A Review on stability of switched systems for arbitrary switchings

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Dedicated to the memory of Wijesuriya P. Dayawansa

Abstract In this paper we review some recent results on stability of multilinear switched systems, under arbitrary switchings. An open problems is stated.

1 Introduction

By a switched system, we mean a family of continuous–time dynamical systems and a rule that determines at each time which dynamical system is responsible of the time evolution. More precisely, let $\{f_u : u \in U\}$ (where U is a subset of $\mathbb{R}^m$) be a finite or infinite set of sufficiently regular vector fields on a manifold M, and consider the family of dynamical systems,

$$\dot{x} = f_u(x), \quad x \in M. \tag{1}$$

The rule is given by assigning the so-called switching function, i.e., a function $u(.) : [0, \infty[\to U \subset \mathbb{R}^m$. Here, we consider the situation in which the switching function is not known a priori and represents some phenomenon (e.g., a disturbance) that is not possible to control. Therefore, the dynamics defined in (1) also fits into the framework of uncertain systems (cf. for instance [6]). In the sequel, we use the notations $u \in U$ to label a fixed individual system and $u(.)$ to indicate the switching function. These kind of systems are sometimes called "n-modal systems", "dynamical polysystems", "input systems". The term "switched system" is often reserved to situations in which the switching function $u(.)$ is piecewise continuous or the set U is finite. For the purpose of this paper, we only require $u(.)$ to be a measurable function. When all the vector fields f_u, $u \in U$ are linear, the switched system is called multilinear. For a discussion of various issues related to switched systems, we refer the reader to [5, 13, 14, 17].

A typical problem for switched systems goes as follows. Assume that, for every fixed $u \in U$, the dynamical system $\dot{x} = f_u(x)$ satisfies a given property (P). Then one can investigate conditions under which property (P) still holds for $\dot{x} = f_{u(t)}(x)$, where $u(.)$ is an arbitrary switching function. In this paper, we focus on multilinear systems and (P) is the asymptotic stability property.

The structure of the paper is the following. In Section 2 we give the definitions of stability we need, we state the stability problem and we recall some sufficient conditions for stability in dimension n due to Agrachev, Hespanha, Liberzon and Morse [1, 7, 16]. In Section 3 we discuss the problem of existence of Lyapunov functions in certain functional classes (in particular in the class of polynomials). This problem was first studied by Molchanov and Pyatnitskii [19, 20, 21]. More recently new results were obtained by Dayawansa, Martin, Blanchini, Miani [4, 5, 13], and in [18] in collaboration with Chitour and Mason. In Section 4, we discuss the necessary and sufficient conditions for asymptotic stability for bidimensional single input systems (bilinear systems) that were found in [7] (see also [18]) and, in collaboration with Balde, in [8].

In Section 5 we state an open problem.

2 General properties of multilinear systems

By a multilinear switched systems (that more often are simply called switched linear systems) we mean a system of the form,

$$\dot{x}(t) = A_{u(t)}x(t), \quad x \in \mathbb{R}^n, \quad \{A_u\}_{u \in U} \subset \mathbb{R}^{n \times n}, \tag{2}$$

here $U \subset \mathbb{R}^m$ is a compact set, $u(.) : [0, \infty[\to U$ is a (measurable) switching function, and the map $u \mapsto A_u$ is continuous (so that $\{A_u\}_{u \in U}$ is a compact set of matrices). For these systems, the problem of asymptotic stability of the origin, uniformly with respect to switching functions was investigated, in [1, 7, 8, 13, 16].

A particular interesting class is the one of *bilinear* (or single-input) systems,

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \quad x \in \mathbb{R}^n, \qquad A, B \in \mathbb{R}^{n \times n}. \tag{3}$$

Here the set U is equivalently $[0, 1]$ or $\{0, 1\}$ (see Remark 2 below).

Let us state the notions of stability that are used in the following.

Definition 1 *For $\delta > 0$ let B_δ be the unit ball of radius δ, centered in the origin. Denote by $\mathcal{U}$ the set of measurable functions defined on $[0, \infty[$ and taking values on the compact set U. Given $x_0 \in \mathbb{R}^n$, we denote by $\gamma_{x_0,u(.)}(.)$ the trajectory of (2) based in x_0 and corresponding to the control $u(.)$. The accessible set from x_0, denoted by $\mathcal{A}(x_0)$, is*

$$\mathcal{A}(x_0) = \cup_{u(.) \in \mathcal{U}} \mathrm{Supp}(\gamma_{x_0,u(.)}(.)) .$$

We say that the system (2) is

- **unbounded** *if there exist* $x_0 \in \mathbb{R}^n$ *and* $u(.) \in \mathcal{U}$ *such that* $\gamma_{x_0,u(.)}(t)$ *goes to infinity as* $t \to \infty$*;*
- **uniformly stable** *if, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $\mathcal{A}(x_0) \subset B_\varepsilon$ *for every* $x_0 \in B_\delta$*;*
- **globally uniformly asymptotically stable** *(***GUAS***, for short) if it is uniformly stable and globally uniformly attractive, i.e., for every* $\delta_1, \delta_2 > 0$*, there exists* $T > 0$ *such that* $\gamma_{x_0,u(.)}(T) \in B_{\delta_1}$ *for every* $u(.) \in \mathcal{U}$ *and every* $x_0 \in B_{\delta_2}$*;*

In this paper we focus on the following problem.

Problem 1 For the system (2) (resp. for the system (3)), find under which conditions on the compact set $\{A_u\}_{u\in U}$ (resp. on the pair (A, B)) the system is **GUAS**.

Next we always assume the following hypothesis,

(H0) for the system (2) (resp. for the system (3)) all the matrices of the compact set $\{A_u\}_{u\in U}$ (resp. A, and B) have eigenvalues with strictly negative real part (Hurwitz in the following),

otherwise Problem 1 is trivial.

<u>*Remark*</u> **1** Under our hypotheses (multilinearity and compactness) there are many notions of stability equivalent to the ones of Definition 1. More precisely since the system is multilinear, local and global notions of stability are equivalent. Moreover, since $\{A_u\}_{u\in U}$ is compact, all notions of stability are automatically uniform with respect to switching functions (see for instance [3]). Finally, thanks to the multilinearity, the GUAS property is equivalent to the more often quoted property of GUES (global exponential stability, uniform with respect to switching), see for example [2] and references therein.

<u>*Remark*</u> **2** Whether systems of type (3) are GUAS or not is independent on the specific choice $U = [0, 1]$ or $U = \{0, 1\}$. In fact, this is a particular instance of a more general result stating that the stability properties of systems (2) depend only on the convex hull of the set $\{A_u\}_{u\in U}$; see for instance [18].

Example One can find many examples of systems such that each element of the family $\{A_u\}_{u\in U}$ is GUAS, while the switched system is not. Consider for instance a bidimensional system of type (3) where,

$$A = \begin{pmatrix} -.03 & -2 \\ 1/2 & -.03 \end{pmatrix}, \quad B = \begin{pmatrix} -.03 & -1/2 \\ 2 & -.03 \end{pmatrix}. \tag{4}$$

Notice that both A and B are Hurwitz. However it is easy to build trajectories of the switched systems that are unbounded, as shown in Figure 1.

Let us recall some results about stability of systems of type (2), subject to **(H0)**. In [1, 16], it is shown that the structure of the Lie algebra generated by the matrices A_u,

$$\mathbf{g} = \{A_u : \ u \in U\}_{L.A.},$$

is crucial for the stability of the system (2). For instance one can easily prove that if all the matrices of the family $\{A_u\}_{u\in U}$ commute, then the switched system is GUAS. The main result of [16] is the following.

Theorem 1 **(Hespanha, Morse, Liberzon)** *If* $\mathbf{g}$ *is a solvable Lie algebra, then the switched system (2) is GUAS.*

In [1] a generalization was given. Let $\mathbf{g} = \mathbf{r} \ltimes \mathbf{s}$ be the Levi decomposition of $\mathbf{g}$ in its radical (i.e., the maximal solvable ideal of $\mathbf{g}$) and a semi–simple sub–algebra, where the symbol $\ltimes$ indicates the semidirect sum.

Theorem 2 **(Agrachev, Liberzon)** *If* $\mathbf{s}$ *is a compact Lie algebra then the switched system (2) is GUAS.*

Theorem 2 contains Theorem 1 as a special case. Anyway the converse of Theorem 2 is not true in general: if $\mathbf{s}$ is non compact, the system can be stable or unstable. This case was also investigated. In particular, if $\mathbf{g}$ has dimension at most 4 as Lie algebra, Agrachev and Liberzon were able to reduce the problem of the asymptotic stability of the system (2) to the problem of the asymptotic stability of an auxiliary bidimensional system. We refer the reader to [1] for details. For this reason the bidimensional problem assumes particularly interest and, for the single input case, was solved in [7] (see also [18]) and, in collaboration with Balde, in [8].

Before stating the stability conditions found in [7, 8, 18], let us recall the advantages and disadvantage of the most used method to check stability, i.e. the method of common Lyapunov functions.

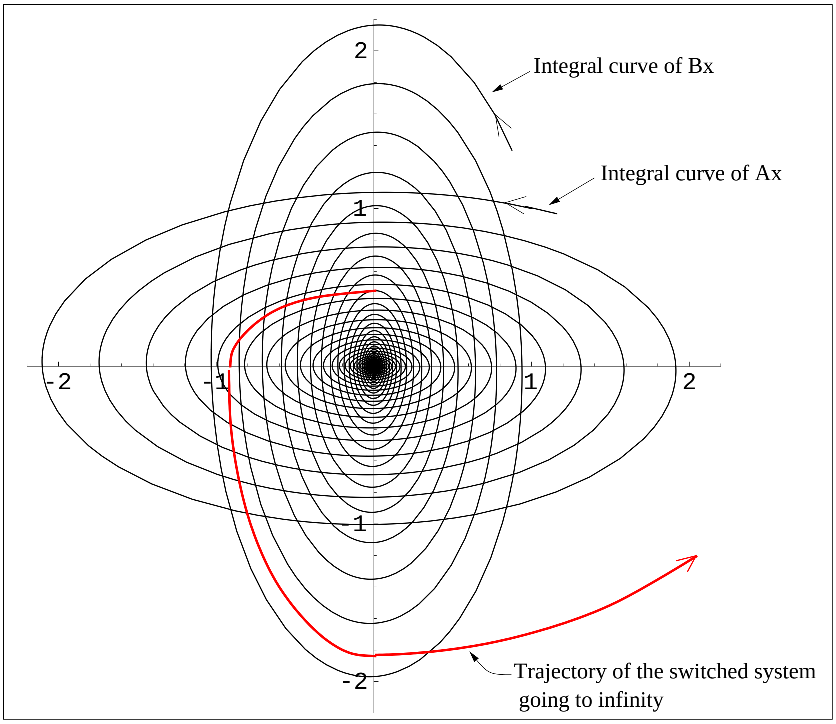

Figure 1: For a system of type (3), where A and B are given by formula (4), this picture shows an integral curve of Ax, an integral curve of Bx, and a trajectory of the switched system, that is unbounded.

3 Common Lyapunov functions

For a system (2), it is well known that the GUAS property is a consequence of the existence of a common Lyapunov function.

Definition 2 *A common Lyapunov function (LF for short)* $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+$*, for a switched system* (S) *of the type (2), is a continuous function such that* V *is positive definite (i.e.* $V(x) > 0$, $\forall x \neq 0$, $V(0) = 0$*) and* V *is strictly decreasing along nonconstant trajectories of* (S).

Vice-versa, it is known that, given a GUAS system of the type (2) subject to **(H0)**, it is always possible to build a $\mathcal{C}^\infty$ common Lyapunov function (see for instance [13, 19, 20, 21] and the bibliographical note in [14]).

Clearly is much more natural to use LFs to prove that a given system is GUAS (one has just to find one LF), than to prove that a system is unstable (i.e. proving that a LF does not exist). Indeed this is the reason why, usually, is much more easy to find (nontrivial) sufficient conditions for GUAS than necessary conditions. (Notice that all stability results given in the previous section in terms of the Lie algebra **g**, are, in fact, sufficient conditions for GUAS.)

Indeed the concept of LF is useful for practical purposes when one can prove that, for a certain class of systems, if a LF exists, then it is possible to find one of a certain type and possibly as simple as possible (e.g. polynomial with a bound on the degree, piecewise quadratic etc.). Typically one would like to work with a class of functions identified by a finite number of parameters. Once such a class of functions is identified, then in order to verify GUAS, one could use numerical algorithms to check (by varying the parameters) whether a LF exists (in which case the system is GUAS) or not (meaning that the system is not GUAS).

The idea of identifying a class of function where to look for a LF (i.e. sufficient to check GUAS), was first formalized by Blanchini and Miani in [5]. They called such a class a "universal class of Lyapunov functions."

For instance, a remarkable result for a given class C of systems of type (2) (for instance the class of single input system (3), in dimension n) could be the following.

Claim: there exists a positive integer m (depending on n) such that, whenever a system of C admits a LF, then it admits one that is polynomial of degree less than or equal to m. In other words, the class of polynomials of degree at most m is sufficient to check GUAS (i.e. the class of polynomials of degree at most m is universal, in the language of Blanchini and Miani).

The problem of proving if this claim is true or false attracted some attention from the community.

For single input bidimensional systems of type (3), with $n = 2$, Shorten and K. Narendra provided in [23] a necessary and sufficient condition on the pair (A, B) for the existence of a *quadratic* LF, but it is known (see for instance [13, 22, 11, 24]) that there exist GUAS bilinear bidimensional systems not admitting a quadratic LF. See for instance the paper by Dayawansa and Martin [13] for a nice example.

In [13], for systems of type (3), Dayawansa and Martin assumed that the Claim above is true and posed the problem of finding the minimum m. More precisely, the problem posed by Dayawansa and Martin is the following:

Problem 2 (Dayawansa and Martin): Let Ξ be the set of systems of type (3) in dimension n satisfying **(H0)**. Define $\Xi_{GUAS} \subset \Xi$ as the set of GUAS systems. Find the minimal integer m such that every system of Ξ_{GUAS} admits a polynomial LF of degree less or equal than m.

Remark **3** In the problem posed by Dayawansa and Martin, it is implicitly assumed that a GUAS system always admits a polynomial common Lyapunov function. This fact was first proved by Molchanov and Pyatnitskii in [19, 20], for systems of type 2, under the assumption that the set $\{A_u\}_{u\in U}$ is of the form $\{(a_{ij})_{i,j=1,\dots n} \ : \ a_{ij}^- \leq a_{ij} \leq a_{ij}^+\}$. In [21] Molchanov and Pyatnitskii state the result, with no further details, under the more general hypothesis $\{A_u\}_{u\in U}$ just compact. In the case in which the convex hull of $\{A_u\}_{u\in U}$ is finitely generated, the existence of a polynomial common Lyapunov function for GUAS systems was proved by Blanchini and Miani [4, 5], in the context of uncertain systems. In [18], in collaboration with Chitour and Mason, a simple proof for a set $\{A_u\}_{u\in U}$ satisfying the weaker hypothesis that its convex hull is compact, (without necessarily requiring U to be compact) is provided.

The core of the paper [18] consists of showing that the Problem of Dayawansa and Martin does not have a solution, i.e. the minimum degree of a polynomial LF cannot be uniformly bounded over the set of all GUAS systems of the form (3). More precisely, we have the following:

Theorem 3 *If (A, B) is a pair of $n \times n$ real matrices giving rise to a system of Ξ_{GUAS}, let $m(A, B)$ be the minimum value of the degree of any polynomial LF associated to that system. Then $m(A, B)$ cannot be bounded uniformly over Ξ_{GUAS}.*

In other words the set of polynomials of fixed degree is not a universal class of Lyapunov functions. Finding which is the right functional class where to look for LFs is indeed a very difficult task. Sometimes, it is even easier to prove directly that a system is GUAS or unstable. Indeed this is the case for bidimensional single-input switched systems.

4 Two-dimensional bilinear systems

In [7] (see also [18]) and [8], we studied conditions on A and B for the following property to be true:

($\mathcal{P}$) The switched system given by

$$\dot{x}(t) = u(t)Ax(t) + (1 - u(t))Bx(t), \quad x \in \mathbb{R}^2, \quad A, B \in \mathbb{R}^{2\times 2}, \quad u(.) : [0, \infty[\to \{0, 1\}, \tag{5}$$

is GUES at the origin.

The idea (coming from optimal control, see for instance [9]) is that many information on the stability of (5) are contained in the set $\mathcal{Z}$ where the two vector fields Ax and Bx are linearly dependent. This set is the set of zeros of the function $Q(x) := \det(Ax, Bx)$. Since Q is a quadratic form, we have the following cases (depicted in Figure 2):

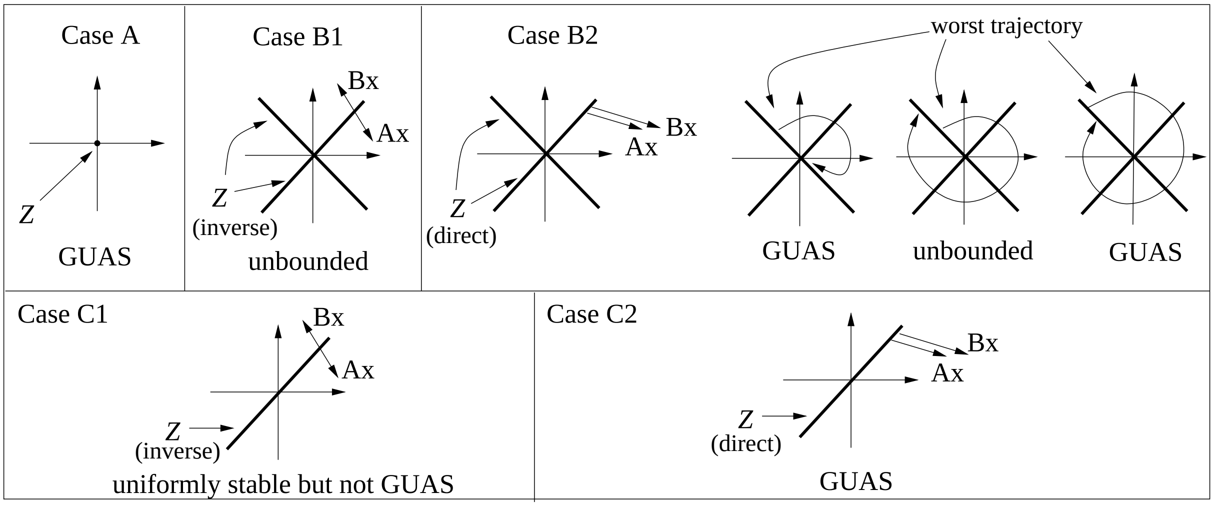

Figure 2:

A. $\mathcal{Z} = \{0\}$ (i.e., Q is positive or negative definite). In this case the vector fields preserve always the same orientation and the system is GUAS. This fact can be proved in several way (for instance building a common quadratic Lyapunov function) and it is true in much more generality (even for nonlinear systems, see the paper [10], in collaboration with Charlot and Sigalotti).

B. $\mathcal{Z}$ is the union of two noncoinciding straight lines passing through the origin (i.e., Q is sign indefinite). Take a point $x \in \mathcal{Z} \setminus \{0\}$. We say that $\mathcal{Z}$ is *direct* (respectively, *inverse*) if Ax and Bx have the same (respectively, opposite) versus. One can prove that this definition is independent of the choice of x on $\mathcal{Z}$. Then we have the two subcases:

- **B1.** $\mathcal{Z}$ is inverse. In this case one can prove that there exists $u_0 \in]0, 1[$ such that the matrix $u_0 Ax + (1 - u_0)Bx$ has an eigenvalue with positive real part. In this case the system is unbounded since it is possible to build a trajectory of the convexified system going to infinity with constant control. (This type of instability is called *static instability.*)
- **B2.** $\mathcal{Z}$ is direct. In this case one can reduce the problem of the stability of (5) to the problem of the stability of a single trajectory called *worst-trajectory*. Fixed $x_0 \in \mathbb{R}^2 \setminus \{0\}$, the worst-trajectory γ_{x_0} is the trajectory of (5), based at x_0, and having the following property. At each time t, $\dot{\gamma}_{x_0}(t)$ forms the smallest angle (in absolute value) with the (exiting) radial direction (see Figure 3). Clearly the worst-trajectory switches among the two vector fields on the set $\mathcal{Z}$. If it does not rotate around the origin (i.e., if it crosses the set $\mathcal{Z}$ a finite number of times) then the system is GUAS. On the other side, if it rotates around the origin, the system is GUAS if and only if after one turn the distance from the origin is decreased. (see Figure 2, Case B2). If after one turn the distance from the origin is increased then the system is unbounded (in this case, since there are no trajectories of the convexified system going to infinity with constant control, we call this instability *dynamic instability*). If γ_{x_0} is periodic then the system is uniformly stable, but not GUAS.

C. In the degenerate case in which the two straight lines of $\mathcal{Z}$ coincide (i.e., when Q is sign semi-definite), one see that the system is GUAS (resp. uniformly stable, but not GUAS) if and only if $\mathcal{Z}$ is direct (resp. inverse). We call these cases respectively **C2** and **C1**.

A consequence of these ideas is that the stability properties of the system (5), depend only on the shape of the integral curves of Ax and Bx and not on the way in which they are parameterized. More precisely we have:

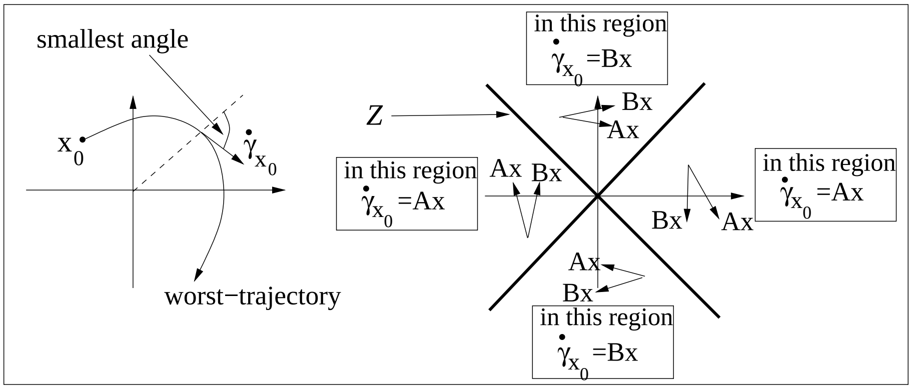

Figure 3:

Lemma 1 *If the switched system* $\dot{x} = u(t)Ax + (1 - u(t))Bx, \quad u(.) : [0, \infty[\to \{0, 1\}$, *has one of the stability properties given in Definition 1, then the same stability property holds for the system* $\dot{x} = u(t)(A/\alpha_A)x + (1 - u(t))(B/\alpha_B)x$, *for every* $\alpha_A, \alpha_B > 0$.

The main point to get stability conditions is to translate the ideas above in terms of coordinate invariant parameters. To this purpose one have first to find good normal forms for the two matrices A and B, in which these parameters appear explicitly.

We treat separately the case in which the two matrices are diagonalizable (called diagonalizable case) and the case in which one or both are not (called nondiagonalizable case).

4.1 The diagonalizable case

In the case in which both A and B are diagonalizable, we assume

H1: Let λ_1, λ_2 (resp., λ_3, λ_4) be the eigenvalues of A (resp., B). Then $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2)$, $\mathrm{Re}(\lambda_3)$, $\mathrm{Re}(\lambda_4) < 0$.

H2: $[A, B] \neq 0$ (that implies that neither A nor B is proportional to the identity).

H3: A and B are diagonalizable in $\mathbb{C}$. (Notice that if **(H2)** and **(H3)** hold, then $\lambda_1 \neq \lambda_2$, $\lambda_3 \neq \lambda_4$.)

H4: Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{C}P^1$ (resp., $\mathbf{V}_3, \mathbf{V}_4 \in \mathbb{C}P^1$) be the eigenvectors of A (resp., B). Then $\mathbf{V}_i \neq \mathbf{V}_j$ for $i \in \{1, 2\}$, $j \in \{3, 4\}$. (Notice that, from **(H2)** and **(H3)**, the V_i are uniquely defined, $\mathbf{V}_1 \neq \mathbf{V}_2$ and $\mathbf{V}_3 \neq \mathbf{V}_4$, and **(H4)** can be violated only when both A and B have real eigenvalues.)

Condition **(H1)** is just the condition that A and B are Hurwitz (cf. condition **(H0)** in Section 2). Condition **(H2)** is required otherwise the system is GUAS as a consequence of Theorem 1. The case in which **(H1)** and **(H2)** hold but **(H3)** does not is treated in the next section. The case in which **(H1)**, **(H2)** and **(H3)** hold but **(H4)** does not can be treated with arguments similar to those of [7], and it possible to conclude that $(\mathcal{P})$ is true.

Theorem 4 below, gives necessary and sufficient conditions for the stability of the system (5) in terms of three (coordinates invariant) parameters given in Definition 3 below. The first two parameters, ρ_A and ρ_B, depend on the eigenvalues of A and B, respectively, and the third parameter $\mathcal{K}$ depends on $Tr(AB)$, which is a Killing-type pseudoscalar product in the space of 2×2 matrices. As explained in [7], the parameter $\mathcal{K}$ contains the interrelation between the two systems $\dot{x} = Ax$ and $\dot{x} = Bx$, and it has a precise geometric meaning. It is in 1 : 1 correspondence with the cross ratio of the four points in the projective line $\mathbb{C}P^1$ that corresponds to the four eigenvectors of A and B. For more details, see [7].

Definition 3 *Let A and B be two 2×2 real matrices and suppose that* **(H1)**, **(H2)**, **(H3)**, *and* **(H4)** *hold. Moreover, choose the labels* $(_1)$ *and* $(_2)$ *(resp.,* $(_3)$ *and* $(_4)$*) so that* $|\lambda_2| > |\lambda_1|$ *(resp.,* $|\lambda_4| > |\lambda_3|$*) if they are real or* $\mathrm{Im}(\lambda_2) < 0$ *(resp.,* $\mathrm{Im}(\lambda_4) < 0$*) if they are complex. Define*

$$\rho_A := -i\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}; \qquad \rho_B := -i\frac{\lambda_3+\lambda_4}{\lambda_3-\lambda_4}; \qquad \mathcal{K} := 2\frac{Tr(AB)-\frac{1}{2}Tr(A)Tr(B)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}.$$

Moreover, define the following function of $\rho_A, \rho_B, \mathcal{K}$:

$$\mathcal{D} := \mathcal{K}^2 + 2\rho_A\rho_B\mathcal{K} - (1+\rho_A^2+\rho_B^2). \tag{6}$$

Notice that ρ_A is a positive real number if and only if A has nonreal eigenvalues and $\rho_A \in i\mathbb{R}$, $\rho_A/i > 1$ if and only if A has real eigenvalues. The same holds for B. Moreover, $\mathcal{D} \in \mathbb{R}$.

4.1.1 Normal forms in the diagonalizable case

Under hypotheses **(H1)** to **(H4)**, using a suitable 3-parameter changes of coordinates, it is always possible to put the matrices A and B, up the their norm, in the normal forms given in the following Proposition, where $\rho_A, \rho_B, \mathcal{K}$ appear explicitly. (See [7] and [18] for the proof.) We will call, respectively, **(CC)** the case where both matrices have nonreal eigenvalues, **(RR)** the case where both matrices have real eigenvalues, and **(RC)** the case where one matrix has real eigenvalues and the other nonreal eigenvalues.

Proposition 1 *Let A, B be two 2×2 real matrices satisfying conditions* **(H1)**, **(H2)**, **(H3)**, *and* **(H4)** . *In the case in which one of the two matrices has real and the other nonreal eigenvalues (i.e., the* **(RC)** *case), assume that A is the one having real eigenvalues. Then there exists a 3-parameter change of coordinates and two constant $\alpha_A, \alpha_B > 0$ such that the matrices A/α_A and B/α_B (still denoted below by A and B) are in the following normal forms.*

Case in which *A* and *B* have both nonreal eigenvalues ((CC) case):

$$A = \begin{pmatrix} -\rho_A & -1/E \\ E & -\rho_A \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B & -1 \\ 1 & -\rho_B \end{pmatrix},$$

where $\rho_A, \rho_B > 0$, $|E| > 1$. In this case, $\mathcal{K} = \frac{1}{2}(E + \frac{1}{E})$. Moreover, the eigenvalues of A and B are, respectively, $-\rho_A \pm i$ and $-\rho_B \pm i$.

Case in which *A* has real and *B* nonreal eigenvalues ((RC) case):

$$A = \begin{pmatrix} -\rho_A/i+1 & 0 \\ 0 & -\rho_A/i-1 \end{pmatrix}, \qquad B = \begin{pmatrix} -\rho_B-\mathcal{K}/i & -\sqrt{1-\mathcal{K}^2} \\ \sqrt{1-\mathcal{K}^2} & -\rho_B+\mathcal{K}/i \end{pmatrix},$$

where $\rho_B > 0$, $\rho_A/i > 1$, $\mathcal{K} \in i\mathbb{R}$. In this case, the eigenvalues of A and B are, respectively, $-\rho_A/i \pm 1$ and $-\rho_B \pm i$.

Case in which *A* and *B* have both real eigenvalues ((RR) case):

$$A = \begin{pmatrix} -\rho_A/i+1 & 0 \\ 0 & -\rho_A/i-1 \end{pmatrix}, \qquad B = \begin{pmatrix} \mathcal{K}-\rho_B/i & 1-\mathcal{K} \\ 1+\mathcal{K} & -\mathcal{K}-\rho_B/i \end{pmatrix},$$

where $\rho_A/i, \rho_B/i > 1$ and $\mathcal{K} \in \mathbb{R}\setminus\{\pm 1\}$. In this case, the eigenvalues of A and B are, respectively, $-\rho_A/i \pm 1$ and $-\rho_B/i \pm 1$.

Using these normal forms, following the ideas presented at the beginning of this section, one gets the following stability conditions (see [7, 18] for the proof).

4.1.2 Stability conditions in the diagonalizable case

Theorem 4 *Let A and B be two real matrices such that* **(H1)**, **(H2)**, **(H3)**, *and* **(H4)**, *given in section 4, hold and define $\rho_A, \rho_B, \mathcal{K}, \mathcal{D}$ as in Definition 3. We have the following stability conditions.*

Case (CC) *If A and B have both complex eigenvalues, then*

Case (CC.1). *if $\mathcal{D} < 0$, then $(\mathcal{P})$ is true;*

Case (CC.2). *if $\mathcal{D} > 0$, then*

Case (CC.2.1). *if $\mathcal{K} < -1$, then $(\mathcal{P})$ is false;*

Case (CC.2.2). *if $\mathcal{K} > 1$, then $(\mathcal{P})$ is true if and only if it holds the following condition:*

$$\begin{aligned}\rho_{CC} \quad := \quad & \exp\left[-\rho_A \arctan\left(\frac{-\rho_A\mathcal{K}+\rho_B}{\sqrt{\mathcal{D}}}\right)\right. \\ & \left.-\rho_B \arctan\left(\frac{\rho_A-\rho_B\mathcal{K}}{\sqrt{\mathcal{D}}}\right) - \frac{\pi}{2}(\rho_A+\rho_B)\right] \\ & \times\sqrt{\frac{(\rho_A\rho_B+\mathcal{K})+\sqrt{\mathcal{D}}}{(\rho_A\rho_B+\mathcal{K})-\sqrt{\mathcal{D}}}} < 1.\end{aligned} \tag{7}$$

Case (CC.3). *If $\mathcal{D} = 0$, then $(\mathcal{P})$ holds true or false whether $\mathcal{K} > 1$ or $\mathcal{K} < -1$.*

Case (RC). *If A and B have one of them complex and the other real eigenvalues, define $\chi := \rho_A\mathcal{K} - \rho_B$, where ρ_A and ρ_B are chosen in such a way $\rho_A \in i\mathbb{R}$, $\rho_B \in \mathbb{R}$. Then*

Case (RC.1). *if $\mathcal{D} > 0$, then $(\mathcal{P})$ is true;*

Case (RC.2). *if $\mathcal{D} < 0$, then $\chi \neq 0$ and we have:*

Case (RC.2.1). *if $\chi > 0$, then (P) is false. Moreover, in this case $\mathcal{K}/i < 0$;*

Case (RC.2.2). *if $\chi < 0$, then*

Case (RC2.2.A). *if $\mathcal{K}/i \leq 0$, then $(\mathcal{P})$ is true;*

Case (RC2.2.B). *if $\mathcal{K}/i > 0$, then $(\mathcal{P})$ is true if and only if it holds the following condition:*

$$\begin{aligned}\rho_{RC} := & \left(\frac{m^+}{m^-}\right)^{-\frac{1}{2}(\rho_A/i-1)} e^{-\rho_B\bar{t}} \\ & \times\left(\sqrt{1-\mathcal{K}^2}\, m^- \sin\bar{t} - \left(\cos\bar{t} - \frac{\mathcal{K}}{i}\sin\bar{t}\right)\right) < 1,\end{aligned} \tag{8}$$

where

$$m^{\pm} := \frac{-\chi \pm \sqrt{-\mathcal{D}}}{(-\rho_A/i-1)\mathcal{K}/i}, \qquad \bar{t} = \arccos\frac{-\rho_A/i+\rho_B\mathcal{K}/i}{\sqrt{(1-\mathcal{K}^2)(1+\rho_B^2)}}.$$

Case (RC.3). *If $\mathcal{D} = 0$, then $(\mathcal{P})$ holds true whether $\chi < 0$ or $\chi > 0$.*

Case (RR). *If A and B have both real eigenvalues, then*

Case (RR.1). *if $\mathcal{D} < 0$, then $(\mathcal{P})$ is true; moreover we have $|\mathcal{K}| > 1$;*

Case (RR.2). *if $\mathcal{D} > 0$, then $\mathcal{K} \neq -\rho_A\rho_B$ (notice that $-\rho_A\rho_B > 1$) and*

Case (RR.2.1). *if $\mathcal{K} > -\rho_A\rho_B$, then (P) is false;*

Case (RR.2.2). *if $\mathcal{K} < -\rho_A\rho_B$, then*

Case (RR.2.2.A). *if $\mathcal{K} > -1$, then (P) is true;*

Case (RR.2.2.B). *if $\mathcal{K} < -1$, then (P) is true if and only if the following condition holds:*

$$\begin{aligned}\rho_{RR} := & -f^{sym}(\rho_A,\rho_B,\mathcal{K})f^{asym}(\rho_A,\rho_B,\mathcal{K}) \\ & \times f^{asym}(\rho_B,\rho_A,\mathcal{K}) < 1,\end{aligned} \tag{9}$$

where

$$f^{sym}(\rho_A,\rho_B,\mathcal{K}) := \frac{1+\rho_A/i+\rho_B/i+\mathcal{K}-\sqrt{\mathcal{D}}}{1+\rho_A/i+\rho_B/i+\mathcal{K}+\sqrt{\mathcal{D}}};$$
$$f^{aym}(\rho_A,\rho_B,\mathcal{K}) := \left(\frac{\rho_B/i-\mathcal{K}\rho_A/i-\sqrt{\mathcal{D}}}{\rho_B/i-\mathcal{K}\rho_A/i+\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}(\rho_A/i-1)}.$$

Case (RR.3). *If $\mathcal{D}=0$, then $(\mathcal{P})$ holds true or false whether $\mathcal{K}<-\rho_A\rho_B$ or $\mathcal{K}>-\rho_A\rho_B$.*

Finally, if $(\mathcal{P})$ is not true, then in case CC.2.2 *with $\rho_{CC}=1$, case* (RC.2.2.B), *with $\rho_{RC}=1$, case* (RR.2.2.B), *with $\rho_{RR}=1$, case* (CC.3) *with $\mathcal{K}<-1$, case* (RC.3) *with $\chi>0$ and case* (RR.3) *with $\mathcal{K}>-\rho_A\rho_B$, the origin is just stable. In the other cases, the system is unstable.*

Remark **4** Notice that cases **(CC.2.1)**, **(RC.2.1)** and **(RR.2.1)** correspond to $\mathcal{Z}$ inverse, while cases **(CC.2.2)**, **(RC.2.2)** and **(RR.2.2)** correspond to $\mathcal{Z}$ direct.

4.2 The nondiagonalizable case

In the case in which one or both the matrices are nondiagonalizable, we assume

(H5) A and B are two 2×2 real Hurwitz matrices. Moreover A is nondiagonalizable and $[A,B]\neq 0$.

In this case new difficulties arises. The first is due to the fact that eigenvectors of A and B are at most 3 noncoinciding points on $\mathbb{C}P^1$. As a consequence the cross ratio is not anymore the right parameter describing the interrelation among the systems. It is either not defined or completely fixed. For this reason new coordinate-invariant parameters should be identified and new normal forms for A and B should be constructed. These coordinate invariant parameters are the three real parameters defined in Definition 4 below. One (η) is, up to time reparametrization, the (only) eigenvalue of A, the second (ρ) depends on the eigenvalues of B and the third (k) plays the role of the cross ratio of the diagonalizable case. For $x\in\mathbb{R}$ define

$$sign(x)=\begin{cases}+1 \text{ if } x>0\\ 0 \text{ if } x=0\\ -1 \text{ if } x<0.\end{cases}$$

Definition 4 *Assume* **(H5)** *and let δ be the discriminant of the equation $det(B-\lambda Id)=0$. Define the following invariant parameters:*

$$\eta=\begin{cases}\dfrac{Tr(A)}{\sqrt{|\delta|}} & \text{if } \delta\neq 0\\ \dfrac{Tr(A)}{2} & \text{if } \delta=0,\end{cases}\qquad \rho=\begin{cases}\dfrac{Tr(B)}{\sqrt{|\delta|}} & \text{if } \delta\neq 0\\ \dfrac{Tr(B)}{2} & \text{if } \delta=0,\end{cases}$$

$$k=\begin{cases}\dfrac{4}{|\delta|}\left(Tr(AB)-\dfrac{1}{2}Tr(A)Tr(B)\right) & \text{if } \delta\neq 0\\ Tr(AB)-\dfrac{1}{2}Tr(A)Tr(B) & \text{if } \delta=0.\end{cases}$$

Remark **5** Notice that $\delta=(\lambda_1-\lambda_2)^2\in\mathbb{R}$, where λ_1 and λ_2 are the eigenvalues of B. Notice moreover that B has non real eigenvalues if and only if $\delta<0$. Finally observe that $\eta,\rho<0$ and $k\in\mathbb{R}$.

Definition 5 *In the following, under the assumption* **(H5)**, *we call regular case (***R***-case for short), the case in which $k\neq 0$ and singular case (***S***-for short), the case in which $k=0$.*

4.2.1 Normal forms in the nondiagonalizable case

We have the following (see [8] for the proof):

Lemma 2 **(R-case)** *Assume* **(H5)** *and* $k \neq 0$*. Then it is always possible to find a linear change of coordinates and a constant* $\tau > 0$ *such that* A/τ *and* B/τ *(that we still call* A *and* B*) have the following form:*

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \tag{10}$$

$$B = \begin{pmatrix} \rho & sign(\delta)/k \\ k & \rho \end{pmatrix}. \tag{11}$$

Moreover in this case $[A, B] \neq 0$ *is automatically satisfied.*

Lemma 3 **(S-case)** *Assume* **(H5)** *and* $k = 0$*. Then* $\delta > 0$ *and it is always possible to find a linear change of coordinates and a constant* $\tau > 0$ *such that* A/τ *and* B/τ *(that we still call* A *and* B*) have the following form,*

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad B = \begin{pmatrix} \rho - 1 & 0 \\ 0 & \rho + 1 \end{pmatrix}, \ \textit{called } \mathbf{S}_1\textit{-case}, \tag{12}$$

or the form,

$$A = \begin{pmatrix} \eta & 1 \\ 0 & \eta \end{pmatrix}, \quad B = \begin{pmatrix} \rho + 1 & 0 \\ 0 & \rho - 1 \end{pmatrix}, \ \textit{called } \mathbf{S}_{-1}\textit{-case}. \tag{13}$$

4.2.2 Stability conditions in the nondiagonalizable case

First we need to define some functions of the invariants η, ρ, k. Set $\Delta = k^2 - 4\eta\rho k + sign(\delta)4\eta^2$. By direct computation one gets that if $k = 2\eta\rho$ then $\Delta = -4 \det A \det B < 0$. It follows,

Lemma 4 *Assume* **(H5)***. Then* $\Delta \geq 0$ *implies* $k \neq 2\eta\rho$*.*

Moreover, when $\Delta > 0$ and $k < 0$, define

$$\mathcal{R} = \begin{cases} \left|\left(\frac{-k+\sqrt{\Delta}}{-k-\sqrt{\Delta}}\right)^{\frac{2k\rho^2 - sign(\delta)(k+2\eta\rho+\sqrt{\Delta})}{2k(\rho - sign(\delta)\frac{\eta}{k})\sqrt{\rho^2 - sign(\delta)}}}\right| \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{sign(\delta)}\right), & \text{if } \ \rho - sign(\delta)\frac{\eta}{k} \neq 0, \\ \frac{-2\eta}{\sqrt{k^2+\eta^2}} \exp\left(\frac{\sqrt{\Delta}}{k} + \rho\theta_{-1}\right), & \text{if } \ \rho - sign(\delta)\frac{\eta}{k} = 0 \ \text{ (which implies } \ sign(\delta) = -1). \end{cases} \tag{14}$$

where

$$\begin{aligned} \theta_{-1} &= \begin{cases} \arctan \frac{\sqrt{\Delta}}{k(\rho + \frac{\eta}{k}) + \eta} & \text{if } \ k(\rho + \frac{\eta}{k}) + \eta \neq 0 \\ \pi/2 & \text{if } \ k(\rho + \frac{\eta}{k}) + \eta = 0, \end{cases} \\ \theta_1 &= \operatorname{arctanh} \frac{\sqrt{\Delta}}{k(\rho - \frac{\eta}{k}) - \eta}, \\ \theta_0 &= \frac{\sqrt{\Delta}}{k\rho}. \end{aligned}$$

Notice that when $k < 0$, then $k(\rho - \frac{\eta}{k}) - \eta > 0$ and $k\rho > 0$. Hence θ_1 and θ_0 are well defined.

The following Theorem states the stability conditions in the case in which A is nondiagonalizable. For the proof see [8]. The letters **A.**, **B.**, and **C.** refer to the cases described at the beginning of this section and in Figure 2. Recall Lemma 4.

Theorem 5 *Assume* **(H5)***. We have the following stability conditions for the system (5).*

A. *If* $\Delta < 0$*, then the system is GUAS.*

B. *If* $\Delta > 0$, *then:*

B1. *if* $k > 2\eta\rho$, *then the system is unbounded,*

B2. *if* $k < 2\eta\rho$, *then*

- *in the regular case* ($k \neq 0$)*, the system is GUAS, uniformly stable (but not GUAS) or unbounded respectively if*

$$\mathcal{R} < 1, \mathcal{R} = 1, \mathcal{R} > 1.$$

- *In the singular case* ($k = 0$)*, the system is GUAS.*

C. *If* $\Delta = 0$, *then:*

C1 *If* $k > 2\eta\rho$, *then the system is uniformly stable (but not GUAS),*

C2 *if* $k < 2\eta\rho$, *then the system is GUAS.*

5 An Open problem

Many problems connected to the ideas presented in this paper as still open. In the following we present a problem that, in our opinion, is of great interest.

Open Problem Find necessary and sufficient conditions for the stability of a bilinear switched system in dimension 3. This problem seems to be quite difficult, and the techniques presented in this paper cannot be applyed. (The reason is that these techniques use implicitly the Jordan separation lemma.) Even if studying all possible cases is probably too complicated, one would like to see if the set of all pairs of 3×3 matrices giving rise to a a GUAS system can be defined with a finite number of inequalities involving analytic functions, exponentials and logarithm. This was the case for bilinear planar systems, and it would be already very interesting to see if the same holds in dimension 3.

Another formulation of this problem is to extend to bilinear systems in dimension 3 the following corollary of the results given in the previous section.

Corollary 1 *The set of all pairs of* 2×2 *real matrices giving rise to a GUAS system is a set in the log-exp category.*

For the precise definition of the log-exp category, see for instance [12, 15].

<u>*Remark*</u> **6** Notice that the topological boundary of the set described by Corollary 1 is related to the problem of finding the right functional class where to look for LFs. Another interesting problem is to clarify this relation. This problem is already interesting in dimension 2.

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