t)$, solution of the equation

$$\dot{q}(t) = f_{u(t)}(q(t)), \qquad q(0) = q_0, \tag{2.42}$$

Recall that $P_{0,t}$ denotes the local flow associated with the optimal control $\widetilde{u}(\cdot)$ and that $\gamma(t) = P_{0,t}(q_0)$ is the optimal admissible curve. We stress that in general, for q different from q_0, the curve $t \mapsto P_{0,t}(q)$ is not optimal. Let us introduce the curve $x(t)$ defined by

$$q(t) := P_{0,t}(x(t)). \tag{2.43}$$ eq:change0

In other words $x(t) = P_{0,t}^{-1}(q(t))$ is obtained by applying the inverse of the flow of $\widetilde{u}(\cdot)$ to the solution associated with the new control $u(\cdot)$ (see Figure 2.5). Notice that if $v(\cdot) = 0$, then $x(t) \equiv q_0$.

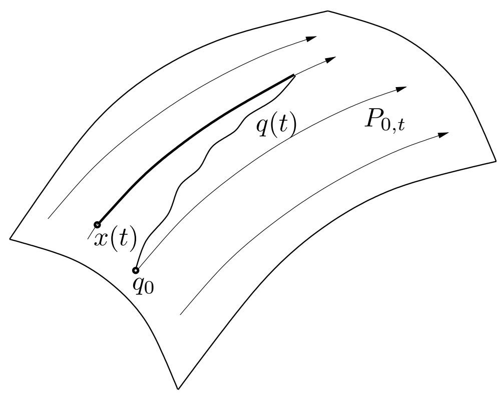


Figure 2.5: The trajectories $q(t)$, associated with $u(\cdot) = \widetilde{u}(\cdot) + v(\cdot)$, and the corresponding $x(t)$. f-3-pmp1

The next step is to write an ODE satisfied by $x(t)$. Differentiating (2.43) we get

$$\begin{aligned} \dot{q}(t) &= f_{\widetilde{u}(t)}(q(t)) + (P_{0,t})_*(\dot{x}(t)) \qquad (2.44)\\ &= f_{\widetilde{u}(t)}(P_{0,t}(x(t)) + (P_{0,t})_*(\dot{x}(t)) \qquad (2.45) \end{aligned}$$ eq:pmpp

and using that $\dot{q}(t) = f_{u(t)}(q(t)) = f_{u(t)}(P_{0,t}(x(t))$ we can invert (2.45) with respect to $\dot{x}(t)$ and rewrite it as follows

$$\begin{aligned} \dot{x}(t) &= (P_{0,t}^{-1})_* \left[(f_{u(t)} - f_{\widetilde{u}(t)})(P_{0,t}(x(t)))\right] \\ &= \left[(P_{0,t}^{-1})_*(f_{u(t)} - f_{\widetilde{u}(t)})\right](x(t)) \\ &= \left[(P_{0,t}^{-1})_*(f_{u(t)-\widetilde{u}(t)})\right](x(t)) \\ &= \left[(P_{0,t}^{-1})_* f_{v(t)}\right](x(t)) \qquad (2.46) \end{aligned}$$ eq:aus1

If we define the nonautonomous vector field $g_{v(t)}^t = (P_{0,t}^{-1})_* f_{v(t)}$ we finally obtain by (2.46) the following Cauchy problem for $x(t)$

$$\dot{x}(t) = g_{v(t)}^t(x(t)), \qquad x(0) = q_0. \tag{2.47}$$ eq:nuova

Notice that the vector field g_v^t is linear with respect to v, since f_u is linear with respect to u. Now we fix the control $v(t)$ and consider the map

$$s \in \mathbb{R} \mapsto \begin{pmatrix} J(\widetilde{u}+sv) \\ x(T;\widetilde{u}+sv) \end{pmatrix} \in \mathbb{R} \times M$$

where $x(T;\widetilde{u}+sv)$ denote the solution at time T of (2.47), starting from q_0, corresponding to control $\widetilde{u}(\cdot)+sv(\cdot)$, and $J(\widetilde{u}+sv)$ is the associated cost.

l:pmp **Lemma 2.48.** *There exists $\bar{\lambda} \in (\mathbb{R} \oplus T_{q_0}M)^*$, with $\bar{\lambda} \neq 0$, such that for all $v \in L^\infty([0,T],\mathbb{R}^m)$*

$$\bar{\lambda} \perp \left(\frac{\partial J(\widetilde{u}+sv)}{\partial s}\Big|_{s=0}, \frac{\partial x(T;\widetilde{u}+sv)}{\partial s}\Big|_{s=0} \right). \tag{2.48}$$ eq:perp

Proof of Lemma 2.48. We argue by contradiction: if (2.48) is not true then there exist $v_0, \dots, v_n \in L^\infty([0,T],\mathbb{R}^m)$ such that the vectors in $\mathbb{R} \oplus T_{q_0}M$

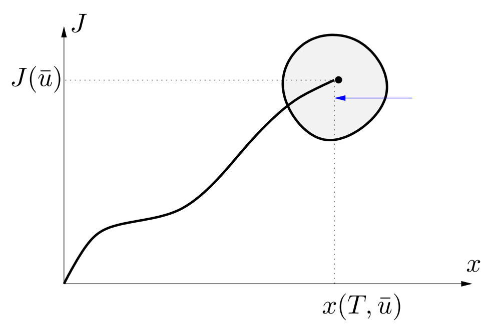
$$\begin{pmatrix} \dfrac{\partial J(\widetilde{u}+sv_0)}{\partial s}\Big|_{s=0} \\ \dfrac{\partial x(T;\widetilde{u}+sv_0)}{\partial s}\Big|_{s=0} \end{pmatrix}, \dots, \begin{pmatrix} \dfrac{\partial J(\widetilde{u}+sv_n)}{\partial s}\Big|_{s=0} \\ \dfrac{\partial x(T;\widetilde{u}+sv_n)}{\partial s}\Big|_{s=0} \end{pmatrix} \tag{2.49}$$ eq:indv

are linearly independent. Let us now consider the map

$$\Phi : \mathbb{R}^{n+1} \to \mathbb{R} \times M, \qquad \Phi(s_0,\dots,s_n) = \begin{pmatrix} J(\widetilde{u}+\sum_{i=0}^n s_i v_i) \\ x(T;\widetilde{u}+\sum_{i=0}^n s_i v_i) \end{pmatrix}. \tag{2.50}$$ eq:maplemma

By differentiability properties of solution of smooth ODEs with respect to parameters, the map (2.50) is smooth. Moreover, since the vectors (2.49) are the components of the differential of Φ and they are independent, then the inverse function theorem implies that Φ is a local diffeomorphism sending a neighborhood of 0 in $\mathbb{R}^{n+1}$ in a neighborhood of $(J(\widetilde{u}), q_0)$ in $\mathbb{R} \times M$. As a result we can find $v(\cdot) = \sum_i s_i v_i(\cdot)$ such that (see also Figure 2.4.2)

$$x(T;\widetilde{u}+v) = q_0, \qquad J(\widetilde{u}+v) < J(\widetilde{u}).$$



f-3-meglio

In other words the curve $t \mapsto q(t;\widetilde{u}+v)$ join $q(0,\widetilde{u}+v) = q_0$ to

$$q(T;\widetilde{u}+v) = P_{0,T}(x(T;\widetilde{u}+v)) = P_{0,T}(q_0) = q_1,$$

with a cost smaller that the cost of $\gamma(t) = q(t,\widetilde{u})$, which is a contradiction $\square$

Notice that if $\bar\lambda$ satisfies (2.48), then for every $\alpha \in \mathbb{R}$, with $\alpha \neq 0$, $\alpha\bar\lambda$ satisfies (2.48) too. Thus we can normalize $\bar\lambda$ to be $(-1, \lambda_0)$ or $(0, \lambda_0)$, with $\lambda_0 \in T^*_{q_0}M$, and $\lambda_0 \neq 0$ in the second case (since $\bar\lambda$ is non zero).

Hence condition (2.48) implies that there exists $\lambda_0 \in T^*_{q_0}M$ such that one of the following identities is satisfied for all $v \in L^\infty([0,T],\mathbb{R}^m)$:

$$\frac{\partial J(\widetilde u + sv)}{\partial s}\Big|_{s=0} = \left\langle \lambda_0, \frac{\partial x(T;\widetilde u + sv)}{\partial s}\Big|_{s=0} \right\rangle, \tag{2.51}$$

$$0 = \left\langle \lambda_0, \frac{\partial x(T;\widetilde u + sv)}{\partial s}\Big|_{s=0} \right\rangle. \tag{2.52}$$

with $\lambda_0 \neq 0$ in the second case. To end the proof we have to show that identities (2.51) and (2.52) are equivalent to conditions (N) and (A) of Theorem 2.41. Let us show that

$$\frac{\partial J(\widetilde u + sv)}{\partial s}\Big|_{s=0} = \int_0^T \sum_{i=1}^m \widetilde u_i(t) v_i(t)dt, \tag{2.53}$$

$$\frac{\partial x(T;\widetilde u + sv)}{\partial s}\Big|_{s=0} = \int_0^T g^t_{v(t)}(q_0)dt = \int_0^T \sum_{i=1}^m ((P_{0,t}^{-1})_* f_i)(q_0) v_i(t)dt. \tag{2.54}$$

Identity (2.53) follows from the definition of J

$$J(\widetilde u + sv) = \frac{1}{2}\int_0^T |\widetilde u + sv|^2 dt, \tag{2.55}$$

while (2.54) can be proved in coordinates. Indeed by (2.47) and the linearity of g_v with respect to v we have

$$x(T;\widetilde u + sv) = q_0 + s\int_0^T g^t_{v(t)}(x(t;\widetilde u + sv))dt,$$

and differentiating with respect to s at $s = 0$ one gets (2.54).

Let us show that (2.51) is equivalent to (N) of Theorem 2.41. Similarly, one gets that (2.52) is equivalent to (A). Using (2.53) and (2.54), equation (2.51) is rewritten as

$$\begin{aligned}\int_0^T \sum_{i=1}^m \widetilde u_i(t) v_i(t)dt &= \int_0^T \sum_{i=1}^m \left\langle \lambda_0, ((P_{0,t}^{-1})_* f_i)(q_0) \right\rangle v_i(t)dt \\ &= \int_0^T \sum_{i=1}^m \langle \lambda(t), f_i(\gamma(t))\rangle v_i(t)dt, \end{aligned} \tag{2.56}$$

where we used, for every $i = 1, \dots, m$, the identities

$$\left\langle \lambda_0, ((P_{0,t}^{-1})_* f_i)(q_0) \right\rangle = \left\langle \lambda_0, (P_{0,t}^{-1})_* f_i(\gamma(t)) \right\rangle = \left\langle (P_{0,t}^{-1})^* \lambda_0, f_i(\gamma(t)) \right\rangle = \langle \lambda(t), f_i(\gamma(t))\rangle.$$

Since $v_i(\cdot) \in L^\infty([0,T],\mathbb{R}^m)$ are arbitrary, we get $\widetilde u_i(t) = \langle \lambda(t), f_i(\gamma(t))\rangle$ for a.e. $t \in [0,T]$.

2.A Measurability of the minimal control

s:measlemma

In this appendix we prove a technical lemma about measurability of solutions to minimization problems. This lemma when specified to the sub-Riemannian context, implies that the minimal control associated with an admissible curve is measurable.

Let us fix an interval $I = [a, b] \subset \mathbb{R}$ and a compact set $U \subset \mathbb{R}^m$. Consider two functions $g : I \times U \to \mathbb{R}^n$, $v : I \to \mathbb{R}^n$ such that

(M1) $g(t, u)$ is measurable with respect to t and continuous with respect to u.

(M2) $v(t)$ is measurable with respect to t.

Moreover we assume that

(M3) for every fixed $t \in I$, the problem $\min\{|u| : g(t, u) = v(t), u \in U\}$ has a unique solution.

Let us denote by $u^*(t)$ the solution of (M3) for fixed $t \in I$.

Lemma 2.49. *The scalar function $t \mapsto |u^*(t)|$ is measurable on I.*

Proof. Denote $\varphi(t) := |u^*(t)|$. To prove the lemma we show that for every fixed $r > 0$ the set

$$A = \{t \in I : \varphi(t) \leq r\}$$

is measurable in $\mathbb{R}$. By our assumptions

$$A = \{t \in I : \exists\, u \in U \text{ s.t. } |u| \leq r, g(t, u) = v(t)\}$$

Let us fix $r > 0$ and a countable dense set $\{u_i\}_{i\in\mathbb{N}}$ in the ball of radius r in U. Let show that

$$A = \bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \underbrace{\bigcup_{i\in\mathbb{N}} A_{i,n}}_{:=A_n} \tag{2.57}$$ eq:1

where

$$A_{i,n} := \{t \in I : \ |g(t, u_i) - v(t)| < 1/n\}$$

Notice that the set $A_{i,n}$ is measurable by construction and if (2.57) is true, A is also measurable.

$\subset$ inclusion. Let $t \in A$. This means that there exists $\bar{u} \in U$ such that $|\bar{u}| \leq r$ and $g(t, \bar{u}) = v(t)$. Since g is continuous with respect to u and $\{u_i\}_{i\in\mathbb{N}}$ is a dense, for each n we can find u_{i_n} such that $|g(t, u_{i_n}) - v(t)| < 1/n$, that is $t \in A_n$ for all n.

$\supset$ inclusion. Assume $t \in \bigcap_{n\in\mathbb{N}} A_n$. Then for every n there exists i_n such that the corresponding u_{i_n} satisfies $|g(t, u_{i_n}) - v(t)| < 1/n$. From the sequence u_{i_n}, by compactness, it is possible to extract a convergent susequence $u_{i_n} \to \bar{u}$. By continuity of g with respect to u one easily gets that $g(t, \bar{u}) = v(t)$. That is $t \in A$. $\square$

Next we exploit the fact that the function $\varphi(t) := |u^*(t)|$ is measurable to show that the vector function $u^*(t)$ is measurable.

ble curve

Lemma 2.50. *The vector function $t \mapsto u^*(t)$ is measurable on I.*

Proof. It is sufficient to prove that, for every closed ball O in $\mathbb{R}^n$ the set

$$B := \{t \in I : u^*(t) \in O\}$$

is measurable. Since the minimum in (M3) is uniquely determined, this is equivalent to

$$B = \{t \in I : \exists\, u \in O \ s.t.\ |u| = \varphi(t), g(t,u) = v(t)\}$$

Let us fix the ball O and a countable dense set $\{u_i\}_{i\in\mathbb{N}}$ in O. Let show that

$$B = \bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \underbrace{\bigcup_{i\in\mathbb{N}} B_{i,n}}_{:=B_n} \tag{2.58}$$

eq:2

where

$$B_{i,n} := \{t \in I :\ |u_i| < \varphi(t) + 1/n, |g(t,u_i) - v(t)| < 1/n; \}$$

Notice that the set $B_{i,n}$ is measurable by construction and if (eq:2 2.58) is true, B is also measurable.

$\subset$ inclusion. Let $t \in B$. This means that there exists $\bar{u} \in O$ such that $|\bar{u}| = \varphi(t)$ and $g(t,\bar{u}) = v(t)$. Since g is continuous with respect to u and $\{u_i\}_{i\in\mathbb{N}}$ is a dense in O, for each n we can find u_{i_n} such that $|g(t,u_{i_n}) - v(t)| < 1/n$ and $|u_{i_n}| < \varphi(t) + 1/n$, that is $t \in B_n$ for all n.

$\supset$ inclusion. Assume $t \in \bigcap_{n\in\mathbb{N}} B_n$. Then for every n it is possible to find i_n such that the corresponding u_{i_n} satisfies $|g(t,u_{i_n}) - v(t)| < 1/n$ and $|u_{i_n}| < \varphi(t) + 1/n$. From the sequence u_{i_n}, by compactness of the closed ball O, it is possible to extract a convergent susequence $u_{i_n} \to \bar{u}$. By continuity of f in u one easily gets that $g(t,\bar{u}) = v(t)$. Moreover $|\bar{u}| \le \varphi(t)$. Hence $|\bar{u}| = \varphi(t)$. That is $t \in B$. $\square$

2.A.1 Proof of Lemma l:measlemma 2.11

Consider an admissible curve $\gamma : [0,T] \to M$ and set $g(t,u) = f(\gamma(t),u)$, $v(t) = \dot{\gamma}(t)$.

Notice that assumptions (M1)-(M3) are satisfied. Indeed (M1) and (M2) follow from the fact that $g(t,u)$ is linear with respect to u and measurable in t. Moreover (M3) is also satisfied by linearity with respect to u of f.

2.B Lipschitz vs Absolutely continuous admissible curves

In these lecture notes sub-Riemannian geometry is developed in the framework of Lipschitz admissible curves (that correspond to the choice of L^∞ controls). However, the theory can be equivalently developed in the framework of H^1 admissible curves (corresponding to L^2 controls) or in the framework of absolutely continuous admissible curves (corresponding to L^1 controls).

Definition 2.51. An absolutely continuous curve $\gamma : [0,T] \to M$ is said to be *AC-admissible* if there exists an L^1 function $u : t \in [0,T] \mapsto u(t) \in U_{\gamma(t)}$ such that $\dot{\gamma}(t) = f(\gamma(t),u(t))$, for a.e. $t \in [0,T]$. We define H^1-admissible curves similarly.

Being the set of absolutely continuous curve bigger than the set of Lipschitz ones, one could expect that the sub-Riemannian distance between two points is smaller when computed among all absolutely continuous admissible curves. However this is not the case thanks to the invariance by reparametrization. Indeed Lemmas 2.14 and 2.15 can be rewritten in the absolutely continuous framework in the following form.

l:linvariantAC **Lemma 2.52.** *The length of an AC-admissible curve is invariant by AC reparametrization.*

l:reparamAC **Lemma 2.53.** *Any AC-admissible curve of positive length is a AC reparametrization of a length-parametrized admissible one.*

The proof of Lemma 2.52 differs from the one of Lemma 2.14 only by the fact that, if $u^* \in L^1$ is the minimal control of γ then $(u^* \circ \varphi)\dot{\varphi}$ is the minimal control associated with $\gamma \circ \varphi$. Moreover $(u^* \circ \varphi)\dot{\varphi} \in L^1$, using the monotonicity of φ. Under these assumptions the change of variables formula (2.14) still holds.

The proof of Lemma 2.53 is unchanged. Notice that the statement of Exercise 2.16 remains true if we replace Lipschitz with absolutely continuous. We stress that the curve γ built in the proof is Lipschitz (since it is length-parametrized).

As a consequence of these results, if we define

$$\mathsf{d}_{AC}(q_0, q_1) = \inf\{\ell(\gamma),\ \gamma\ AC\text{-admissible},\ \gamma(0) = q_0,\ \gamma(T) = q_1\}, \tag{2.59}$$ def:dist

we have the following proposition.

p:ddac **Proposition 2.54.** $\mathsf{d}_{AC}(q_0, q_1) = \mathsf{d}(q_0, q_1)$

Since $L^2([0,T]) \subset L^1([0,T])$, Lemmas 2.52, 2.53 and Proposition 2.54 are valid also in the framework of admissible curves associated with L^2 controls.

Bibliographical notes

Sub-Riemannian manifolds have been introduced, even if with different terminology, in several contexts starting from the end of 60s, see for instance [?, ?, ?, ?, ?]. However, some pioneering ideas were already present in the work of Carathéodory and Cartan. The name sub-Riemannian geometry first appeared in [?].

Classical general references for sub-Riemannian geometry are: [?, ?, ?, ?, ?].

The definition of sub-Riemannian manifold using the language of bundles dates back to [?] ma chiedi agrachev, jean. For the original proof of the Raschevski-Chow theorem see [?, ?]. The proof of existence of sub-Riemannian length minimizer presented here is an adaptation of the proof of Filippov theorem in optimal control. The fact that in sub-Riemannian geometry there exist abnormal length minimizers is due to Montgomery [?, ?]. The fact that the theory can be equivalently developed in AC or Lipschitz setting is well known, a discussion can be found in [?].

The definition of the length by using the minimal control is, up to our best knowledge, original. The problem of the measurability of the minimal control can be seen as a problem of differential inclusion [?]. The characterization of Pontryagin extremals given in Theorem 2.41 is a simplified version of the Pontryagin Maximum Priciple (PMP) [?]. The proof presented here is original and adapted to this setting. For more general versions of PMP see [?, ?]. The fact that every sub-Riemannian structure is equivalent to a free one (cf. Section 2.1.4) is a consequence of classical results on fiber bundles. A different proof in the case of classical (constant rank) distribution was also considered in [?, ?].

Chapter 3

Characterization and local minimality of Pontryagin extremals

:hamiltonian

This chapter is devoted to the study of geometric properties of Pontryagin extremals. To this purpose we first rewrite Theorem p:pmp 2.41 in a more geometric setting, which permits to write a differential equation in T^*M satisfied by Pontryagin extremals and to show that they do not depend on the choice of a generating family. Finally we prove that small pieces of normal extremal trajectories minimize the length.

To this aim, all along this chapter we develop the language of symplectic geometry, starting by the key concept of Poisson bracket.

3.1 Geometric characterization of Pontryagin extremals

In the previuos chapter we proved that if $\gamma : [0,T] \to M$ is a length minimizer on a sub-Riemannian manifold, associated with a control $u(\cdot)$, then there exists $\lambda_0 \in T^*_{\gamma(0)}M$ such that defining

$$\lambda(t) = (P_{0,t}^{-1})^*\lambda_0, \qquad \lambda(t) \in T^*_{\gamma(t)}M, \tag{3.1}$$ eq:lambatH

we have that one of the following conditions is satisfied:

(N) $u_i(t) \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle\,, \quad \forall\, i = 1, \ldots, m,$

(A) $0 \equiv \langle \lambda(t), f_i(\gamma(t)) \rangle\,, \quad \forall\, i = 1, \ldots, m, \qquad \lambda_0 \neq 0.$

Here $P_{0,t}$ denotes the flow associated with the nonautonomous vector field $f_{u(t)} = \sum_{i=1}^m u_i(t) f_i$ and

$$(P_{0,t}^{-1})^* : T_q^*M \to T^*_{P_{0,t}(q)}M. \tag{3.2}$$ eq:inducedfl

is the induced flow on the cotangent space.

The goal of is section is to characterize the curve (eq:lambatH 3.1) as the integral curve of a suitable (non-autonomous) vector field on T^*M. To this purpose, we first show that a vector field on T^*M is completely characterized by its action on function that are affine on fibers. To fix the ideas, we first focus on the case in which $P_{0,t} : M \to M$ is the flow associated with an autonomous vector field $X \in \mathrm{Vec}(M)$, namely $P_{0,t} = e^{tX}$.

3.1.1 Lifting a vector field from M to T^*M

sec:sympl

We start by some preliminary considerations on the algebraic structure of smooth functions on T^*M. As usual $\pi : T^*M \to M$ denotes the canonical projection.

Functions in $\mathcal{C}^\infty(M)$ are in a one-to-one correspondence with functions in $\mathcal{C}^\infty(T^*M)$ that are *constant on fibers* via the map $\alpha \mapsto \pi^*\alpha = \alpha \circ \pi$. In other words we have the isomorphism of algebras

$$\mathcal{C}^\infty(M) \simeq \mathcal{C}^\infty_{\mathsf{cst}}(T^*M) := \{\pi^*\alpha \,|\, \alpha \in \mathcal{C}^\infty(M)\} \subset \mathcal{C}^\infty(T^*M). \tag{3.3}$$

In what follows, with abuse of notation, we often identify the function $\pi^*\alpha \in \mathcal{C}^\infty(T^*M)$ with the function $\alpha \in \mathcal{C}^\infty(M)$.

In a similar way smooth vector fields on M are in a one-to-one correspondence with functions in $\mathcal{C}^\infty(T^*M)$ that are *linear on fibers* via the map $Y \mapsto a_Y$, where $a_Y(\lambda) := \langle \lambda, Y(q)\rangle$ and $q = \pi(\lambda)$.

$$\mathrm{Vec}(M) \simeq \mathcal{C}^\infty_{\mathsf{lin}}(T^*M) := \{a_Y \,|\, Y \in \mathrm{Vec}(M)\} \subset \mathcal{C}^\infty(T^*M). \tag{3.4}$$

Notice that this is an isomorphism as modules over $C^\infty(M)$. Indeed, as $\mathrm{Vec}(M)$ is a module over $C^\infty(M)$, we have that $\mathcal{C}^\infty_{\mathsf{lin}}(T^*M)$ is a module over $C^\infty(M)$ as well. For any $\alpha \in C^\infty(M)$ and $a_X \in \mathcal{C}^\infty_{\mathsf{lin}}(T^*M)$ their product is defined as $\alpha a_X := (\pi^*\alpha)a_X = a_{\alpha X} \in \mathcal{C}^\infty_{\mathsf{lin}}(T^*M)$.

Definition 3.1. We say that a function $a \in \mathcal{C}^\infty(T^*M)$ is *affine on fibers* if there exists two functions $\alpha \in \mathcal{C}^\infty_{\mathsf{cst}}(T^*M)$ and $a_X \in \mathcal{C}^\infty_{\mathsf{lin}}(T^*M)$ such that $a = \alpha + a_X$. In other words

$$a(\lambda) = \alpha(q) + \langle \lambda, X(q)\rangle\,, \qquad q = \pi(\lambda).$$

We denote by $\mathcal{C}^\infty_{\mathsf{aff}}(T^*M)$ the set of affine function on fibers.

r:affine0 *Remark* 3.2. Linear and affine functions on T^*M are particularly important since they reflects the linear structure of the cotangent bundle. In particular every vector field on T^*M, as a derivation of $\mathcal{C}^\infty(T^*M)$, is completely characterized by its action on affine functions,

Indeed for a vector field $V \in \mathrm{Vec}(T^*M)$ and $f \in \mathcal{C}^\infty(T^*M)$, one has that

$$(Vf)(\lambda) = \frac{d}{dt}\bigg|_{t=0} f(e^{tV}(\lambda)) = \langle d_\lambda f, V(\lambda)\rangle\,, \qquad \lambda \in T^*M. \tag{3.5}$$ eq:fVfV

which depends only on the differential of f at the point λ. Hence, for each fixed $\lambda \in T^*M$, to compute (3.5) one can replace the function f with any affine function whose differential at λ coincide with $d_\lambda f$. Notice that such a function is not unique.

Let us now consider the generator of the flow $(P_{0,t}^{-1})^* = (e^{-tX})^*$. Since it satisfies the group law

$$(e^{-tX})^* \circ (e^{-sX})^* = (e^{-(t+s)X})^* \qquad \forall\, t, s \in \mathbb{R},$$

by Lemma 1.10 its generator is an autonomous vector field V_X on T^*M. In other words we have $(e^{-tX})^* = e^{tV_X}$ for all t.

Let us then compute the right hand side of (3.5) when $V = V_X$ and f is either a function constant on fibers or a function linear on fibers.

The action of V_X on functions that are constant on fibers, of the form $\beta \circ \pi$ with $\beta \in \mathcal{C}^\infty(M)$, coincides with the action of X. Indeed we have for all $\lambda \in T^*M$

$$\frac{d}{dt}\bigg|_{t=0} \beta \circ \pi((e^{-tX})^*\lambda)) = \frac{d}{dt}\bigg|_{t=0} \beta(e^{tX}(q)) = (X\alpha)(q), \qquad q = \pi(\lambda). \tag{3.6}$$ eq:V1

For what concerns the action of V_X on functions that are linear on fibers, of the form $a_Y(\lambda) = \langle \lambda, Y(q)\rangle$, we have for all $\lambda \in T^*M$ Poisson bracke

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} a_Y((e^{-tX})^*\lambda) &= \left.\frac{d}{dt}\right|_{t=0} \left\langle (e^{-tX})^*\lambda, Y(e^{tX}(q))\right\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \left\langle \lambda, (e_*^{-tX}Y)(q)\right\rangle = \langle \lambda, [X,Y](q)\rangle \qquad (3.7) \\
&= a_{[X,Y]}(\lambda).
\end{aligned}$$

eq:V2

Hence, by linearity, one gets that the action of V_X on functions of $\mathcal{C}^\infty_{\mathfrak{aff}}(T^*M)$ is

$$V_X(\beta + a_Y) = X\beta + a_{[X,Y]}. \qquad (3.8)$$

eq:V3

As explained in Remark 3.2, formula (3.8) characterizes completely the generator V_X of $(P_{0,t}^{-1})^*$. To find its explicit form we introduce the notion of Poisson bracket.

3.1.2 The Poisson bracket

The purpose of this section is to introduce an operation $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(T^*M)$, called *Poisson bracket*. First we introduce it in $\mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$, where it can be seen as the Lie bracket of vector fields in $\mathrm{Vec}(M)$, seen as elements of $\mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$. Then it is uniquely extended to $\mathcal{C}^\infty_{\mathfrak{aff}}(T^*M)$ and $\mathcal{C}^\infty(T^*M)$ by requiring that it is a derivation of the algebra $\mathcal{C}^\infty(T^*M)$ in each argument.

More precisely we start by the following definition.

Definition 3.3. Let $a_X, a_Y \in \mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$ be associated with vector fields $X, Y \in \mathrm{Vec}(M)$. Their *Poisson bracket* is defined by

$$\{a_X, a_Y\} := a_{[X,Y]}, \qquad (3.9)$$

eq:poisslin

where $a_{[X,Y]}$ is the function in $\mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$ associated with the vector field $[X,Y]$.

Remark 3.4. Recall that the Lie bracket is a bilinear, skew-symmetric map defined on $\mathrm{Vec}(M)$, that satisfies the Leibnitz rule for $X, Y \in \mathrm{Vec}(M)$:

$$[X, \alpha Y] = \alpha[X,Y] + (X\alpha)Y, \qquad \forall\, \alpha \in \mathcal{C}^\infty(M). \qquad (3.10)$$

eq:lieleibni

As a consequence, the Poisson bracket is bilinear, skew-symmetric and satisfies the following relation

$$\{a_X, \alpha\, a_Y\} = \{a_X, a_{\alpha Y}\} = a_{[X,\alpha Y]} = \alpha\, a_{[X,Y]} + (X\alpha)\, a_Y, \qquad \forall\, \alpha \in \mathcal{C}^\infty(M). \qquad (3.11)$$

eq:b1

Notice that this relation makes sense since the product between $\alpha \in \mathcal{C}^\infty_{\mathfrak{cst}}(T^*M)$ and $a_X \in \mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$ belong to $\mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$, i.e. $\alpha a_X = a_{\alpha X}$.

Now we extend this definition on the whole $\mathcal{C}^\infty(T^*M)$.

p:expoisson **Proposition 3.5.** *There exists a unique bilinear and skew-simmetric map*

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(T^*M) \times \mathcal{C}^\infty(T^*M) \to \mathcal{C}^\infty(T^*M)$$

that extends (3.9) on $\mathcal{C}^\infty(T^*M)$, and that is a derivation in each argument, i.e. it satisfies

$$\{a, bc\} = \{a,b\}c + \{a,c\}b, \qquad \forall\, a,b,c \in \mathcal{C}^\infty(T^*M). \qquad (3.12)$$

eq:poissder0

We call this operation the Poisson bracket on $\mathcal{C}^\infty(T^*M)$.

Proof. We start by proving that, as a consequence of the requirement that $\{\cdot,\cdot\}$ is a derivation in each argument, it is uniquely extended to $\mathcal{C}^\infty_{\mathfrak{aff}}(T^*M)$.

By linearity and skew-symmetry we are reduced to compute Poisson brackets of kind $\{a_X,\alpha\}$ and $\{\alpha,\beta\}$, where $a_X \in \mathcal{C}^\infty_{\mathfrak{lin}}(T^*M)$ and $\alpha,\beta \in \mathcal{C}^\infty_{\mathfrak{cst}}(T^*M)$. Using that $a_{\alpha Y} = \alpha a_Y$ and (3.12) one gets

$$\begin{aligned}\{a_X, a_{\alpha Y}\} &= \{a_X, \alpha\, a_Y\} \\ &= \alpha\{a_X, a_Y\} + \{a_X, \alpha\}a_Y. \end{aligned}\tag{3.13}$$

Comparing (3.11) and (3.13) one gets

$$\{a_X, \alpha\} = X\alpha \tag{3.14}$$

Next, using (3.12) and (3.14), one has

$$\{a_{\alpha Y}, \beta\} = \{\alpha\, a_Y, \beta\} = \alpha\{a_Y, \beta\} + \{\alpha,\beta\}a_Y \tag{3.15}$$

$$= \alpha Y\beta + \{\alpha,\beta\}a_Y. \tag{3.16}$$

Using again (3.14) one also has $\{a_{\alpha Y},\beta\} = \alpha Y\beta$, hence $\{\alpha,\beta\} = 0$.

Combining the previous formulas one obtains the following expression for the Poisson bracket between two affine functions on T^*M

$$\{a_X + \alpha, a_Y + \beta\} := a_{[X,Y]} + X\beta - Y\alpha. \tag{3.17}$$

From the explicit formula (3.17) it is easy to see that the Poisson bracket computed at a fixed $\lambda \in T^*M$ depends only on the differential of the two functions $a_X + \alpha$ and $a_Y + \beta$ at λ.

Next we extend this definition to $\mathcal{C}^\infty(T^*M)$ in such a way that it is still a derivation. For $f, g \in \mathcal{C}^\infty(T^*M)$ we define

$$\{f,g\}|_\lambda := \{a_{f,\lambda}, a_{g,\lambda}\}|_\lambda \tag{3.18}$$

where $a_{f,\lambda}$ and $a_{g,\lambda}$ are two functions in $\mathcal{C}^\infty_{\mathfrak{aff}}(T^*M)$ such that $d_\lambda f = d_\lambda(a_{f,\lambda})$ and $d_\lambda g = d_\lambda(a_{g,\lambda})$.

The definition (3.18) is well posed, since if we take two different affine functions $a_{f,\lambda}$ and $a'_{f,\lambda}$ their difference satisfy $d_\lambda(a_{f,\lambda} - a'_{f,\lambda}) = d_\lambda(a_{f,\lambda}) - d_\lambda(a'_{f,\lambda}) = 0$, hence by bilinearity of the Poisson bracket

$$\{a_{f,\lambda}, a_{g,\lambda}\}|_\lambda = \{a'_{f,\lambda}, a_{g,\lambda}\}|_\lambda.$$

Let us now compute the coordinate expression of the Poisson bracket. In canonical coordinates (p,x) in T^*M, if

$$X = \sum_{i=1}^n X_i(x)\frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n Y_i(x)\frac{\partial}{\partial x_i},$$

we have

$$a_X(p,x) = \sum_{i=1}^n p_i X_i(x), \quad a_Y(p,x) = \sum_{i=1}^n p_i Y_i(x).$$

Hamiltonian!ve
vector field!Har

and, denoting $f = a_X + \alpha$, $g = a_Y + \beta$ we have

$$
\begin{aligned}
\{f, g\} &= a_{[X,Y]} + X\beta - Y\alpha \\
&= \sum_{i,j=1}^{n} p_j \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) + X_i \frac{\partial \beta}{\partial p_i} - Y_i \frac{\partial \alpha}{\partial p_i} \\
&= \sum_{i,j=1}^{n} X_i \left(p_j \frac{\partial Y_j}{\partial x_i} + \frac{\partial \beta}{\partial p_i} \right) - Y_i \left(p_j \frac{\partial X_j}{\partial x_i} + \frac{\partial \alpha}{\partial p_i} \right) \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i}.
\end{aligned}
$$

From these computations we get the formula for Poisson brackets of two functions $a, b \in \mathcal{C}^\infty(T^*M)$

$$
\{a, b\} = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial p_i}, \qquad a, b \in \mathcal{C}^\infty(T^*M). \tag{3.19}
$$

eq:poisscoor

eq:poisscoord

The explicit formula (3.19) shows that the extension of the Poisson bracket to $\mathcal{C}^\infty(T^*M)$ is still a derivation. □

:poisslambda

Remark 3.6. We stress that the value $\{a, b\}|_\lambda$ at a point $\lambda \in T^*M$ depends only on $d_\lambda a$ and $d_\lambda b$. Hence the Poisson bracket computed at the point $\lambda \in T^*M$ can be seen as a skew-symmetric and nondegenerate bilinear form

$$
\{\cdot, \cdot\}_\lambda : T^*_\lambda(T^*M) \times T^*_\lambda(T^*M) \to \mathbb{R}.
$$

3.1.3 Hamiltonian vector fields

By construction, the linear operator defined by

$$
\vec{a} : \mathcal{C}^\infty(T^*M) \to \mathcal{C}^\infty(T^*M) \qquad \vec{a}(b) := \{a, b\} \tag{3.20}
$$

eq:hvf

is a derivation of the algebra $\mathcal{C}^\infty(T^*M)$, therefore can be identified with an element of $\mathrm{Vec}(T^*M)$.

eq:hvf

Definition 3.7. The vector field $\vec{a}$ on T^*M defined by (3.20) is called the *Hamiltonian vector field* associated with the smooth function $a \in \mathcal{C}^\infty(T^*M)$.

eq:poisscoord

From (3.19) we can easily write the coordinate expression of $\vec{a}$ for any arbitrary function $a \in \mathcal{C}^\infty(T^*M)$

$$
\vec{a} = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial}{\partial p_i}. \tag{3.21}
$$

The following proposition gives the explicit form of the vector field V on T^*M generating the flow $(P_{0,t}^{-1})^*$.

p:pmpaux1

Proposition 3.8. *Let $X \in \mathrm{Vec}(M)$ be complete and let $P_{0,t} = e^{tX}$. The flow on T^*M defined by $(P_{0,t}^{-1})^* = (e^{-tX})^*$ is generated by the Hamiltonian vector field $\vec{a}_X$, where $a_X(\lambda) = \langle \lambda, X(q) \rangle$ and $q = \pi(\lambda)$.*

Proof. To prove that the generator V of $(P_{0,t}^{-1})^*$ coincides with the vector field $\vec{a}_X$ it is sufficient to show that their action is the same. Indeed, by definition of Hamiltonian vector field, we have

$$\vec{a}_X(\alpha) = \{a_X, \alpha\} = X\alpha$$
$$\vec{a}_X(a_Y) = \{a_X, a_Y\} = a_{[X,Y]}.$$

Hence this action coincides with the action of V as in (3.6) and (3.7). $\square$

Remark 3.9. In coordinates (p, x) if the vector field X is written $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ then $a_X(p, x) = \sum_{i=1}^n p_i X_i$ and the Hamitonian vector field $\vec{a}_X$ is written as follows

$$\vec{a}_X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} - \sum_{i,j=1}^n p_i \frac{\partial X_i}{\partial x_j} \frac{\partial}{\partial p_j}. \tag{3.22}$$

Notice that the projection of $\vec{a}_X$ onto M coincides with X itself, i.e., $\pi_*(\vec{a}_X) = X$.

This construction can be extended to the case of nonautonomous vector fields.

Proposition 3.10. *Let X_t be a nonautonomous vector field and denote by $P_{0,t}$ the flow of X_t on M. Then the nonautonomous vector field on T^*M*

$$V_t := \overrightarrow{a_{X_t}}, \qquad a_{X_t}(\lambda) = \langle \lambda, X_t(q) \rangle ,$$

is the generator of the flow $(P_{0,t}^{-1})^*$.

3.2 The symplectic structure

In this section we introduce the symplectic structure of T^*M following the classical construction. In subsection 3.2.1 we show that the symplectic form can be interpreted as the "dual" of the Poisson bracket, in a suitable sense.

Definition 3.11. The *tautological* (or *Liouville*) 1-form $s \in \Lambda^1(T^*M)$ is defined as follows:

$$s : \lambda \mapsto s_\lambda \in T_\lambda^*(T^*M), \qquad \langle s_\lambda, w \rangle := \langle \lambda, \pi_* w \rangle , \quad \forall\, \lambda \in T^*M,\, w \in T_\lambda(T^*M),$$

where $\pi : T^*M \to M$ denotes the canonical projection.

The name "tautological" comes from its expression in coordinates. Recall that, given a system of coordinates $x = (x_1, \ldots, x_n)$ on M, canonical coordinates (p, x) on T^*M are coordinates for which every element $\lambda \in T^*M$ is written as follows

$$\lambda = \sum_{i=1}^n p_i dx_i.$$

For every $w \in T_\lambda(T^*M)$ we have the following

$$w = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial p_i} + \beta_i \frac{\partial}{\partial x_i} \quad \Longrightarrow \quad \pi_* w = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i},$$

hence we get symplectic stru

$$\langle s_\lambda, w\rangle = \langle \lambda, \pi_* w\rangle = \sum_{i=1}^n p_i\beta_i = \sum_{i=1}^n p_i \langle dx_i, w\rangle = \left\langle \sum_{i=1}^n p_i dx_i, w\right\rangle.$$

In other words the coordinate expression of the Liouville form s at the point λ coincides with the one of λ itself, namely

$$s_\lambda = \sum_{i=1}^n p_i dx_i. \tag{3.23}$$ eq:tautcoord

Exercise 3.12. Let $s \in \Lambda^1(T^*M)$ be the tautological form. Prove that

$$\omega^* s = \omega, \qquad \forall\, \omega \in \Lambda^1(M).$$

(Recall that a 1-form ω is a section of T^*M, i.e. a map $\omega : M \to T^*M$ such that $\pi \circ \omega = id_M$).

Definition 3.13. The differential of the tautological 1-form $\sigma := ds \in \Lambda^2(T^*M)$ is called the *canonical symplectic structure* on T^*M.

By construction σ is a closed 2-form on T^*M. Moreover its expression in canonical coordinates (p, x) shows immediately that is a nondegenerate two form

$$\sigma = \sum_{i=1}^n dp_i \wedge dx_i. \tag{3.24}$$ eq:symplcoor

rem:coord *Remark* 3.14 (The symplectic form in non-canonical coordinates). Given a basis of 1-forms $\omega_1, \ldots, \omega_n$ in $\Lambda^1(M)$, one can build coordinates on the fibers of T^*M as follows.

Every $\lambda \in T^*M$ can be written uniquely as $\lambda = \sum_{i=1}^n h_i\omega_i$. Thus h_i become coordinates on the fibers. Notice that these coordinates are not related to any choice of coordinates on the manifold, as the p were. By definition, in these coordinates, we have

$$s = \sum_{i=1}^n h_i\omega_i, \qquad \sigma = ds = \sum_{i=1}^n dh_i \wedge \omega_i + h_i d\omega_i. \tag{3.25}$$

Notice that, with respect to (3.24) in the expression of σ an extra term appears since, in general, the 1-forms ω_i are not closed.

3.2.1 The symplectic form vs the Poisson bracket

ss:sfvspb

Let V be a finite dimensional vector space and V^* denotes its dual (i.e. the space of linear forms on V). By classical linear algebra arguments one has the following identifications

$$\left\{\begin{matrix}\text{non degenerate}\\ \text{bilinear forms on } V\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{linear invertible maps}\\ V \to V^*\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{non degenerate}\\ \text{bilinear forms on } V^*\end{matrix}\right\}. \tag{3.26}$$ eq:iso123

Indeed to every bilinear form $B : V \times V \to \mathbb{R}$ we can associate a linear map $L : V \to V^*$ defined by $L(v) = B(v, \cdot)$. On the other hand, given a linear map $L : V \to V^*$, we can associate with it a bilinear map $B : V \times V \to \mathbb{R}$ defined by $B(v, w) = \langle L(v), w\rangle$, where $\langle\cdot,\cdot\rangle$ denotes as usual the

n!ODE
n!system

pairing between a vector space and its dual. Moreover B is non-degenerate if and only if the map $B(v\cdot)$ is an isomorphism for every $v \in V$, that is if and only if L is invertible.

The previous argument shows how to identify a bilinear form on B on V with an invertible linear map L from V to V^*. Applying the same reasoning to the linear map L^{-1} one obtain a bilinear map on V^*.

Exercise 3.15. (a). Let $h \in \mathcal{C}^\infty(T^*M)$. Prove that the Hamiltonian vector field $\vec{h} \in \mathrm{Vec}(T^*M)$ satisfies the following identity

$$\sigma(\cdot, \vec{h}(\lambda)) = d_\lambda h, \qquad \forall\, \lambda \in T^*M.$$

(b). Prove that, for every $\lambda \in T^*M$ the bilinear forms σ_λ on $T_\lambda(T^*M)$ and $\{\cdot,\cdot\}_\lambda$ on $T^*_\lambda(T^*M)$ (cf. Remark 3.6) are dual under the identification (3.26). In particular show that

$$\{a, b\} = \vec{a}(b) = \langle db, \vec{a}\rangle = \sigma(\vec{a}, \vec{b}), \qquad \forall\, a, b \in \mathcal{C}^\infty(T^*M). \tag{3.27}$$ eq:defpoiss

Remark 3.16. Notice that σ is nondegenerate, which means that the map $w \mapsto \sigma_\lambda(\cdot, w)$ defines a linear isomorphism between the vector spaces $T_\lambda(T^*M)$ and $T^*_\lambda(T^*M)$. Hence $\vec{h}$ is the vector field canonically associated by the symplectic structure with the differential dh. For this reason $\vec{h}$ is also called *symplectic gradient* of h.

From formula (3.24) we have that in canonical coordinates (p, x) the Hamiltonian vector filed associated with h is expressed as follows

$$\vec{h} = \sum_{i=1}^n \frac{\partial h}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial p_i},$$

and the *Hamiltonian system* $\dot{\lambda} = \vec{h}(\lambda)$ is rewritten as

$$\begin{cases} \dot{x}_i = \dfrac{\partial h}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial h}{\partial x_i} \end{cases}, \qquad i = 1, \ldots, n.$$

We conclude this section with two classical but rather important results:

p:noether **Proposition 3.17.** *A function $a \in \mathcal{C}^\infty(T^*M)$ is a constant of the motion of the Hamiltonian system associated with $h \in \mathcal{C}^\infty(T^*M$ if and only if $\{h, a\} = 0$.*

Proof. Let us consider a solution $\lambda(t) = e^{t\vec{h}}(\lambda_0)$ of the Hamiltonian system associated with $\vec{h}$, with $\lambda_0 \in T^*M$. Let us prove the following formula for the derivative of the function a along the solution

$$\frac{d}{dt}a(\lambda(t)) = \{h, a\}(\lambda(t)). \tag{3.28}$$ eq:derivham

By (3.28) it is easy to see that, if $\{h, a\} = 0$, then the derivative of the function a along the flow vanishes for all t and then a is constant. Conversely, if a is constant along the flow then its derivative vanishes and the Poisson bracket is zero. □

The skew-simmetry of the Poisson brackets immediately implies the following corollary.

c:hconst **Corollary 3.18.** *A function $h \in \mathcal{C}^\infty(T^*M)$ is a constant of the motion of the Hamiltonian system defined by $\vec{h}$.*

3.3 Characterization of normal and abnormal extremals

sec:norabn

extremal!normal sub-Riemannian!Hamiltonian!sub-Riemannian

Now we can rewrite the Pontryagin Maximum Principle (see Theorem 2.41) using the symplectic language developed in the last section.

Given a sub-Riemannian structure on M with generating family $\{f_1, \ldots, f_m\}$, and define the fiberwise linear functions on T^*M associated with these vector fields

$$h_i : T^*M \to \mathbb{R}, \qquad h_i(\lambda) := \langle \lambda, f_i(q) \rangle, \quad i = 1, \ldots, m.$$

p:hampmp **Theorem 3.19** (PMP)**.** *Let $\gamma : [0,T] \to M$ be an admissible curve which is a length-minimizer, parametrized by constant speed. Let $\widetilde{u}(\cdot)$ be the corresponding minimal control. Then there exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma(t)}M$ such that*

$$\dot{\lambda}(t) = \sum_{i=1}^{m} \widetilde{u}_i(t)\vec{h}_i(\lambda(t)), \qquad a.e.\ t \in [0,T], \tag{3.29}$$

eq:hampmp

and one of the following conditions is satisfied:

(N) $h_i(\lambda(t)) \equiv \widetilde{u}_i(t), \quad i = 1, \ldots, m, \ \forall\, t,$

(A) $h_i(\lambda(t)) \equiv 0, \qquad i = 1, \ldots, m, \ \forall\, t.$

Moreover in case (A) one has $\lambda(t) \neq 0$ for all $t \in [0,T]$.

Proof. The statement is a rephrasing of Theorem 2.41, combining Proposition 3.8 and Exercise 3.10. □

Notice that Theorem 3.19 says that normal and abnormal extremals appear as solution of an Hamiltonian system. Nevertheless, this Hamiltonian system is non autonomous and depends on the trajectory itself by the presence of the control $\widetilde{u}(t)$ associated with the extremal trajectory.

Moreover, the actual formulation of Theorem 3.19 for the necessary condition for optimality still does not clarify if the extremals depend on the generating family $\{f_1, \ldots, f_m\}$ for the sub-Riemannian structure. The rest of the section is devoted to the geometric intrinsic description of normal and abnormal extremals.

3.3.1 Normal extremals

In this section we show that normal extremals are characterized as solutions of an *smooth autonomous* Hamiltonian system on T^*M, where the Hamiltonian H is a function that encodes all the informations on the sub-Riemannian structure.

Definition 3.20. Let M be a sub-Riemannian manifold. The *sub-Riemannian Hamiltonian* is the smooth function on T^*M defined as follows

$$H : T^*M \to \mathbb{R}, \qquad H(\lambda) = \max_{u \in U_q} \left(\langle \lambda, f_u(q) \rangle - \frac{1}{2}|u|^2 \right), \quad q = \pi(\lambda). \tag{3.30}$$

eq:srham0

Proposition 3.21. *The sub-Riemannian Hamiltonian H is quadratic on fibers. Moreover, for every generating family $\{f_1, \dots, f_m\}$ of the sub-Riemannian structure, the sub-Riemannian Hamiltonian H is written as follows*

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^{k} \langle \lambda, f_i(q) \rangle^2, \qquad \lambda \in T_q^*M, \quad q = \pi(\lambda). \tag{3.31}$$

Proof. In terms of a generating family $\{f_1, \dots, f_m\}$, the sub-Riemannian Hamiltonian (2.56) is written as follows

$$H(\lambda) = \max_{u \in \mathbb{R}^m} \left(\sum_{i=1}^{m} u_i \langle \lambda, f_i(q) \rangle - \frac{1}{2} \sum_{i=1}^{m} u_i^2 \right). \tag{3.32}$$

Differentiating (3.32) with respect to u_i, one gets that the maximum is attained at $u_i = \langle \lambda, f_i(q) \rangle$, from which formula (3.31) follows. The fact that H is quadratic on fibers then easily follows from (3.31). $\square$

Exercise 3.22. Prove that two equivalent sub-Riemannian structures $(\mathbf{U}, f)$ and $(\mathbf{U}', f')$ on a manifold M define the same Hamiltonian.

Theorem 3.23. *Every normal extremal is a solution of the Hamiltonian system $\dot{\lambda}(t) = \vec{H}(\lambda(t))$. In particular, every normal extremal trajectory is smooth.*

Proof. Denoting, as usual, $h_i(\lambda) = \langle \lambda, f_i(q) \rangle$ for $i = 1, \dots, m$, the functions linear on fibers associated with a generating family and using the identity $\overrightarrow{h_i^2} = 2h_i\vec{h}_i$ (see (3.12)), it follows that

$$\vec{H} = \overrightarrow{\frac{1}{2}\sum_{i=1}^{m} h_i^2} = \sum_{i=1}^{m} h_i \vec{h}_i.$$

In particular, since along a normal extremal $h_i(\lambda(t)) = \widetilde{u}_i(t)$ by condition (N) of Theorem 3.19, one gets

$$\vec{H}(\lambda(t)) = \sum_{i=1}^{m} h_i(\lambda(t)) \vec{h}_i(\lambda(t)) = \sum_{i=1}^{m} \widetilde{u}_i(t) \vec{h}_i(\lambda(t)).$$

$\square$

Remark 3.24. In canonical coordinates $\lambda = (p, x)$, H is quadratic with respect to p and

$$H(p, x) = \frac{1}{2} \sum_{i=1}^{m} \langle p, f_i(x) \rangle^2$$

The Hamiltonian system associated with H, in these coordinates, is written as follows

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \sum_{i=1}^{m} \langle p, f_i(x) \rangle f_i(x) \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -\sum_{i=1}^{m} \langle p, f_i(x) \rangle \langle p, D_x f_i(x) \rangle \end{cases} \tag{3.33}$$

From here it is easy to see that if $\lambda(t) = (p(t), x(t))$ is a solution of (3.33) then also the rescaled extremal $\alpha\lambda(\alpha t) = (\alpha\, p(\alpha t), x(\alpha t))$ is a solution of the same Hamiltonian system, for every $\alpha > 0$.

Lemma 3.25. *Let $\lambda(t)$ be a normal extremal and $\gamma(t) = \pi(\lambda(t))$ be the corresponding normal extremal trajectory. Then for all $t \in [0,T]$* characteristic c

$$\frac{1}{2}\|\dot\gamma(t)\|^2 = H(\lambda(t)).$$

Proof. For every normal extremal $\lambda(t)$ associated with the (minimal) control $u(\cdot)$ we have

$$\frac{1}{2}\|\dot\gamma(t)\|^2 = \frac{1}{2}|u(t)|^2 = \frac{1}{2}\sum_{i=1}^{k} u_i(t)^2 = H(\lambda(t)) \tag{3.34}$$

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where we used the fact that, along a normal extremal, we have the relations for all $t \in [0,T]$

$$u_i(t) = \langle \lambda(t), f_i(\gamma(t))\rangle . \tag{3.35}$$

□

Corollary 3.26. *A normal extremal trajectory is parametrized by constant speed. In particular it is length parametrized if and only if its extremal lift is contained in the level set $H^{-1}(1/2)$.*

Proof. The fact that H is constant along $\lambda(t)$, easily implies by (3.34) that $\|\dot\gamma(t)\|^2$ is constant. Moreover one easily gets that $\|\dot\gamma(t)\| = 1$ if and only if $H(\lambda(t)) \equiv 1/2$.

Moreover, by Remark 3.24, all normal extremal trajectories are reparametrization of length parametrized ones. □

Let $\lambda(t)$ be a normal extremal such that $\lambda(0) = \lambda_0 \in T^*_{q_0}M$. The corresponding normal extremal path $\gamma(t) = \pi(\lambda(t))$ can be written in the exponential notation

$$\gamma(t) = \pi \circ e^{t\vec{H}}(\lambda_0).$$

By the previous discussion length parametrized normal extremal trajectories corresponds to the choice of $\lambda_0 \in H^{-1}(1/2)$.

We end this section by characterizing normal extremal trajectory as characteristic curves of the canonical symplectic form contained in the level sets of H.

Definition 3.27. Let M be a smooth manifold and $\Omega \in \Lambda^k M$ a 2-form. A Lipschitz curve $\gamma : [0,T] \to M$ is said *characteristic* for Ω if for almost every $t \in [0,T]$ it holds

$$\dot\gamma(t) \in \operatorname{Ker}\Omega_{\gamma(t)}, \quad (\text{i.e. } \Omega_{\gamma(t)}(\dot\gamma(t), \cdot) = 0) \tag{3.36}$$

eq:defcaratt

Notice that this notion is independent on the parametrization of the curve.

Proposition 3.28. *Let H be the sub-Riemannian Hamiltonian and assume that $c > 0$ is a regular value of H. Then a curve γ is a characteristic curve of $\sigma|_{H^{-1}(c)}$ if and only if it is the reparametrization of a normal extremal on $H^{-1}(c)$.*

Proof. Recall that if c is a regular value of H, then the set $H^{-1}(c)$ is a smooth $(2n-1)$-dimensional manifold in T^*M.[1] For every $\lambda \in H^{-1}(c)$ let us denote by $E_\lambda = T_\lambda H^{-1}(c)$ its tangent space at this

[1]by Sard Theorem almost every $c > 0$ is regular value.

normal

point. Notice that, by construction, E_λ is an hyperplane (i.e., $\dim E_\lambda = 2n-1$) and $d_\lambda H\big|_{E_\lambda} = 0$. The restriction $\sigma|_{H^{-1}(c)}$ is computed by $\sigma_\lambda|_{E_\lambda}$, for each $\lambda \in H^{-1}(c)$.

One one hand $\ker \sigma_\lambda|_{E_\lambda}$ is non trivial since the dimension of E_λ is odd. On the other hand the symplectic 2-form σ is nondegenerate on T^*M, hence the dimension of $\ker \sigma_\lambda|_{E_\lambda}$ cannot be greater than one. It follows that $\dim \ker \sigma_\lambda|_{E_\lambda} = 1$.

We are left to show that $\ker \sigma_\lambda|_{E_\lambda} = \vec{H}(\lambda)$. Assume that $\ker \sigma_\lambda|_{E_\lambda} = \mathbb{R}\xi$, for some $\xi \in T_\lambda(T^*M)$. By construction, E_λ coincides with the subspace that is skew-orthogonal to ξ, namely

$$E_\lambda = \{w \in T_\lambda(T^*M)) \,|\, \sigma_\lambda(\xi, w) = 0\} = \xi^{\angle}.$$

Since, by antisymmetricity, $\sigma_\lambda(\xi,\xi) = 0$, it follows that $\xi \in E_\lambda$. Moreover, by definition of Hamiltonian vector field $\sigma(\cdot, \vec{H}) = dH$, hence for the restriction to E_λ one has

$$\sigma_\lambda(\cdot, \vec{H}(\lambda))\big|_{E_\lambda} = d_\lambda H\big|_{E_\lambda} = 0. \qquad \square$$

es:Hamdistr **Exercise 3.29.** The sub-Riemannian Hamiltonian encodes all the informations about the distribution and the metric defined on it.

(a) Prove that a vector $v \in T_qM$ is *sub-unit*, i.e. it satisfies $v \in \mathcal{D}_q$ and $\|v\| \leq 1$ if and only if

$$\frac{1}{2}|\langle \lambda, v\rangle|^2 \leq H(\lambda), \qquad \forall \lambda \in T_q^*M.$$

(b) Show that this implies the following characterization for the sub-Riemannian Hamiltonian

$$H(\lambda) = \frac{1}{2}\|\lambda\|^2, \qquad \|\lambda\| = \sup_{v \in \mathcal{D}_q, |v|=1} |\langle \lambda, v\rangle|.$$

When the structure is Riemannian, H is the "inverse" norm defined on the cotangent space.

3.3.2 Abnormal extremals

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In this section we provide a geometric characterization of abnormal extremals. Even if for abnormal extremals it is not possible to determine their a priori regularity, we show that they can be characterized as characteristic curves of the symplectic form. This gives an unified point of view of both class of extremals.

We recall that an abnormal extremal is a non zero solution of the following equations

$$\dot{\lambda}(t) = \sum_{i=1}^m u_i(t)\vec{h}_i(\lambda(t)), \qquad h_i(\lambda(t)) = 0, \; i = 1, \ldots, m.$$

where $\{f_1, \ldots, f_m\}$ is a generating family for the sub-Riemannian structure and $h_1, \ldots, h_m$ are the corresponding functions on T^*M linear on fibers. In particular every abnormal extremal is contained in the set

$$H^{-1}(0) = \{\lambda \in T^*M, \langle \lambda, f_i(q)\rangle = 0, i = 1, \ldots, m, \, q = \pi(\lambda)\}. \tag{3.37}$$ eq:Hlevel0

where H denotes the sub-Riemannian Hamiltonian (eq:Hfi2 3.31).

Proposition 3.30. *Let H be the sub-Riemannian Hamiltonian and assume that $H^{-1}(0)$ is a smooth manifold. Then a curve γ is a characteristic curve of $\sigma|_{H^{-1}(0)}$ if and only if it is the reparametrization of a normal extremal on $H^{-1}(0)$.*

Proof. In this proof we denote for simplicity $N := H^{-1}(0) \subset T^*M$. For every $\lambda \in N$ we have the identity

$$\operatorname{Ker} \sigma_\lambda|_N = T_\lambda N^\angle = \operatorname{span}\{\vec{h}_i(\lambda),\, i = 1, \dots, m\}. \tag{3.38}$$

Indeed, from the definition of N, it follows that

$$\begin{aligned} T_\lambda N &= \{w \in T_\lambda(T^*M) |\ \langle d_\lambda h_i,\, w\rangle = 0,\, i = 1, \dots, m\} \\ &= \{w \in T_\lambda(T^*M) |\, \sigma(w, \vec{h}_i(\lambda)) = 0,\, i = 1, \dots, m\} \\ &= \operatorname{span}\{\vec{h}_i(\lambda),\, i = 1, \dots, m\}^\angle. \end{aligned}$$

and (3.38) follows by taking the skew-orthogonal. Thus $w \in T_\lambda H^{-1}(0)$ if and only if w is a linear combination of the vectors $\vec{h}_i(\lambda)$. This implies that $\lambda(t)$ is a characteristic curve for $\sigma|_{H^{-1}(0)}$ if and only if there exists controls $u_i(\cdot)$ for $i = 1, \dots, m$ such that

$$\dot{\lambda}(t) = \sum_{i=1}^m u_i(t)\vec{h}_i(\lambda(t)). \qquad \square \tag{3.39}$$

The following exercise shows that the assumption of Proposition 3.30 is always satisfied in the case of a regular sub-Riemannian structure.

Exercise 3.31. Assume that the sub-Riemannian structure is *regular*, namely the following assumption holds

$$\dim \mathcal{D}_q = \dim \operatorname{span}_q\{f_1, \dots, f_m\} = \text{const}. \tag{3.40}$$

Shows that, under this assumption, the set $H^{-1}(0)$ defined by (3.37) is a smooth submanifold of T^*M. Notice, however, that 0 is never a regular value of H.

Remark 3.32. From Proposition 3.30 it follows that abnormal extremals *do not* depend on the sub-Riemannian metric, but only on the distribution. Indeed the set $H^{-1}(0)$ is characterized as the annihilator of the distribution

$$H^{-1}(0) = \{\lambda \in T^*M \mid \langle \lambda, v\rangle = 0,\ \forall\, v \in \mathcal{D}_{\pi(\lambda)}\} = \mathcal{D}^\perp \subset T^*M,$$

Here the orthogonal is meant in the duality sense.

Under the regularity assumption (3.40) we can select (at least locally) a basis of 1-forms $\omega_1, \dots, \omega_m$ for the dual of the distribution

$$\mathcal{D}_q^\perp = \operatorname{span}\{\omega_i(q),\, i = 1, \dots, m\}, \tag{3.41}$$

Let us complete this set of 1-forms to a basis $\omega_1, \dots, \omega_n$ of T^*M and consider the induced coordinates $h_1, \dots, h_n$ as defined in Remark 3.14. In these coordinates the restriction of the symplectic structure $\mathcal{D}^\perp$ to is expressed as follows

$$\sigma|_{\mathcal{D}^\perp} = d(s|_{\mathcal{D}^\perp}) = \sum_{i=1}^m dh_i \wedge \omega_i + h_i d\omega_i, \tag{3.42}$$

We stress that the restriction $\sigma|_{\mathcal{D}^\perp}$ can be written only in terms of the elements $\omega_1, \dots, \omega_m$ (and not of a full basis of 1-forms) since the differential d commutes with the restriction.

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3.3.3 Example: codimension one distribution and contact distributions

Let M be a n-dimensional manifold endowed with a constant rank distribution $\mathcal{D}$ of codimension one, i.e., $\dim \mathcal{D}_q = n-1$ for every $q \in M$. In this case $\mathcal{D}$ and $\mathcal{D}^\perp$ are sub-bundles of TM and T^*M respectively and their dimension, as smooth manifolds, are

$$\dim\, \mathcal{D} = \dim M + \operatorname{rank} \mathcal{D} = 2n-1,$$
$$\dim\, \mathcal{D}^\perp = \dim M + \operatorname{rank} \mathcal{D}^\perp = n+1.$$

Since the symplectic form σ is skew-symmetric, by a dimensional argument we easily get that for n even, the restriction $\sigma|_{\mathcal{D}^\perp}$ has always a nontrivial kernel, hence there always exist characteristic curves of $\sigma|_{\mathcal{D}^\perp}$, that correspond to reparametrized abnormal extremals by Proposition 3.30.

Let us consider in more detail the simplest case $n=3$. Assume that there exists a one form $\omega_0 \in \Lambda^1(M)$ such that $\mathcal{D} = \ker \omega$ (this is not restrictive, at least for a local description). Consider a basis of one forms $\omega_0, \omega_1, \omega_2$ such that $\omega_0 := \omega$ and the associated coordinates h_0, h_1, h_2 the coordinate associated to these forms (see Remark 3.14). By (3.42)

$$\sigma|_{\mathcal{D}^\perp} = dh_0 \wedge \omega + h_0\, d\omega, \tag{3.43}$$

and we can easily compute (recall that $\mathcal{D}^\perp$ is 4-dimensional)

$$\sigma \wedge \sigma|_{\mathcal{D}^\perp} = 2h_0\, dh_0 \wedge \omega \wedge d\omega. \tag{3.44}$$ eq:sigma2

l:sigmak **Lemma 3.33.** *Let N be a smooth $2k$-dimensional manifold and $\Omega \in \Lambda^2 M$. Then Ω is nondegenerate on N if and only if $\wedge^k \Omega \neq 0$.*[2]

Definition 3.34. Let M be a three dimensional manifold. We say that a constant rank distribution $\mathcal{D}$ on M of corank one is a *contact distribution* if $\omega \wedge d\omega \neq 0$.

Since M is three dimensional, the differential form $\omega \wedge d\omega$ is a top dimensional form, hence it is meaningful to consider the set, called *Martinet set*

$$\mathfrak{M} = \{q \in M \,|\, (\omega \wedge d\omega)|_q = 0\} \subset M.$$

c:noabn **Corollary 3.35.** *Under the previous assumptions all nontrivial abnormal extremal trajectories are contained in the Martinet set $\mathfrak{M}$. In particular if the structure is contact, there are no nontrivial abnormal extremal trajectories.*

Proof. Assume that the structure is contact. Then $\omega \wedge d\omega \neq 0$ and, thanks to (3.44), it follows that $\sigma \wedge \sigma|_{\mathcal{D}^\perp} \neq 0$. By Lemma 3.33 $\sigma|_{\mathcal{D}^\perp}$ is non degenerate (notice that dh_0 is always independent on $\omega \wedge d\omega$ since they depend on coordinates on the fibers and on the manifold, respectively). This shows that, under the contact assumption, the set $\mathfrak{M}$ is empty and there exists no nontrivial characteristic curve of $\sigma|_{\mathcal{D}^\perp}$. The first part of the statement follows by analogue arguments. $\square$

Remark 3.36. Since M is three dimensional, we can write

$$\omega \wedge d\omega = a dV$$

[2] Here $\wedge^k \Omega = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{k}$.

where $a \in C^\infty(M)$ and dV is some smooth volume form on M, that is a never vanishing 3-form on M. 2D Riemannian

In particular the Martinet set is $\mathfrak{M} = a^{-1}(0)$ and the distribution is *contact* if and only if a is never vanishing. If 0 is a regular value of a, the set $a^{-1}(0)$ defines a two dimensional surface on M, called the *Martinet surface*. Recall that this condition is true for a generic choice of the distribution.

In this case abnormal extremal trajectories can be precisely characterized as the horizontal curves that are contained in the Martinet surface $\mathfrak{M}$. The intersection of the tangent bundle to the surface $\mathfrak{M}$ and the 2-dimensional distribution of admissible velocities defines, generically, a line field on $\mathfrak{M}$. Abnormal extremal trajectories are exactly (reparametrized) integral curves of this line field.

es:hlevel **Exercise 3.37.** Prove that if two smooth Hamiltonians $h_1, h_2 : T^*M \to \mathbb{R}$ define the same level set, i.e. $E = \{h_1 = c_1\} = \{h_2 = c_2\}$ for some $c_1, c_2 \in \mathbb{R}$, then their Hamiltonian flow $\vec{h}_1, \vec{h}_2$ coincide on E, up to a reparametrization.

3.4 Examples

3.4.1 2D Riemannian Geometry

Let M be a 2-dimensional manifold and $f_1, f_2 \in \mathrm{Vec}(M)$ a local orthonormal frame for the Riemannian structure. The problem of finding geodesics in M could be described as the optimal control problem

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q),$$

where length and energy are expressed as

$$\ell(q(\cdot)) = \int_0^T \sqrt{u_1^2 + u_2^2}\, dt, \qquad J(q(\cdot)) = \frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt.$$

Equations of geodesics are projections of integral curves of the sub-Riemannian Hamiltonian in T^*M

$$H = \frac{1}{2}(h_1^2 + h_2^2), \qquad h_i(\lambda) = \langle \lambda, f_i(q) \rangle, \quad i = 1, 2.$$

Now we consider coordinates (q, h_1, h_2) on T^*M. Using the fact that $u_i(t) = h_i(\lambda_t)$ we find the equation on the base

$$\dot{q} = h_1 f_1(q) + h_2 f_2(q). \tag{3.45}$$ eq:2dq

For the equation on the fiber we have (remember that along solutions $\dot{a} = \{H, a\}$)

$$\begin{cases} \dot{h}_1 = \{H, h_1\} = -\{h_1, h_2\} h_2 \\ \dot{h}_2 = \{H, h_2\} = \{h_1, h_2\} h_1. \end{cases} \tag{3.46}$$ eq:2dsist

From here one can see directly that H is constant along solutions. Indeed

$$\dot{H} = h_1 \dot{h}_1 + h_2 \dot{h}_2 = 0.$$

If we require that extremals are parametrized by arclength $u_1(t)^2 + u_2(t)^2 = 1$, we have

$$H = \frac{1}{2} \iff h_1^2 + h_2^2 = 1.$$

It is then convenient to restrict to the spherical bundle SM (see Example ex:sphb 1.43) of coordinates (q, θ) setting

$$h_1 = \cos\theta, \qquad h_2 = \sin\theta.$$

Then equations (eq:2dq 3.45) and (eq:2dsist 3.46) become,

$$\begin{cases} \dot\theta = \{h_1, h_2\} \\ \dot q = \cos\theta f_1(q) + \sin\theta f_2(q). \end{cases} \tag{3.47}$$

eq:sist2

Now, since $\{h_1, h_2\}(\lambda) = \langle \lambda, [f_1, f_2] \rangle$ and setting

$$[f_1, f_2] = a_1 f_1 + a_2 f_2, \qquad a_1, a_2 \in \mathcal{C}^\infty(M),$$

we have $\{h_1, h_2\} = a_1 h_1 + a_2 h_2$ and,

$$\begin{cases} \dot\theta = a_1(q)\cos\theta + a_2(q)\sin\theta \\ \dot q = \cos\theta f_1(q) + \sin\theta f_2(q) \end{cases} \tag{3.48}$$

eq:sist3

In other words we are saying that an arc-length parametrized curve on M (i.e. a curve which satisfies the second equation) is a geodesic if and only if it satisfies the first! Heuristically this suggests that the quantity

$$\dot\theta - a_1(q)\cos\theta - a_2(q)\sin\theta,$$

has some relation with the geodesic curvature on M.

Let μ_1, μ_2 the dual frame of f_1, f_2 (so that $dV = \mu_1 \wedge \mu_2$) and consider the Hamiltonian field in these coordinates

$$\vec{H} = \cos\theta f_1 + \sin\theta f_2 + (a_1\cos\theta + a_2\sin\theta)\partial_\theta. \tag{3.49}$$

eq:hamfield

The Levi-Civita connection on M is expressed by some coefficients (see Chapter ch:surfaces ??)

$$\omega = d\theta + b_1\mu_1 + b_2\mu_2,$$

where $b_i = b_i(q)$. On the other hand geodesics are projections of integral curves of $\vec{H}$ so that

$$\langle \omega, \vec{H} \rangle = 0 \implies b_1 = -a_1, \quad b_2 = -a_2.$$

In particular if we apply $\omega = d\theta - a_1\mu_1 - a_2\mu_2$ to a generic curve (not necessarily a geodesic)

$$\lambda = \cos\theta f_1 + \sin\theta f_2 + \dot\theta\,\partial_\theta,$$

which projects on γ we find geodesic curvature

$$\kappa_g(\gamma) = \dot\theta - a_1(q)\cos\theta - a_2(q)\sin\theta,$$

as we infer above. To end this section we prove a useful formula for the Gaussian curvature of M

Corollary 3.38. *If κ denotes the Gaussian curvature of M we have*

$$\kappa = f_1(a_2) - f_2(a_1) - a_1^2 - a_2^2.$$

Proof. From (eq:gausskappa ??) we have $d\omega = -\kappa dV$ where $dV = \mu_1 \wedge \mu_2$ is the Riemannian volume form. On the other hand, using the following identities isoperimetric p

$$d\mu_i = -a_i \mu_1 \wedge \mu_2, \qquad da_i = f_1(a_i)\mu_1 + f_2(a_i)\mu_2, \quad i = 1, 2.$$

we can compute

$$\begin{aligned} d\omega &= -da_1 \wedge \mu_1 - da_2 \wedge \mu_2 - a_1 d\mu_1 - a_2 d\mu_2 \\ &= -(f_1(a_2) - f_2(a_1) - a_1^2 - a_2^2)\mu_1 \wedge \mu_2. \end{aligned}$$

□

soperimetric

3.4.2 Isoperimetric problem

Let M be a 2-dimensional orientable Riemannian manifold and ν its volume form. Fix $A \in \Lambda^1 M$ and $c \in \mathbb{R}$.
Problem. Fixed $q_0, q_1 \in M$, find (if exists) the minimum:

$$\min \left\{ \ell(\gamma) : \gamma(0) = q_0, \gamma(T) = q_1, \int_\gamma A = c \right\}. \tag{3.50}$$

eq:iso2d

Remark 3.39. Local minimizers depend only on dA, i.e. if we add an exact term to A we will find same minima for the problem (with a different value of c).

Problem 1 can be reformulated as a sub-Riemannian problem on the extended manifold

$$\widehat{M} = \mathbb{R} \times M,$$

in the sense that solutions of the problem (eq:iso2d 3.50) turns to be geodesics for a suitable sub-Riemannian structure on $\widehat{M}$, that we are going to construct.

Define on the extended manifold the 1-form:

$$\omega = dy - A, \qquad \widehat{M} = \{(y, q), y \in \mathbb{R}, q \in M\}.$$

Admissible curves are pairs $z(t) = (y(t), \gamma(t))$ such that $\dot{z}(t) \in \Delta_{z(t)}$, i.e. $\omega(\dot{z}(t)) = 0$. This implies

$$\omega(\dot{z}(t)) = \dot{y}(t) - \langle A, \dot{\gamma}(t) \rangle = 0.$$

In other words $\gamma(t)$ is a curve on M and $y(t)$ satisfies the identity

$$y(t) = y_0 + \int_{\gamma_t} A, \qquad \text{where } \gamma_t = \gamma|_{[0,t]}.$$

In particular we can recover a basis for the distribution

$$\begin{cases} \dot{\gamma} = u_1 f_1 + u_2 f_2 \\ \dot{y} = u_1 \langle A, f_1 \rangle \partial_y + u_2 \langle A, f_2 \rangle \partial_y \end{cases} \Rightarrow \begin{pmatrix} \dot{\gamma} \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} f_1 \\ \langle A, f_1 \rangle \partial_y \end{pmatrix} + u_2 \begin{pmatrix} f_2 \\ \langle A, f_2 \rangle \partial_y \end{pmatrix}, \tag{3.51}$$

and $\mathcal{D} = \operatorname{span}(F_1, F_2)$ where

$$F_1 = f_1 + \langle A, f_1 \rangle \partial_y, \qquad F_2 = f_2 + \langle A, f_2 \rangle \partial_y.$$

Remark 3.40. Notice that the projection of the control system

$$\dot{z} = u_1 F_1(z) + u_2 F_2(z),$$

on the manifold M is

$$\dot{\gamma} = u_1 f_1(\gamma) + u_2 f_2(\gamma),$$

from which follows that the sub-Riemannian length on $\widehat{M}$ coincides exactly with the Riemannian length on M.

We denote with $h_i = \langle \lambda, F_i(q) \rangle$ the Hamiltonians linear on fibers of $T^*\widehat{M}$ and we want to compute normal and abnormal geodesics of this problem.

Normal geodesics

With computations analogous to the 2D case we get MA QUI SIAMO SOTTO O SOPRA? definiamo pure H?

$$\begin{cases} \dot{q} = \cos\theta F_1(q) + \sin\theta F_2(q) \\ \dot{\theta} = \{h_1, h_2\} \end{cases} \tag{3.52}$$

eq:sistiso

where we have to compute $\{h_1, h_2\} = \langle \lambda, [F_1, F_2] \rangle$. We set, as in the previous paragraph:

$$[f_1, f_2] = a_1 f_1 + a_2 f_2, \qquad a_1, a_2 \in \mathcal{C}^\infty(M). \tag{3.53}$$

eq:comma1a2

so that

$$\begin{aligned} [F_1, F_2] &= [f_1 + \langle A, f_1 \rangle \partial_y, f_2 + \langle A, f_2 \rangle \partial_y] \\ &= [f_1, f_2] + (f_1 \langle A, f_2 \rangle - f_2 \langle A, f_1 \rangle)\partial_y \\ \text{(by (3.53))} \quad &= a_1(F_1 - \langle A, f_1 \rangle) + a_2(F_2 - \langle A, f_2 \rangle) + f_1 \langle A, f_2 \rangle - f_2 \langle A, f_1 \rangle)\partial_y \\ &= a_1 F_1 + a_2 F_2 + dA(f_1, f_2)\partial_y. \end{aligned}$$

eq:comma1a2

where in the last equality we use (3.66). eq:cartandw

Let μ_1, μ_2 be the dual forms to f_1 and f_2. We can write $dA = b\mu_1 \wedge \mu_2$, for some $b \in \mathcal{C}^\infty(M)$. Then

$$[F_1, F_2] = a_1 F_1 + a_2 F_2 + b\partial_y.$$

Set now $h_0 := \langle \lambda, \partial_y \rangle$. He have

$$\{h_1, h_2\} = \langle \lambda, [F_1, F_2] \rangle = a_1 h_1 + a_2 h_2 + b h_0.$$

It follows that

$$\begin{cases} \dot{\theta} = a_1 \cos\theta + a_2 \sin\theta + b h_0 \\ \dot{h}_0 = 0 \Rightarrow h_0 = \text{ const.} METTIAMOILCALCOLETTO \end{cases} \tag{3.54}$$

eq:sistiso2

In other words

$$\kappa_g(\gamma) = \dot{\theta} - a_1(q)\cos\theta - a_2(q)\sin\theta = h_0 b. \tag{3.55}$$

eq:curviso2d

C"E" CONFUZIONE TRA LA NOTAZIONE SOTTO E SOTTOSOPRA. MEGLIO PURE METTERE AL DIPENDENZA DAL TEMPO Normal geodesics are curves with geodesic curvature proportional to the function b at every point. mettiamo

Abnormal Extremals Abnormal geodesics are contained in the set of points where $\omega \wedge d\omega = 0$.

$$\begin{aligned}\omega \wedge d\omega &= (dy - A) \wedge (b\mu_1 \wedge \mu_2)\\ &= b dy \wedge \mu_1 \wedge \mu_2.\end{aligned}$$

In other words abnormal geodesics are connected components of $b^{-1}(0)$. They are *independent* on the metric and, in general, they are not normal geodesics.

NON CAPISCO PERCHE' USIAMO omega? viene da prima?

3.4.3 Heisenberg group

In the case $M = \mathbb{R}^2$ and $b = b_0$ costant we have that normal geodesics of this problem are circles on M (and helix on $\widehat{M}$).

The Heisenberg group is a basic example in sub-Riemannian geometry. It is classically defined by the sub-Riemannian structure $(\mathbb{R}^3, \mathcal{D}, \langle\cdot|\cdot\rangle)$ defined by the distribution $\mathcal{D} = \operatorname{span}\{X_1, X_2\}$ given by

$$X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_y, \qquad X_2 = \partial_{x_2} + \frac{x_1}{2}\partial_y.$$

Another possibility is to introduce it as the sub-Riemannian structure defined by the ispoerimetric problem in $M = \mathbb{R}^2$ endowed with the 1-form $A = \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$ (cf. previous section). Notice that $dA = dx_1 \wedge dx_2$ defines the area form on $\mathbb{R}^2$.

On the extended manifold

$$\widehat{M} = \mathbb{R}^3 = \{(x_1, x_2, y)\}$$

the one-form ω takes the form

$$\omega = dy - \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$$

Following the notation of the previous paragraph we can choose as an orthonormal frame for the base $\mathbb{R}^2$ the frame $f_1 = \partial_{x_1}$ and $f_2 = \partial_{x_2}$ so that

$$F_1 = \partial_{x_1} - \frac{x_2}{2}\partial_y, \qquad F_2 = \partial_{x_2} + \frac{x_1}{2}\partial_y.$$

together with

$$[F_1, F_2] = \partial_y, \qquad b = 1$$

Hence, defining $h_i = \langle \lambda, F_i(q) \rangle$ the Hamiltonians linear on fibers of $T^*\widehat{M}$.

$$\{h_1, h_2\} = h_0.$$

The equation of normal geodesics

$$\begin{cases}\dot{q} = \cos\theta F_1(q) + \sin\theta F_2(q)\\ \dot{\theta} = \{h_1, h_2\} = h_0\end{cases} \tag{3.56}$$ eq:sistiso22

It follows that

$$\begin{cases}\dot{\theta} = h_0\\ \dot{h}_0 = 0\end{cases} \qquad \Leftrightarrow \qquad \begin{cases}\theta(t) = \theta_0 + h_0 t\\ h_0(t) = h_0\end{cases} \tag{3.57}$$ eq:sistiso22

Then $u_1(t) = h_1(t) = \cos(\theta_0 + h_0 t)$ and $u_2(t) = h_2(t) = \sin(\theta_0 + h_0 t)$ and by integrating ($x_{1,0} = x_{2,0} = 0$)

$$x_1(t) = \frac{1}{h_0}(\sin(\theta_0 + h_0 t) - \sin(\theta_0)) \qquad x_2(t) = \frac{1}{h_0}(\cos(\theta_0 + h_0 t) - \cos(\theta_0)) \tag{3.58}$$

and integrating to find y

$$formulapery$$

Normal geodesics are curves with constant geodesic curvature, i.e., straight lines or circles on $\mathbb{R}^2$ (and helix on $\widehat{M}$). There are no non trivial abnormal geodesics since $\omega \wedge d\omega neq 0$ ($b = 1$).

$$\begin{aligned} \omega \wedge d\omega &= (dy - A) \wedge (b\mu_1 \wedge \mu_2) \\ &= b dy \wedge \mu_1 \wedge \mu_2. \end{aligned}$$

3.5 Lie derivative

sec:lieder

In this section we extend the notion of Lie derivative, already introduced for vector fields in Section 2.2), to differential forms. Recall that if $X, Y \in \mathrm{Vec}(M)$ are two vector fields we define

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} e_*^{-tX} Y = [X, Y].$$

If $P : M \to M$ is a diffeomorphism we can consider the pullback $P^* : T^*_{P(q)}M \to T^*_q M$ and extend its action to k-forms. Let $\omega \in \Lambda^k M$, we define $P^*\omega \in \Lambda^k M$ in the following way:

$$(P^*\omega)_q(\xi_1, \ldots, \xi_k) := \omega_{P(q)}(P_*\xi_1, \ldots, P_*\xi_k), \quad q \in M, \quad \xi_i \in T_q M. \tag{3.59}$$ eq:azionekfc

It is an easy check that this operation is linear and satisfies the two following properties

$$P^*(\omega_1 \wedge \omega_2) = P^*\omega_1 \wedge P^*\omega_2, \tag{3.60}$$ eq:propP*

$$P^* \circ d = d \circ P^*. \tag{3.61}$$ eq:propP*2

def:lieder **Definition 3.41.** Let $X \in \mathrm{Vec}(M)$ and $\omega \in \Lambda^k M$. We define the *Lie derivative* of ω with respect to X as

$$\mathcal{L}_X : \Lambda^k M \to \Lambda^k M, \qquad \mathcal{L}_X \omega = \left.\frac{d}{dt}\right|_{t=0} (e^{tX})^*\omega. \tag{3.62}$$

From (3.60) and (3.61), we easily deduce the following properties of the Lie derivative:

(i) $\mathcal{L}_X(\omega_1 \wedge \omega_2) = (\mathcal{L}_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathcal{L}_X \omega_2)$,

(ii) $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.

The first of these properties can be also expressed by saying that $\mathcal{L}_X$ is a *derivation* of the exterior algebra of k-forms.

The Lie derivative combines together a k-form and a vector field defining a new k-form. A second way of combining these two object is to define their inner product, by defining a $(k-1)$-form.

Definition 3.42. Let $X \in \mathrm{Vec}(M)$ and $\omega \in \Lambda^k M$. We define the *inner product of* ω *and* X as the operator $i_X : \Lambda^k M \to \Lambda^{k-1} M$, where we set *Cartan's formu*

$$(i_X\omega)(Y_1, \dots, Y_{k-1}) := \omega(X, Y_1, \dots, Y_{k-1}), \quad Y_i \in \mathrm{Vec}(M). \tag{3.63}$$

One can show that the operator i_X is an *anti-derivation*, in the following sense:

$$i_X(\omega_1 \wedge \omega_2) = (i_X\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge (i_X\omega_2), \quad \omega_i \in \Lambda^{k_i} M, \quad i = 1, 2. \tag{3.64}$$ eq:innantide

We end this section proving two classical formulas linking together these notions, and usually referred as Cartan's formulas.

Proposition 3.43 (Cartan's formula)**.** *The following identity holds true*

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \tag{3.65}$$ eq:cartanL

Proof. Define $D_X := i_X \circ d + d \circ i_X$. It is easy to check that D_X is a derivation on the algebra of k-forms, since i_X and d are anti-derivations. Let us show that D_X commutes with d. Indeed, using that $d^2 = 0$, one can write

$$d \circ D_X = d \circ i_X \circ d = D_X \circ d.$$

Moreover, since any k-form can be expressed in coordinates as $\omega = \sum \omega_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}$, it is sufficient to prove that $\mathcal{L}_X$ coincide with D_X on functions. This last property is easily checked by

$$D_X f = i_X(df) + \underbrace{d(i_X f)}_{=0} = \langle df, X\rangle = Xf = \mathcal{L}_X f. \qquad \square$$

Corollary 3.44. *Let $X, Y \in \mathrm{Vec}(M)$ and $\omega \in \Lambda^1 M$, then*

$$d\omega(X, Y) = X \langle \omega, Y\rangle - Y \langle \omega, X\rangle - \langle \omega, [X, Y]\rangle. \tag{3.66}$$ eq:cartandw

Proof. On one hand Definition 3.41 implies, by Leibnitz rule

$$\begin{aligned}
\langle \mathcal{L}_X\omega, Y\rangle_q &= \left.\frac{d}{dt}\right|_{t=0} \left\langle (e^{tX})^*\omega, Y\right\rangle_q \\
&= \left.\frac{d}{dt}\right|_{t=0} \left\langle \omega, e^{tX}_* Y\right\rangle_{e^{tX}(q)} \\
&= X \langle \omega, Y\rangle - \langle \omega, [X, Y]\rangle.
\end{aligned}$$

On the other hand, Cartan's formula (3.65) gives

$$\begin{aligned}
\langle \mathcal{L}_X\omega, Y\rangle &= \langle i_X(d\omega), Y\rangle + \langle d(i_X\omega), Y\rangle \\
&= d\omega(X, Y) + Y \langle \omega, X\rangle.
\end{aligned}$$

Comparing the two identities one gets (3.66). $\square$

manifold
orphism

3.6 Symplectic geometry

In this section we generalize some of the construction we considered on the cotangent bundle T^*M to the case of a general symplectic manifold.

def:symman **Definition 3.45.** A *symplectic manifold* (N, σ) is a smooth manifold N endowed with a closed, non degenerate 2-form $\sigma \in \Lambda^2(N)$. A *symplectomorphism* of N is a diffeomorphism $\phi : N \to N$ such that $\phi^*\sigma = \sigma$.

Notice that a symplectic manifold N is necessarily even-dimensional. We stress that, in general, the symplectic form σ is not necessarily exact, as in the case of $N = T^*M$.

The symplectic structure on a symplectic manifold N permits us to define the *Hamiltonian vector field* $\vec{h} \in \mathrm{Vec}(N)$ associated with a function $h \in \mathcal{C}^\infty(N)$ by the formula $i_{\vec{h}}\sigma = -dh$, or equivalently $\sigma(\cdot, \vec{h}) = dh$.

Proposition 3.46. *A diffeomorphism $\phi : N \to N$ is a symplectomorphism if and only if for every $h \in \mathcal{C}^\infty(N)$:*

$$(\phi_*^{-1})\vec{h} = \overrightarrow{h \circ \phi}. \tag{3.67}$$ eq:idsympl

Proof. Assume that ϕ is a symplectomorphism, namely $\phi^*\sigma = \sigma$. More precisely, this means that for every $\lambda \in N$ and every $v, w \in T_\lambda N$ one has

$$\sigma_\lambda(v, w) = (\phi^*\sigma)_\lambda(v, w) = \sigma_{\phi(\lambda)}(\phi_* v, \phi_* w),$$

where the second equality is the definition of $\phi^*\sigma$. If we apply the above equality at $w = \phi_*^{-1}\vec{h}$ one gets, for every $\lambda \in N$ and $v \in T_\lambda N$

$$\begin{aligned}
\sigma_\lambda(v, \phi_*^{-1}\vec{h}) &= (\phi^*\sigma)_\lambda(v, \phi_*^{-1}\vec{h}) = \sigma_{\phi(\lambda)}(\phi_* v, \vec{h}) \\
&= \left\langle d_{\phi(\lambda)}h, \phi_* v \right\rangle = \left\langle \phi^* d_{\phi(\lambda)}h, v \right\rangle . \\
&= \langle d(h \circ \phi), v \rangle
\end{aligned}$$

This shows that $\sigma_\lambda(\cdot, \phi_*^{-1}\vec{h}) = d(h \circ \phi)$, that is exactly (eq:idsympl 3.67). The converse implication follows analogously. ☐

Next we want to characterize those vector fields whose flow generates a one-parametric family of symplectomorphisms.

l:sympl **Lemma 3.47.** *Let $X \in \mathrm{Vec}(N)$ be a complete vector field on a symplectic manifold (N, σ). The following properties are equivalent*

(i) $(e^{tX})^*\sigma = \sigma$ *for every* $t \in \mathbb{R}$,

(ii) $\mathcal{L}_X \sigma = 0$,

(iii) $i_X \sigma$ *is a closed 1-form on* N.

Proof. By the group property $e^{(t+s)X} = e^{tX} \circ e^{sX}$ one has the following identity for every $t \in \mathbb{R}$:

$$\frac{d}{dt}(e^{tX})^*\sigma = \left.\frac{d}{ds}\right|_{s=0} (e^{tX})^*(e^{sX})^*\sigma = (e^{tX})^*\mathcal{L}_X\sigma.$$

This proves the equivalence between (i) and (ii), since the map $(e^{tX})^*$ is invertible for every $t \in \mathbb{R}$.

Recall now that the symplectic form σ is, by definition, a closed form. Then $d\sigma = 0$ and Cartan's formula (3.65) reads as follows

$$\mathcal{L}_X \sigma = d(i_X \sigma) + i_X(d\sigma) = d(i_X \sigma).$$

This proves the the equivalence between (ii) and (iii). □

c:sympl **Corollary 3.48.** *The flow of a Hamiltonian vector field defines a flow of symplectomorphisms.*

Proof. This is a direct consequence of the fact that, for an Hamitonian vector field $\vec{h}$, one has $i_{\vec{h}}\sigma = -dh$. Hence $i_{\vec{h}}\sigma$ is a cloded form (actually exact) and property (iii) of Lemma 3.47 holds. □

Notice that the converse of Corollary 3.48 is true when N is simply connected, since in this case every closed form is exact.

Definition 3.49. Let (N, σ) be a symplectic manifold and $a, b \in \mathcal{C}^\infty(N)$. The Poisson bracket between a and b is defined as $\{a, b\} = \sigma(\vec{a}, \vec{b})$.

We end this section by collecting some properties of the Poisson bracket that follow from the previous results.

p:newpoisson **Proposition 3.50.** *The Poisson bracket satisfies the identities*

(i) $\{a, b\} \circ \phi = \{a \circ \phi, b \circ \phi\}, \qquad \forall\, a, b \in \mathcal{C}^\infty(N), \forall\, \phi \in \mathit{Sympl}(N),$

(ii) $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0, \qquad \forall\, a, b, c \in \mathcal{C}^\infty(N).$

Proof. Property (i) follows from (3.67). Property (ii) follows by considering $\phi = e^{t\vec{c}}$ in (i), for some $c \in \mathcal{C}^\infty(N)$,. and computing the derivative with respect to t at $t = 0$. □

Finally we are able to prove the following generalization of (**??**).

c:idcommpar **Corollary 3.51.** *For every $a, b \in \mathcal{C}^\infty(N)$ we have*

$$\overrightarrow{\{a, b\}} = [\vec{a}, \vec{b}]. \tag{3.68}$$ eq:idcommpar

Proof. Property (ii) of Proposition 3.50 can be rewritten, by skew-symmetry of the Poisson bracket, as follows

$$\{\{a, b\}, c\} = \{a, \{b, c\}\} - \{b, \{a, c\}\}. \tag{3.69}$$ eq:poisson00

Using that $\{a, b\} = \sigma(\vec{a}, \vec{b}) = \vec{a}b$ one can rewrite again (3.69) as

$$\overrightarrow{\{a, b\}}c = \vec{a}(\vec{b}c) - \vec{b}(\vec{a}c) = [\vec{a}, \vec{b}]c.$$ □

Remark 3.52. Property (ii) of Proposition 3.50 says that $\{a, \cdot\}$ is a derivation of the algebra $\mathcal{C}^\infty(N)$. Moreover, the space $\mathcal{C}^\infty(N)$ endowed with $\{\cdot, \cdot\}$ as a product is a Lie algebra isomorphic to a subalgebra of $\mathrm{Vec}(N)$. Indeed, by (3.68), the correspondence $a \mapsto \vec{a}$ is a Lie algebra homomorphism between $\mathcal{C}^\infty(N)$ and $\mathrm{Vec}(N)$.

3.7 Local minimality of normal trajectories

In this section we prove a fundamental result about local optimality of normal trajectories. More precisely we show small pieces of a normal trajectory are length minimizers.

3.7.1 The Poincaré-Cartan one form

Fix a smooth function $a \in \mathcal{C}^\infty(M)$ and consider the smooth submanifold of T^*M defined by the graph of its differential

$$\mathcal{L}_0 = \{d_q a \,|\, q \in M\} \subset T^*M. \tag{3.70}$$ eq:L0

Notice that the restriction of the canonical projection $\pi : T^*M \to M$ to $\mathcal{L}_0$ defines a diffeomorphism between $\mathcal{L}_0$ and M, hence $\dim \mathcal{L}_0 = n$. Let us then consider the image $\mathcal{L}_t$ of $\mathcal{L}_0$ under the Hamiltonian flow

$$\mathcal{L}_t := e^{t\vec{H}}(\mathcal{L}_0), \qquad t > 0, \tag{3.71}$$ eq:Lt

and define the $(n+1)$-dimensional manifold with boundary in $T^*M \times \mathbb{R}$ as follows

$$\mathcal{L} = \{(t, \lambda) \in \mathbb{R} \times T^*M \,|\, \lambda \in \mathcal{L}_t,\, 0 \le t \le T\} \tag{3.72}$$

$$= \{(t, e^{t\vec{H}}\lambda_0) \in \mathbb{R} \times T^*M \,|\, \lambda_0 \in \mathcal{L}_0,\, 0 \le t \le T\}. \tag{3.73}$$

Here we assume that the Hamiltonian flow is defined on the interval $[0, T]$.

Finally, let us introduce the *Poincaré-Cartan* 1-form on $T^*M \times \mathbb{R} \simeq T^*(M \times \mathbb{R})$ defined by

$$s - H dt \in \Lambda^1(T^*M \times \mathbb{R})$$

where $s \in \Lambda^1(T^*M)$ denotes, as usual, the tautological 1-form of T^*M. We start by proving a preliminary lemma.

l:sapi **Lemma 3.53.** $s|_{\mathcal{L}_0} = d(a \circ \pi)|_{\mathcal{L}_0}$

Proof. By definition of tautological 1-form $s_\lambda(w) = \langle \lambda, \pi_* w\rangle$, for every $w \in T_\lambda(T^*M)$. If $\lambda \in \mathcal{L}_0$ then $\lambda = d_q a$, where $q = \pi(\lambda)$. Hence for every $w \in T_\lambda(T^*M)$

$$s_\lambda(w) = \langle \lambda, \pi_* w\rangle = \langle d_q a, \pi_* w\rangle = \langle \pi^* d_q a, w\rangle = \langle d_q(a \circ \pi), w\rangle .$$ □

l:exact **Proposition 3.54.** *The 1-form $(s - Hdt)|_{\mathcal{L}}$ is exact.*

Proof. We divide the proof in two steps: (i) we show that the restriction of the Poincare-Cartan 1-form $(s - Hdt)|_{\mathcal{L}}$ is closed and (ii) that it is exact.

(i). To prove that the 1-form is closed we need to show that the differential

$$d(s - Hdt) = \sigma - dH \wedge dt, \tag{3.74}$$

vanishes when applied to a pair of tangent vectors to $\mathcal{L}$. Since, for each $t \in [0, T]$, the set $\mathcal{L}_t$ has codimension 1 in $\mathcal{L}$, there are only two possibilities for the choice of the two tangent vectors:

(a) both vectors are tangent to $\mathcal{L}_t$, for some $t \in [0, T]$.

(b) one vector is tangent to $\mathcal{L}_t$ while the second one is transversal.

Case (a). Since both tangent vectors are tangent to $\mathcal{L}_t$, it is enough to show that the restriction of the one form $\sigma - dH \wedge dt$ to $\mathcal{L}_t$ is zero. First let us notice that dt vanishes when applied to tangent vectors to $\mathcal{L}_t$, thus $\sigma - dH \wedge dt|_{\mathcal{L}_t} = \sigma|_{\mathcal{L}_t}$. Moreover, since by definition $\mathcal{L}_t = e^{t\vec{H}}(\mathcal{L}_0)$ one has

$$\begin{aligned}\sigma|_{\mathcal{L}_t} &= \sigma|_{e^{t\vec{H}}(\mathcal{L}_0)}\\ &= (e^{t\vec{H}})^*\sigma|_{\mathcal{L}_0} = \sigma|_{\mathcal{L}_0} = ds|_{\mathcal{L}_0} = d^2(a\circ\pi)|_{\mathcal{L}_0} = 0.\end{aligned}$$

where in the last line we used Lemma 3.53 and the fact that $(e^{t\vec{H}})^*\sigma = \sigma$, since $e^{t\vec{H}}$ is an Hamiltonian flow and thus preserves the symplectic form.
Case (b). The manifold $\mathcal{L}$ is, by construction, the image of the smooth mapping

$$\Psi : [0,T]\times\mathcal{L}_0 \to [0,T]\times T^*M, \qquad \Psi(t,\lambda)\mapsto (t, e^{t\vec{H}}\lambda),$$

Thus a tangent vector to $\mathcal{L}$ that is transversal to $\mathcal{L}_t$ can be obtained by differentiating the map Ψ with respect to t:

$$\frac{\partial\Psi}{\partial t}(t,\lambda) = \vec{H}(\lambda) + \frac{\partial}{\partial t} \in T_{(t,\lambda)}\mathcal{L}. \tag{3.75}$$

eq:tantr

It is then sufficient to show that the vector (3.75) is in the kernel of the two form $\sigma - dH\wedge dt$. In other words we have to prove

$$i_{\vec{H}+\partial_t}(\sigma - dH\wedge dt) = 0. \tag{3.76}$$

eq:closed

The last equality follows from the following identities

$$\begin{aligned} &i_{\vec{H}}\sigma = \sigma(\vec{H},\cdot) = -dH, \qquad i_{\partial_t}\sigma = 0,\\ &i_{\vec{H}}(dH\wedge dt) = \underbrace{(i_{\vec{H}}dH)}_{=0}\wedge dt - dH\wedge\underbrace{(i_{\vec{H}}dt)}_{=0} = 0,\\ &i_{\partial_t}(dH\wedge dt) = \underbrace{(i_{\partial_t}dH)}_{=0}\wedge dt - dH\wedge\underbrace{(i_{\partial_t}dt)}_{=1} = -dH.\end{aligned}$$

where we used that $i_{\vec{H}}dH = dH(\vec{H}) = \{H,H\} = 0$.

(ii). Next we show that the form $s - Hdt|_{\mathcal{L}}$ is exact. To this aim we have to prove that, for every closed curve Γ in $\mathcal{L}$ one has

$$\int_\Gamma s - Hdt = 0. \tag{3.77}$$

eq:exact

Every curve Γ in $\mathcal{L}$ can be written as follows

$$\Gamma : [0,T]\to\mathcal{L}, \quad \Gamma(s) = (t(s), e^{t(s)\vec{H}}\lambda(s)), \qquad \text{where } \lambda(s)\in\mathcal{L}_0.$$

Moreover, it is easy to see that the continuous map defined by

$$K : [0,T]\times\mathcal{L}\to\mathcal{L}, \qquad K(\tau,(t,e^{t\vec{H}}\lambda_0)) = (t-\tau, e^{(t-\tau)\vec{H}}\lambda_0)$$

defines an homotopy of $\mathcal{L}$ such that $K(0,(t,e^{t\vec{H}}\lambda_0)) = (t,e^{t\vec{H}}\lambda_0)$ and $K(t,(t,e^{t\vec{H}}\lambda_0)) = (0,\lambda_0)$. Then the curve Γ is homotopic to the curve $\Gamma_0(s) = (0,\lambda(s))$. Since the 1-form $s - Hdt$ is closed, the integral is invariant under homotopy, namely

$$\int_\Gamma s - Hdt = \int_{\Gamma_0} s - Hdt.$$

Moreover, the integral over Γ_0 is computed as follows (recall that $\Gamma_0 \subset \mathcal{L}_0$ and $dt = 0$ on $\mathcal{L}_0$):

$$\int_{\Gamma_0} s - H dt = \int_{\Gamma_0} s = \int_{\Gamma_0} d(a \circ \pi) = 0,$$

where we used Lemma 3.53 and the fact that the integral of an exact form over a closed curve is zero. Then (3.77) follows. $\square$

3.7.2 Normal trajectories are geodesics

Now we are ready to prove a sufficient condition that ensures the optimality of small pieces of normal trajectories. As a corollary we will get that small pieces of normal trajectories are geodesics.

Recall that normal trajectories for the problem

$$\dot{q} = f_u(q) = \sum_{i=1}^{m} u_i f_i(q), \tag{3.78}$$

where $f_1, \dots, f_m$ is a generating family for the sub-Riemannian structure are projections of integral curves of the Hamiltonian vector fields associated with the sub-Riemannian Hamiltonian

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \qquad (\text{i.e. } \lambda(t) = e^{t\vec{H}}(\lambda_0)), \tag{3.79}$$

$$\gamma(t) = \pi(\lambda(t)), \qquad t \in [0, T]. \tag{3.80}$$

where

$$H(\lambda) = \max_{u \in U_q} \left\{ \langle \lambda, f_u(q) \rangle - \frac{1}{2}|u|^2 \right\} = \frac{1}{2} \sum_{i=1}^{m} \langle \lambda, f_i(q) \rangle^2 . \tag{3.81}$$

t:minimal **Theorem 3.55.** *Assume that there exists $a \in \mathcal{C}^\infty(M)$ such that the restriction of the projection $\pi|_{\mathcal{L}_t}$ is a diffeomorphism for every $t \in [0, T]$. Then for any $\lambda_0 \in \mathcal{L}_0$ the normal geodesic*

$$\widetilde{\gamma}(t) = \pi \circ e^{t\vec{H}}(\lambda_0), \qquad t \in [0, T], \tag{3.82}$$

eq:normstric

is a strict length-minimizer among all admissible curves γ with the same boundary conditions.

Proof. Let $\gamma(t)$ be an admissible trajectory, different from $\widetilde{\gamma}(t)$, associated with the control $u(t)$ and such that $\gamma(0) = \widetilde{\gamma}(0)$ and $\gamma(T) = \widetilde{\gamma}(T)$. We denote by $\widetilde{u}(t)$ the control associated with the curve $\widetilde{\gamma}(t)$.

By assumption, for every $t \in [0, T]$ the map $\pi|_{\mathcal{L}_t} : \mathcal{L}_t \to M$ is a local diffeomorphism, thus the trajectory $\gamma(t)$ can be uniquely lifted to a smooth curve $\lambda(t) \in \mathcal{L}_t$. Notice that the corresponding curves Γ and $\widetilde{\Gamma}$ in $\mathcal{L}$ defined by

$$\Gamma(t) = (t, \lambda(t)), \qquad \widetilde{\Gamma}(t) = (t, \widetilde{\lambda}(t)) \tag{3.83}$$

eq:Gammas

have the same boundary conditions, since for $t = 0$ and $t = T$ they project to the same base point on M and their lift is uniquely determined by the diffeomorphisms $\pi|_{\mathcal{L}_0}$ and $\pi|_{\mathcal{L}_T}$, respectively.

Recall now that, by definition of the sub-Riemannian Hamiltonian, we have

$$H(\lambda(t)) \leq \langle \lambda(t), f_{u(t)}(\gamma(t)) \rangle - \frac{1}{2}|u(t)|^2, \qquad \gamma(t) = \pi(\lambda(t)), \tag{3.84}$$

eq:hamiltmax

where $\lambda(t)$ is a lift of the trajectory $\gamma(t)$ associated with a control $u(t)$. Moreover, the equality holds in (3.84) if and only if $\lambda(t)$ is a solution of the Hamiltonian system $\dot{\lambda}(t) = H(\lambda(t))$. For this reason we have the relations

$$H(\lambda(t)) < \left\langle \lambda(t), f_{u(t)}(\gamma(t)) \right\rangle - \frac{1}{2}|u(t)|^2, \tag{3.85}$$ $\boxed{\texttt{eq:hammax1}}$

$$H(\widetilde{\lambda}(t)) = \left\langle \widetilde{\lambda}(t), f_{\widetilde{u}(t)}(\widetilde{\gamma}(t)) \right\rangle - \frac{1}{2}|\widetilde{u}(t)|^2. \tag{3.86}$$ $\boxed{\texttt{eq:hammax2}}$

since $\widetilde{\lambda}(t)$ is a solution of the Hamiltonian equation by assumptions, while $\lambda(t)$ is not. Indeed $\lambda(t)$ and $\widetilde{\lambda}(t)$ have the same initial condition, hence, by uniqueness of the solution of the Cauchy problem, it follows that $\dot{\lambda}(t) = H(\lambda(t))$ if and only if $\lambda(t) = \widetilde{\lambda}(t)$, that implies that $\widetilde{\gamma}(t) = \gamma(t)$.

Let us then show that the energy associated with the curve γ is bigger than the one of the curve $\widetilde{\gamma}$. Actually we prove the following chain of (in)equalities

$$\frac{1}{2}\int_0^T |\widetilde{u}(t)|^2 dt = \int_{\widetilde{\Gamma}} s - H dt = \int_{\Gamma} s - H dt < \frac{1}{2}\int_0^T |u(t)|^2 dt, \tag{3.87}$$ $\boxed{\texttt{eq:normalmin}}$

where Γ and $\widetilde{\Gamma}$ are the curves in $\mathcal{L}$ defined in (3.83).

By Lemma 3.54, the 1-form $s - Hdt$ is exact. Then the integral over the closed curve $\Gamma \cup \widetilde{\Gamma}$ vanishes, and one gets

$$\int_{\widetilde{\Gamma}} s - H dt = \int_{\Gamma} s - H dt.$$

The last inequality in (3.87) can be proved as follows

$$\begin{aligned}
\int_{\Gamma} s - H dt &= \int_0^T \langle \lambda(t), \dot{\gamma}(t) \rangle - H(\lambda(t)) dt \\
&= \int_0^T \left\langle \lambda(t), f_{u(t)}(\gamma(t)) \right\rangle - H(\lambda(t)) dt \\
&< \int_0^T \left\langle \lambda(t), f_{u(t)}(\gamma(t)) \right\rangle - \left(\left\langle \lambda(t), f_{u(t)}(\gamma(t)) \right\rangle - \frac{1}{2}|u(t)|^2 \right) dt \\
&= \frac{1}{2}\int_0^T |u(t)|^2 dt.
\end{aligned} \tag{3.88}$$ $\boxed{\texttt{eq:www}}$

where we used (3.85). A similar computation gives computation, using (3.86), gives

$$\int_{\widetilde{\Gamma}} s - H dt = \frac{1}{2}\int_0^T |\widetilde{u}(t)|^2 dt, \tag{3.89}$$ $\boxed{\texttt{eq:www}}$

that ends the proof of (3.87). □

As a corollary we state a local version of the same theorem, that can be proved by adapting the above technique.

$\boxed{\texttt{c:normal}}$ **Corollary 3.56.** *Assume that there exists $a \in \mathcal{C}^\infty(M)$ and neighborhoods Ω_t of $\widetilde{\gamma}(t)$, such that $\pi \circ e^{t\vec{H}} \circ da|_{\Omega_0} : \Omega_0 \to \Omega_t$ is a diffeomorphism for every $t \in [0,T]$. Then (3.82) is a strict length-minimizer among all admissible trajectories γ with same boundary conditions and such that $\gamma(t) \in \Omega_t$ for all $t \in [0,T]$.*

nian!geodesic

We are in position to prove that small pieces of normal trajectories are global length minimizers.

c:geodesic

Theorem 3.57. *Let $\gamma : [0,T] \to M$ be a sub-Riemannian normal trajectory. Then for every $\tau \in [0,T[$ there exists $\varepsilon > 0$ such that*

(i) $\gamma|_{[\tau,\tau+\varepsilon]}$ *is a length minimizer, i.e.,* $d(\gamma(\tau),\gamma(\tau+\varepsilon)) = \ell(\gamma|_{[\tau,\tau+\varepsilon]})$.

(ii) $\gamma|_{[\tau,\tau+\varepsilon]}$ *is the unique length minimizer joining* $\gamma(\tau)$ *and* $\gamma(\tau+\varepsilon)$*, up to reparametrization.*

Proof. Without loss of generality we can assume that the curve is parametrized by length and prove the theorem for $\tau = 0$. Let $\gamma(t)$ be a normal extremal trajectory, such that $\gamma(t) = \pi(e^{t\vec{H}}(\lambda_0))$, for $t \in [0,T]$. Consider a smooth function $a \in \mathcal{C}^\infty(M)$ such that $d_q a = \lambda_0$ and let $\mathcal{L}_t$ be the family of submanifold of T^*M associated with this function by (3.70) and (3.71). By construction, for the extremal lift associated with γ one has $\lambda(t) = e^{t\vec{H}}(\lambda_0) \in \mathcal{L}_t$ for all t. Moreover the projection $\pi\big|_{\mathcal{L}_0}$ is a diffeomorphism, since $\mathcal{L}_0$ is a section of T^*M.

Hence, for every fixed compact $K \subset M$ containing the curve γ, by continuity there exists $t_0 = t_0(K)$ such that the restriction on K of the map $\pi\big|_{\mathcal{L}_t}$ is also a diffeomorphism, for all $0 \leq t < t_0$. Let us now denote δ_K the positive constant defined in Lemma 2.33 such that every curve starting from $\gamma(0)$ and leaving K is necessary longer than δ_K.

Then, defining $\varepsilon = \varepsilon(K) := \min\{\delta_K, t_0(K)\}$ we have that the curve $\gamma|_{[0,\varepsilon]}$ is contained in K and is shorter than any other curve contained in K with the same boundary condition by Corollary 3.56 (applied to $\Omega_t = K$ for all $t \in [0,T]$). Moreover $\ell(\gamma|_{[0,\varepsilon]}) = \varepsilon$ since γ is length parametrized, hence it is shorter than any admissible curve that is not contained in K. Thus $\gamma|_{[0,\varepsilon]}$ is a global minimizer. Moreover it is unique up to reparametrization by uniqueness of the solution of the Hamiltonian equation (see proof of Theorem 3.55). $\square$

Remark 3.58. When $\mathcal{D}_{q_0} = T_{q_0}M$, as it is the case for a Riemannian structure, the level set of the Hamiltonian

$$\{H = 1/2\} = \{\lambda \in T^*_{q_0}M | H(\lambda) = 1/2\},$$

is diffeomorphic to an ellipsoid, hence compact. Under this assumption, for each $\lambda_0 \in \{H = 1/2\}$, the corresponding geodesic $\gamma(t) = \pi(e^{t\vec{H}}(\lambda_0))$ is optimal up to a time $\varepsilon = \varepsilon(\lambda_0)$, with λ_0 belonging to a compact set. It follows that it is possible to find a common $\varepsilon > 0$ (depending only on q_0) such that each normal trajectory with base point q_0 is optimal on the interval $[0,\varepsilon]$.

Bibliographical notes

The Hamiltonian approach to sub-Riemannian geometry is nowadays classical. However the construction of the symplectic structure, obtained by extending the Poisson bracket from the space of affine functions, is not standard and is inspired by [?].

Historically, in the setting of PDE, the sub-Riemannian distance (also called Carnot-Carathéodory distance) is introduced by means of sub-unit curves, see for instance [?] and references therein. The link between the two definition is clarified in Exercice 3.29

The proof that normal extremal are geodesics is an adaptation of a more general condition for optimality given in [?] for a more general class of problems. This is inspired by the classical idea of "fields of extremals" in classical Calculus of Variation.