The sub-Riemannian heat equation

Andrei Agrachev

Davide Barilari

Ugo Boscain

March 16, 2015

last version available at `http://people.sissa.it/agrachev/agrachev_files/notes.html`

Preprint SISSA 09/2012/M

In this chapter we derive the sub-Riemannian heat equation and we discuss the strictly related question of how to define an intrinsic volume in sub-Riemannian geometry.

0.1 The heat equation

To write the heat equation in a sub-Riemannian manifold, let us recall how to write it in the Riemannian context and let us see which mathematical structures are missing in the sub-Riemannian one.

0.1.1 The heat equation in the Riemannian context

Let (M, g) be an oriented Riemannian manifold of dimension n and let ω be a volume form on M, i.e., a never-vanishing n-form on M.[1] The most natural choice for ω is of course the Riemannian volume defined by[2]

$$\omega(X_1, \ldots, X_n) = 1, \text{ where } \{X_1, \ldots, X_n\} \text{ is a local orthonormal frame.}$$

In coordinates if g is represented by a matrix (g_{ij}), we have

$$\omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \ldots \wedge dx_n.$$

However in the following we write the heat equation for a general volume form that not necessarily coincides with the Riemannian one. This point of view is useful in sub-Riemannian geometry, where a canonical volume exists only in certain cases.

Let ϕ be a quantity (depending on the position q and the time t) subjects to a diffusion process e.g. the temperature of a body, the concentration of a chemical product, the noise etc..... Let $\mathbf{F}$ be a time dependent vector field representing the *flux* of the quantity ϕ, i.e., how much of ϕ is flowing through the unity of surface in unitary time.

Our purpose is to get a partial differential equation describing the evolution of ϕ. The Riemannian heat equation is obtained by postulating the following two facts:

(R1) the flux is proportional to minus the gradient of ϕ i.e., normalizing the proportionality constant to one, we assume that

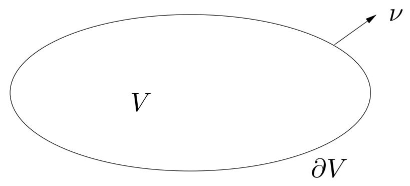
$$\mathbf{F} = -\text{grad}(\phi); \tag{1}$$

(R2) the quantity ϕ satisfies a conservation law, i.e. for every bounded open set V having a smooth boundary ∂V we have the following: the rate of decreasing of ϕ inside V is equal to the rate of flowing of ϕ via $\mathbf{F}$, out of V, through ∂V. In formulas this is written as

$$-\frac{d}{dt}\int_V \phi\, \omega = \int_{\partial V} \mathbf{F} \cdot \nu \, \mathrm{dS}. \tag{2}$$

[1] For simplicity here we assume that M is orientable, but since this construction is essentially local, this hypothesis it is not restrictive

[2] `vedere se mettere come esercizio che non dipende dalla scelta del frame`



Here ν is the external (Riemannian) normal to ∂V and dS is the element of area induced by ω on M, thanks to the Riemannian structure, i.e., $\mathrm{dS} = \omega(\nu, \cdot)$. The quantity $\mathbf{F} \cdot \nu$ is a notation for $g_q(\mathbf{F}(q,t), \nu(q))$.

Applying the Riemannian divergence theorem to (2) and using (1) we have then

$$-\frac{d}{dt}\int_V \phi\,\omega = \int_{\partial V} \mathbf{F}\cdot\nu\,\mathrm{dS} = \int_V \mathrm{div}_\omega(\mathbf{F})\,\omega = -\int_V \mathrm{div}_\omega(\mathrm{grad}(\phi))\,\omega.$$

By the arbitrarity of V and defining the Riemannian Laplace operator as

$$\triangle\phi = \mathrm{div}_\omega(\mathrm{grad}(\phi)) \tag{3}$$

we get the heat equation

$$\frac{\partial}{\partial t}\phi(q,t) = \triangle\phi(q,t).$$

Useful expressions for the Riemannian Laplacian

In this section we get some useful expressions for $\triangle$. To this purpose we have to recall what are grad and div_ω in formula (13).

We recall that the gradient of a smooth function $\varphi : M \to \mathbb{R}$ is a vector field pointing in the direction of the greatest rate of increase of φ and its magnitude is the derivative of φ in that direction. In formulas it is the unique vector field $\mathrm{grad}(\varphi)$ satisfying for every $q \in M$,

$$g_q(\mathrm{grad}(\varphi), v) = d\varphi(v), \quad \text{for every } v \in T_qM. \tag{4}$$

In coordinates, if g is represented by a matrix (g_{ij}), and calling (g^{ij}) its inverse, we have

$$\mathrm{grad}(\varphi)^i = \sum_{j=1}^n g^{ij}\partial_j\varphi. \tag{5}$$

If $\{X_1, \ldots, X_n\}$ is a local orthonormal frame for g, we have the useful formula[3]

$$\mathrm{grad}(\varphi) = \sum_{i=1}^n X_i(\varphi)X_i. \tag{6}$$

Exercise 0.1. Prove that if the Riemannian metric is defined globally via a generating frame $\{X_1, \ldots, X_k\}$ with $k \geq n$, in the sense of Section .... then $\mathrm{grad}(\varphi) = \sum_{i=1}^k X_i(\varphi)X_i$.

[3] `dobbiamo ricordare che i camp vert agiscono sully funzioni????`

Recall that the divergence of a smooth vector field X says how much the flow of X is increasing or decreasing the volume. It is defined in the following way. The Lie derivative in the direction of X of the volume form is still a n-form and hence point-wise proportional to the volume form itself. The "point-wise" constant of proportionality is a smooth function that by definition is the divergence of X. In formulas

$$L_X\omega = \mathrm{div}_\omega(X)\omega.$$

Now using $d\omega = 0$ and the Cartan formula we have that $L_X\omega = i_X d\omega + d(i_X\omega) = d(i_X\omega)$. Hence the divergence of a vector field X can be defined by

$$d(i_X\omega) = \mathrm{div}_\omega(X)\omega. \tag{7}$$

In coordinates, if $\omega = h(x)dx^1 \wedge \dots dx^n$ we have

$$\mathrm{div}_\omega(X) = \frac{1}{h(x)}\sum_{i=1}^{n} \partial_i(h(x)X^i). \tag{8}$$

Remark 0.2. Notice that to define the divergence of a vector field it is not necessary a Riemannian structure, but only a volume form.

If we put together formula 5 and formula 8, with $X = \mathrm{grad}(\varphi)$ we get the well known expression

$$\triangle(\varphi) = \mathrm{div}_\omega(\mathrm{grad}(\varphi)) = \frac{1}{h(x)}\sum_{i,j=1}^{n} \partial_i(h(x)g^{ij}\partial_j\varphi). \tag{9}$$

Combining formula 6 with the property $\mathrm{div}(aX) = a\,\mathrm{div}(X) + X(a)$ where X is a vector field and a a function, we get

$$\triangle(\varphi) = \sum_{i=1}^{n}\left(X_i^2\varphi + \mathrm{div}_\omega(X_i)X_i(\varphi)\right) \quad \text{where } \{X_1, \dots X_n\} \text{ is a local orthonormal frame.} \tag{10}$$

Similarly, defining the Riemannian structure via a generating frame we get

$$\triangle(\varphi) = \sum_{i=1}^{k}\left(X_i^2\varphi + \mathrm{div}_\omega(X_i)X_i(\varphi)\right) \quad \text{where } \{X_1, \dots X_k\},\ k \geq n, \text{ is a generating frame} \tag{11}$$

Remark 0.3. Notice that the choice of the volume form does not affect the second order terms, but only the first order ones.

When $\triangle$ is built with respect to the Riemannian volume form, it is called the Laplace-Beltrami operator.

0.1.2 The heat equation in the sub-Riemannian context

Let M be a sub-Riemannian manifold of dimension n. Let $\mathcal{D}$ be the associated set of horizontal vector fields and g_q the corresponding metric[4] on the distribution $\mathcal{D}_q$.

[4] nel capitol 3 non la abbiamo quasi definita

As in the Riemannian case, we assume by simplicity that M is oriented and we assume that a volume form ω has been assigned on M.[5] In Section .... we show that, in the equiregular case, the sub-Riemannian structure induces, canonically, a volume form on M. For the moment we assume that the volume form is assigned independently of the sub-Riemannian structure.

As in the previous section, we denote by ϕ the quantity subject to the diffusion process, by $\mathbf{F}$ the corresponding flux, and we postulate that:

(SR1) the heat flows in the direction where ϕ is varying more but only among horizontal directions;

(SR2) the quantity ϕ satisfies a conservation law, i.e. for every bounded open set V having a smooth and orientable boundary ∂V we have the following: the rate of decreasing of ϕ inside V is equal to the rate of flowing of ϕ via $\mathbf{F}$, out of V, through ∂V.

To derive the heat equation in the Riemannian case, we have used the following ingredients that are not directly available in the sub-Riemannian context:

- the Riemannian gradient;
- the Riemannian normal to ∂V, and the inner product to define the conservation 2;
- the Riemannian divergence theorem.

Hence the standard Riemannian construction fails in the sub-Riemannian context and we have to reason in a different way to derive the heat equation. Let us analyse one by one the ingredients above and let us see how to generalise them in sub-Riemannian geometry.

The horizontal gradient

In sub-Riemannian geometry the gradient of a smooth function $\varphi : M \to \mathbb{R}$ is a horizontal vector field (called horizontal gradient) pointing in the horizontal direction of the greatest rate of increase of φ and its magnitude is the derivative of φ in that direction. In formulas it is the unique vector field $\mathrm{grad}_H(\varphi)$ satisfying for every $q \in M$,

$$g_q(\mathrm{grad}_H(\varphi), v) = d\varphi(v), \quad \text{for every } v \in \mathcal{D}_q M. \tag{12}$$

If $\{X_1, \ldots, X_k\}$ is a generating frame then[6]

$$\mathrm{grad}_H(\varphi) = \sum_{i=1}^{k} X_i(\varphi) X_i.$$

The postulate (SR1) is then written as

$$\mathbf{F} = -\mathrm{grad}_H(\phi).$$

[5] spiegare been che deve essere liscia e non degenere. Spiegare che bel caso non orientabile basta una density.

[6] notare che questa formula come le precedenti del caso riemmaniano non dipendono dall'orthonormal frame?

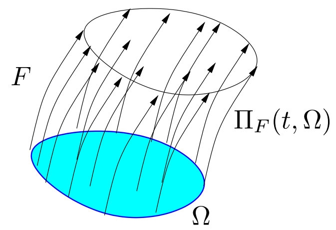


Figure 1:

The conservation of the heat

The next step is to express the conservation of the heat without a Riemannian structure. This can be done thanks to the following Lemma, whose proof is left for exercise.

Lemma 0.4. *Let M be a smooth manifold provided with a smooth volume form ω. Let Ω be an embedded bounded sub-manifold (possible with boundary) of codimension 1. Let F be a (possible time dependent) complete smooth vector field and $P_{0,t}$ be the corresponding flow. Consider the cylinder formed by the images of Ω translated by the flow of F for times between 0 and t (see Figure 1):*

$$\Pi_F(t,\Omega) = \{P_{0,t}(\Omega) \mid s \in [0,t]\}.$$

Then

$$\frac{d}{dt}\bigg|_{t=0} \int_{\Pi_F(t,\Omega)} \omega = \int_\Omega i_F\big|_{t=0}\, \omega.$$

With the notation of this Lemma, the postulate (SR2) is written as

$$-\frac{d}{dt}\int_V \phi\ \omega = \frac{d}{dt}\int_{\Pi_{\mathbf{F}(t,\partial V)}} \omega = \int_{\partial V} i_{\mathbf{F}}\, \omega,$$

where in the last equality we have used the result of the lemma.

Now, using the Stokes theorem, the definition of divergence 7 and using that $\mathbf{F} = -\mathrm{grad}_H \phi$ we have

$$\int_{\partial V} i_{\mathbf{F}}\, \omega = \int_V d(i_{\mathbf{F}}\, \omega) = \int_V \mathrm{div}_\omega(\mathbf{F})\, \omega = -\int_V \mathrm{div}(\mathrm{grad}_H(\phi))\, \omega.$$

By the arbitrarity of V and defining

$$\triangle_H \phi = \mathrm{div}_\omega(\mathrm{grad}_H(\phi)), \tag{13}$$

we get the sub-Riemannian heat equation

$$\frac{\partial}{\partial t}\phi(q,t) = \triangle_H \phi(q,t).$$

Definition 0.5. Let M be a sub-Riemannian manifolds and let ω be a volume on M. The operator $\triangle_H \phi = \mathrm{div}_\omega(\mathrm{grad}_H(\phi))$ is called the *sub-Riemannian Laplacian.*

When it is possible to construct a volume from the sub-Riemannian structure, then the corresponding sub-Riemannian Laplacian is called the *intrinsic sub-Laplacian.* The construction of a canonical volume form in a sub-Riemannian manifold is the purpose of Section **??**.[7] Here let us just remark that in the case of left-invariant structures on Lie groups, a canonical volume can be built naturally from the group structure. This will be done in Section 0.2 for the Heisenberg group.

0.1.3 Few properties of the sub-Riemannian Laplacian: the Hörmander theorem and the existence of the heat kernel

The same computation of the Riemannian case provides the following expression for the sub-Riemannian Laplacian,

$$\triangle_H(\phi) = \sum_{i=1}^{k} \left(X_i^2 \phi + \mathrm{div}_\omega(X_i) X_i(\phi) \right) \quad \text{where } \{X_1, \dots X_k\}, \text{ is a generating frame.} \tag{14}$$

In the Riemannian case, the operator Δ_H is elliptic, i.e., in coordinates it has the expression

$$\triangle_H = \sum_{i,j=0}^{n} a_{ij}(x) \partial_i \partial_j + \text{first order terms},$$

where the matrix (a_{ij}) is symmetric and positive definite for every x.

In the sub-Riemannian (and not-Riemannian) case, Δ_H it is not elliptic since the matrix (a_{ij}) can have several zero eigenvalues. However, a theorem of Hörmander says that thanks to the Lie bracket generating condition Δ_H is hypoelliptic. More precisely we have the following.

Theorem 0.6 (Hörmander)**.** *Let $Y_0, Y_1 \dots Y_k$ be a set of Lie bracket generating vector fields on a smooth manifold M. Then the operator $L = Y_0 + \sum_{i=1}^{k} Y_i^2$ is hypoellptic which means that if φ is a distribution defined on an open set $\Omega \subset M$, such that $L\varphi$ is $\mathcal{C}^\infty$, then φ is $\mathcal{C}^\infty$ in Ω.*

Remark 0.7. Notice that elliptic operators with $\mathcal{C}^\infty$ coefficients are hypoelliptic. The heat operator $\partial_t - \Delta$, where Δ is the standard Laplacian in $\mathbb{R}^n$ is not elliptic (since the matrix of coefficients of the second order derivatives in $\mathbb{R}^{n+1}$ has one zero eigenvalue), but it is hypoelliptic since $\partial_{x_1}, \dots, \partial_{x_n}, \partial_t$ are Lie Bracket generating in $\mathbb{R}^{n+1}$.

One of the most important consequences of the Hörmander theorem is that the heat evolution smooths out immediately every initial condition. Indeed if one can guarantee that a solution of $(\partial_t - \Delta_H)\varphi = 0$ exists in distributional sense in an open set Ω of $\mathbb{R} \times M$, then, being $0 \in \mathcal{C}^\infty$, it follows that φ is $\mathcal{C}^\infty$ in Ω.

A standard result for the existence of a solution in $L^2(M, \omega)$ is given by the following theorem. See for instance [**?**].[8]

Theorem 0.8. *Let M be a smooth manifold and ω a volume on M. If Δ is a non negative and essentially self-adjoint operator on $L^2(M, \omega)$, then, there exists a unique solution to the Cauchy problem*

$$\begin{cases} (\partial_t - \triangle)\phi = 0 \\ \phi(q, 0) = \phi_0(q) \in L^2(M, \omega), \end{cases} \tag{15}$$

on $[0, \infty[\times M$. Moreover for each $t \in [0, \infty[$ this solution belongs to $L^2(M, \omega)$.

[7] citare

[8] `vedere che coda citare.`

It is immediate to prove that Δ_H is non-negative and symmetric on $L^2(M,\omega)$. If in addition one can prove that Δ_H is essentially self-adjoint, then thanks to the Hörmander theorem, one has that the solution of (15) is indeed $\mathcal{C}^\infty$ in $]0,\infty[\times M$.

The discussion of the theory of self-adjoint operators is out of the purpose of this book. However the essential self-adjointness of Δ_H is guaranteed by the completeness of the sub-Riemannian manifold as metric space. This condition guarantees also the existence of the solution to the Cauchy problem in the form of a convolution kernel.

Theorem 0.9 (Strichartz)**.** *Consider a sub-Riemannian manifold that is complete as metric space. Let ω be a volume on M. Then Δ_H is essentially self-adjoint on $L^2(M,\omega)$. Moreover the unique solution to the Cauchy problem*

$$\begin{cases} (\partial_t - \triangle_H)\phi = 0 \\ \phi(q,0) = \phi_0(q) \in L^2(M,\omega), \end{cases} \tag{16}$$

on $[0,\infty[\times M$ can be written as

$$\phi(q,t) = \int_M \phi_0(\bar{q}) K_t(q,\bar{q})\, \omega(\bar{q})$$

where $K_t(q,\bar{q})$ is a positive function defined on $]0,\infty[\times M \times M$ which is smooth, symmetric for the exchange of q and $\bar{q}$ and such that for every fixed t, q, we have $K_t(q,\cdot) \in L^2(M,\omega)$.

Typical cases in which the sub-Riemannian manifold is complete are let-invariant structure on Lie groups, sub-Riemannian structures obtained as restriction of complete Riemannian structures, sub-Riemannian structures defined in $\mathbb{R}^n$ having as generating frame a set of sublLinear vector fields.

Let us just remark that if the sub-Riemannian structure is not Lie-bracket generated,[9] then in general the operator is not hypoelliptic and the heat evolution does not smooth the initial condition.

Consider for example the operator $L = \partial_x^2 + \partial_y^2$ on $\mathbb{R}^3$. This operator is not obtained from Lie-bracket generating vector fields. Consider the corresponding heat operator $\partial_t - L$ on $[0,\infty] \times \mathbb{R}^3$. Since the z direction is not appearing in this operator, any discontinuity in the z variable is not smoothed by the evolution. For instance if $\psi(x,y,t)$ is a solution of the heat equation $\partial_t - L = 0$ on $[0,\infty] \times \mathbb{R}^2$, then $\psi(x,y,t)\theta(z)$ is a solution of the heat equation in $[0,\infty] \times \mathbb{R}^3$, where θ is the Heaviside function.

0.2 The heat-kernel on the Heisenberg group

In this section we construct the heat kernel on the Heisenberg sub-Riemannian structure. To this purpose it is convenient to see this structure as a left-invariant structure on a matrix representation of the Heisenberg group. This point of view is useful to build in a canonical way a volume form and hence the sub-Riemannian Laplacian. Moreover this point of view permits to look for a simplified version of the heat kernel using the group law.

[9] i.e. a proto-sub-Riemannian structure

0.2.1 The Heisenberg group as a group of matrices

The Heisenberg group H_2 can be seen as the 3-dimensional group of matrices

$$H_2 = \left\{ \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

endowed with the standard matrix product. H_2 is indeed $\mathbb{R}^3$, endowed with the group law

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right).$$

This group law comes from the matrix product after making the identification

$$(x, y, z) \sim \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The identity of the group is the element $(0, 0, 0)$ and the inverse element is given by the formula

$$(x, y, z)^{-1} = (-x, -y, -z)$$

A basis of its Lie algebra of H_2 is $\{p_1, p_2, k\}$ where

$$p_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

They satisfy the following commutation rules: $[p_1, p_2] = k$, $[p_1, k] = [p_2, k] = 0$, hence H_2 is a 2-step nilpotent group.

Remark 0.10. Notice that if one write an element of the algebra as $xp_1 + yp_2 + zk$, one has that

$$\exp(xp_1 + yp_2 + zk) = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

Hence the coordinates (x, y, z) are the coordinates on the Lie algebra related to the basis $\{p_1, p_2, k\}$, transported on the group via the exponential map. They are called coordinates of the "first type". As we will see later, coordinate $x, y, w = z + \frac{1}{2}xy$, that are more adapted to the group, are also useful.

The standard sub-Riemannian structure on H_2 is the one having as generating frame:

$$X_1(g) = gp_1, \quad X_2(g) = gp_2.$$

With a straightforward computation one get the following coordinate expression for the generating frame:

$$X_1 = \partial_x - \frac{y}{2}\partial_z, \qquad X_2 = \partial_y + \frac{x}{2}\partial_z,$$

that we already met several times in the previous chapters.

Let L_g (reap. R_g) be the left (resp. right) multiplication on H_2:

$$L_g : H_2 \ni h \mapsto gh \quad (\text{resp. } R_g : H_2 \ni h \mapsto hg).$$

Exercise Prove that, up to a multiplicative constant, there exist one and only one 3-form dh_L on H_2 which is left-invariant, i.e. such that $L_g^* dh = dh$[10] and that in coordinates coincide (up to a constant) with the Lebesgue measure $dx \wedge dy \wedge dz$. Prove the same for a right-invariant 3-form dh_R,

The left- and right-invariant forms built in the exercise above are called the left and right Haar measures. Since they coincide up to a constant the Heisenberg group is said to be "unimodular". In the following we normalise the left and right Haar measures on the sub-Riemannian structure in such a way that

$$dh_L(X_1, X_2, [X_1, X_2]) = dh_R(X_1, X_2, [X_1, X_2]) = 1. \tag{19}$$

The 3-form obtained in this way coincide with the Lebesgue measure and in the following we call it simply the "Haar measure"

$$dh = dx \wedge dy \wedge dz.$$

Exercise Prove that the two conditions (19) are invariant by change of the orthonormal frame.

0.2.2 The heat equation on the Heisenberg group

Given a volume form ω on $\mathbb{R}^3$, the sub-Riemannian Laplacian for the Heisenberg sub-Riemannian structure is given by the formula,

$$\triangle_H(\phi) = \big(X_1^2 + X_2^2 + \operatorname{div}_\omega(X_1)X_1 + \operatorname{div}_\omega(X_2)X_2\big)\phi. \tag{20}$$

If we take as volume the Haar volume dh, and using the fact that X_1 and X_2 are divergence free with respect to dh, we get for the sub-Riemannian Laplacian

$$\triangle_H(\phi) = (X_1)^2 + (X_2)^2 = (\partial_x - \frac{y}{2}\partial_z)^2 + (\partial_y + \frac{x}{2}\partial_z)^2. \tag{21}$$

The heat equation on the Heisenberg group is then

$$\triangle_H(\phi) = \big((\partial_x - \frac{y}{2}\partial_z)^2 + (\partial_y + \frac{x}{2}\partial_z)^2\big)\phi(x, y, z, t) = \partial_t\phi(x, y, z, t).$$

For this equation, we are looking for the heat kernel, namely a function $K_t(q, \bar{q})$ such that the solution to the Cauchy problem[11]

$$\begin{cases} (\partial_t - \triangle_H)\phi = 0 \\ \phi(q, 0) = \phi_0(q) \in L^2(\mathbb{R}^3, dh) \end{cases} \tag{22}$$

can be expressed as

$$\phi(q, t) = \int_{\mathbb{R}^3} K_t(q, \bar{q})\phi_0(\bar{q})dh(\bar{q}). \tag{23}$$

[10] `scrivener meglio`

[11] `vedere se mettere in tutta la teoria sul calorie, la condition initial in` $\phi_0(q) \in L^1(\mathbb{R}^3, \mathbb{R}) \cap L^2(\mathbb{R}^3, dh)$

The existence of a heat kernel that is smooth, positive and symmetric is guaranteed by Theorem 0.9 since the Heisenberg group (as sub-Riemannian structure) is complete.

The construction of the explicit expression of the heat kernel on the Heisenberg group was an important achievement of the end of the seventies. Here we propose an elementary direct method. divided in the following step:

STEP 1. We look for a special form for $K_t(q, \bar{q})$ using the group law.

STEP 2. We make a change of variables in such a way that the coefficients of the heat equation depend only on one variable instead than two.

STEP 3. By using the Fourier transform in two variables, we transform the heat equation (that was a PDE in 3 variable plus the time) in a heat equation with an harmonic potential in one variable plus the time.

STEP 4. We find the kernel for the heat equation with the harmonic potential, thanks to the Mehler formula for Hermite polynomials.[12]

STEP 5. We come back to the original variables.

Let us make these steps one by one.

STEP 1 Due to invariance under the group law, we have that for $K_t(q, \bar{q}) = K_t(p \cdot q, p \cdot \bar{q})$ for every $p \in H_2$. Taking $p = q^{-1}$ we have that $K_t(q, \bar{q}) = K_t(0, q^{-1}\bar{q})$ hence we can write

$$K_t(q, \bar{q}) = p_t(q^{-1} \cdot \bar{q}) = p_t(\bar{x} - x, \bar{y} - y, \bar{z} - z) = p_t(x - \bar{x}, y - \bar{y}, z - \bar{z}),$$

for a suitable function $p_t(\cdot)$ called the *fundamental solution.* In the last equality we have used the symmetry of the heat kernel.

STEP 2 Let us make the change the variable $z \to w$, where

$$w = z + \frac{1}{2}xy$$

(cf. Remark 0.10). In the new variables we have that the Haar measure is $dh = dx \wedge dy \wedge dw$. The generating frame and the sub-Riemannian Laplacian become

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \partial_x \tag{24}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} = \partial_y + x\partial_w \tag{25}$$

$$\triangle_H(\phi) = (X_1)^2 + (X_2)^2 = \partial_x^2 + (\partial_y + x\partial_w)^2. \tag{26}$$

The new coordinates are very useful since now the coefficients of the different terms in $\triangle_H$ depend only on one variable. We are then looking for the solution to the Cauchy problem

$$\begin{cases} \partial_t\varphi(x, y, w, t) = \triangle_H(\varphi(x, y, w, t)) = \left(\partial_x^2 + (\partial_y + x\partial_w)^2\right)\varphi(x, y, w, t) \\ \varphi(x, y, w, 0) = \varphi_0(x, y, w) \in L^2(\mathbb{R}^3, dh) \end{cases} \tag{27}$$

[12] `dire che questo step sara' utile pure per altri kernels`

where $\varphi(x,y,w,t) = \phi(x,y,w-\frac{1}{2}xy)$.

STEP 3 By making the Fourier transform in y and w, we have $\partial_y \to i\mu$, $\partial_w \to i\nu$ and the Cauchy problem become

$$\begin{cases} \partial_t\hat{\varphi}(x,\mu,\nu,t) = \left(\partial_x^2 - (\mu+\nu x)^2\right)\hat{\varphi}(x,\mu,\nu,t) \\ \hat{\varphi}(x,\mu,\nu,0) = \hat{\varphi}_0(x,\mu,\nu). \end{cases} \tag{28}$$

By making the change of variable $x \to \theta$, where $\mu + \nu x = \nu\theta$, i.e., $\theta = x + \frac{\mu}{\nu}$ we get:

$$\begin{cases} \partial_t\bar{\varphi}^{\mu,\nu}(\theta,t) = \left(\partial_\theta^2 - \nu^2\theta^2\right)\bar{\varphi}^{\mu,\nu}(\theta,t) \\ \bar{\varphi}^{\mu,\nu}(\theta,0) = \bar{\varphi}_0^{\mu,\nu}(\theta), \end{cases} \tag{29}$$

where we set $\bar{\varphi}^{\mu,\nu}(\theta,t) := \hat{\varphi}(\theta-\frac{\mu}{\nu},\mu,\nu,t)$, and $\bar{\varphi}_0^{\mu,\nu}(\theta) = \hat{\varphi}_0(\theta-\frac{\mu}{\nu},\mu,\nu)$.

STEP 4. We have the following

Theorem 0.11. *The solution of the Cauchy problem for the evolution of the heat in an harmonic potential, i.e.*

$$\begin{cases} \partial_t\psi(\theta,t) = \left(\partial_\theta^2 - \nu^2\theta^2\right)\psi(\theta,t) \\ \psi(\theta,0) = \psi_0(\theta) \in L^2(\mathbb{R},d\theta) \end{cases} \tag{30}$$

can be written in the form of a convolution kernel

$$\psi(\theta,t) = \int_{\mathbb{R}} Q_t^\nu(\theta,\bar{\theta})\psi_0(\bar{\theta})d\bar{\theta}.$$

where

$$Q_t^\nu(\theta,\bar{\theta}) := \sqrt{\frac{\nu}{2\pi\sinh(2\nu t)}}\exp\left(-\frac{1}{2}\frac{\nu\cosh(2\nu t)}{\sinh(2\nu t)}(\theta^2+\bar{\theta}^2) + \frac{\nu\theta\bar{\theta}}{\sinh(2\nu t)}\right). \tag{31}$$

Remark 0.12. In the case $\nu = 0$ we interpret $Q_t^0(\theta,\bar{\theta})$ as

$$\lim_{\nu\to 0} Q_t^\nu(\theta,\bar{\theta}) = \frac{1}{\sqrt{4\pi t}}\exp[-\frac{(\theta-\bar{\theta})^2}{4t}]. \tag{32}$$

Proof. For $\nu = 0$, equation (30) is the standard heat equation on $\mathbb{R}$ and the heat kernel is given by formula (32). See for instance [?]. In the following we assume $\nu \neq 0$. The eigenvalues and the eigenfunctions of the operator $\partial_\theta^2 - \nu^2\theta^2$ on $\mathbb{R}$ are (see Appendix ....)

$$\begin{aligned} E_j &= -2\nu(j+1/2) \\ \varphi_j^\nu(\theta) &= \frac{1}{\sqrt{2^j j!}}\left(\frac{\nu}{\pi}\right)^{\frac{1}{4}}\exp(-\frac{\nu\theta^2}{2})H_j(\sqrt{\nu}\,\theta) \end{aligned} \tag{33}$$

where H_j are the Hermite polynomials

$$H_j(\theta) = (-1)^j\exp(\theta^2)\frac{d^j}{d\theta^j}\exp(-\theta^2).$$

Being $\{\varphi_j^\nu\}_{j\in\mathbb{N}}$ an orthonormal frame of $L^2(\mathbb{R})$, we can write

$$\psi(\theta,t) = \sum_j C_j(t)\varphi_j^\nu(\theta).$$

Using equation (30), we obtain that

$$C_j(t) = C_j(0)\exp(tE_j)$$

where $C_j(0) = \int_{\mathbb{R}} \varphi_j^{\nu}(\bar{\theta})\psi_0(\bar{\theta})\, d\bar{\theta}$. Hence

$$\psi(\theta, t) = \int_{\mathbb{R}} Q_t^{\nu}(\theta, \bar{\theta})\psi_0(\bar{\theta})\, d\bar{\theta}$$

where

$$Q_t^{\nu}(\theta, \bar{\theta}) = \sum_j \varphi_j^{\nu}(\theta)\varphi_j^{\nu}(\bar{\theta})\exp(tE_j).$$

After some algebraic manipulations and using the Mehler formula for Hermite polynomials

$$\sum_j \frac{H_j(\theta)H_j(\bar{\theta})}{2^j j!}(w)^j = (1-w^2)^{-\frac{1}{2}}\exp\left(\frac{2\theta\bar{\theta}w - (\theta^2+\bar{\theta}^2)w^2}{1-w^2}\right), \quad \forall\, w \in \mathbb{R}$$

with $\theta \to \sqrt{\nu}\theta$, $\bar{\theta} \to \sqrt{\nu}\bar{\theta}$, $w \to \exp(-2\nu t)$, one get formula (31). $\square$

Using Theorem 0.11 we can write the solution to 30 as

$$\bar{\varphi}^{\mu,\nu}(\theta, t) = \int_{\mathbb{R}} Q_t^{\nu}(\theta, \bar{\theta})\bar{\varphi}_0^{\mu,\nu}(\bar{\theta})d\bar{\theta}.$$

STEP 5 We now come back to the original variables step by step. We have

$$\hat{\varphi}(x,\mu,\nu,t) = \bar{\varphi}^{\mu,\nu}(x+\frac{\mu}{\nu},t) = \int_{\mathbb{R}} Q_t^{\nu}(x+\frac{\mu}{\nu},\bar{\theta})\bar{\varphi}_0^{\mu,\nu}(\bar{\theta})d\bar{\theta} = \int_{\mathbb{R}} Q_t^{\nu}(x+\frac{\mu}{\nu},\bar{x}+\frac{\mu}{\nu})\hat{\varphi}_0(\bar{x},\mu,\nu)d\bar{x}.$$

In the last equality we made the change of integration variable $\bar{\theta} \to \bar{x}$ with $\bar{\theta} = \bar{x} + \frac{\mu}{\nu}$ and we used the fact that $\hat{\varphi}_0^{\mu,\nu}(\bar{x}+\frac{\mu}{\nu}) = \hat{\varphi}_0(\bar{x},\mu,\nu)$.

Now, using the fact that $\hat{\varphi}_0(\bar{x},\mu,\nu)$ is the Fourier transform of the initial condition, i.e.

$$\hat{\varphi}_0(\bar{x},\mu,\nu) = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_0(\bar{x},\bar{y},\bar{w})e^{-i\mu\bar{y}}e^{-i\nu\bar{w}}d\bar{y}\, d\bar{w},$$

and making the inverse Fourier transform we get

$$\begin{aligned}\varphi(x,y,w,t) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\varphi}(x,\mu,\nu,t)e^{i\mu y}e^{i\nu w}d\mu\, d\nu \\ &= \int_{\mathbb{R}^3}\left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} Q_t^{\nu}(x+\frac{\mu}{\nu},\bar{x}+\frac{\mu}{\nu})e^{i\mu(y-\bar{y})}e^{i\nu(w-\bar{w})}d\mu\, d\nu\right)\varphi_0(\bar{x},\bar{y},\bar{w})d\bar{x}\, d\bar{y}\, d\bar{w}.\end{aligned}$$

Coming back to the variable x, y, z, we have

$$\phi(x,y,z,t) = \varphi(x,y,z+\frac{1}{2}xy) = \int_{\mathbb{R}^3} K_t(x,y,z,\bar{x},\bar{y},\bar{z})\phi_0(\bar{x},\bar{y},\bar{z})d\bar{x}\, d\bar{y}\, d\bar{z}.$$

where

$$K_t(x,y,z,\bar{x},\bar{y},\bar{z}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} Q_t^{\nu}(x+\frac{\mu}{\nu},\bar{x}+\frac{\mu}{\nu})e^{i\mu(y-\bar{y})}e^{i\nu(z-\bar{z}+\frac{1}{2}(xy-\bar{x}\bar{y}))}d\mu\, d\nu.$$

Setting $\bar{x},\bar{y},\bar{z}$ to zero and after some algebraic manipulations we get for the fundamental solution[13]

$$p_t(x,y,z) = \frac{1}{(2\pi t)^2}\int_{\mathbb{R}} \frac{2\tau}{\sinh(2\tau)}\exp\left(-\frac{\tau(x^2+y^2)}{2t\tanh(2\tau)}\right)\cos(2\frac{z\tau}{t})d\tau. \tag{34}$$

The integral representation (34) can be computed explicitly on the origin and on the z axis. Indeed we have[14]

$$K_t(0,0,0;0,0,0) = p_t(0,0,0) = \frac{1}{16t^2} \tag{35}$$

$$K_t(0,0,0;0,0,z) = p_t(0,0,z) = \frac{1}{8t^2\left(1+\cosh\left(\frac{\pi z}{t}\right)\right)} = \frac{1}{4t^2}\exp\left(-\frac{d^2(0,0,0;0,0,z)}{4t}\right)f(t) \tag{36}$$

In the last equality we have used the fact that for the Heisenberg group $d(0,0,0;0,0,z) = \sqrt{4\pi z}$. Here $f(t)$ is a smooth function of t such that $f(0) = 1$ (here $z \neq 0$ is fixed). A more detailed analysis permits to get for every fixed (x,y,z) such that $x^2+y^2 \neq 0$

$$K_t(0,0,0;x,y,z) = p_t(x,y,z) = \frac{C+O(t)}{t^{3/2}}\exp\left(-\frac{d^2(0,0,0;x,y,z)}{4t}\right). \tag{37}$$

Notice that the asymptotics (35), (36), (37) are deeply different with respect to those in the Euclidean case. Indeed the heat kernel for the standard heat equation in $\mathbb{R}^n$ is given by the formula

$$K_t(0,0,0;x,y,z) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{x^2+y^2+z^2}{4t}\right). \tag{38}$$

Comparing (38) with (35), (36), (37), one has the impression that the heat diffusion on the Heisenberg group at the origin and on the points on the z axis, is similar to the one in an Euclidean space of dimension 4. While on all the other points it is similar to to the one in an Euclidean space of dimension 3. Indeed the difference of asymptotics between the Heisenberg and the Euclidean case at the origin is related to the fact that the Hausdorff dimension of the Heisenberg group is 4, while its topological dimension is 3 (See Chapter .....). While the difference of asymptotics on the z axis (without the origin) is related to the fact that these are points reached a one parameter family of optimal geodesics starting from the origin and hence they are at the same time cut and conjugate points. For more details see [?]. It is interesting to remark that on a Riemannian manifold of dimension n the asymptotics are similar to the Euclidean ones for points close enough. Indeed for every $\bar{q}$ close enough to q we have $K_t(q,\bar{q}) = \frac{1+O(t)}{(4\pi t)^{n/2}}\exp\left(-\frac{d^2(q,\bar{q})}{4t}\right)$.
[15]

[13] fare ancora qualche dettaglio
[14] vedere se meter il calcolo de second integral
[15] citare rosemberg