Partial Differential Equations. — *Gaussian estimates for hypoelliptic operators via optimal control.* Nota di Ugo Boscain e Sergio Polidoro.

Abstract. — We obtain Gaussian lower bounds for the fundamental solution of a class of hypoelliptic equations, by using repeatedly an invariant Harnack inequality. Our main result is given in terms of the value function of a suitable optimal control problem.

Key words: hypoelliptic equations, Lie groups, Gaussian bounds, optimal control theory.

Sunto. — *Stime Gaussiane per operatori ipoellittici mediante controllo ottimo.* Si dimostrano stime Gaussiane dal basso della soluzione fondamentale per una classe di equazioni ipoellittiche, mediante l'uso ripetuto di una disuguaglianza di Harnack invariante. Il nostro principale risultato è espresso in termini della funzione valore di un opportuno problema di controllo ottimo.

1 Introduction

We consider a class of linear second order operators in $\mathbb{R}^{N+1}$ of the form

$$L := \sum_{k=1}^{m} X_k^2 + X_0 - \partial_t. \tag{1.1}$$

In (1.1) the X_k's are smooth vector fields on $\mathbb{R}^N$, i.e. denoting $z = (x,t)$ the point in $\mathbb{R}^{N+1}$

$$X_k(x) = \sum_{j=1}^{N} a_j^k(x)\partial_{x_j}, \qquad k = 0, \dots, m.$$

In the sequel we will also consider the X_k's as vector fields in $\mathbb{R}^{N+1}$ and denote

$$Y = X_0 - \partial_t. \tag{1.2}$$

Our main assumption on the operators L is the invariance with respect to a homogeneous Lie group structure, and a controllability condition:

Hypothesis [H] *There exists a homogeneous Lie group* $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ, \delta_\lambda\right)$ *such that*

(i) $X_1, \dots, X_m, Y$ *are left translation invariant on* $\mathbb{G}$*;*

(ii) $X_1, \dots, X_m$ *are* δ_λ*-homogeneous of degree one and* Y *is* δ_λ*-homogeneous of degree two.*

HYPOTHESIS [C] *For every* $(x,t),(y,s) \in \mathbb{R}^{N+1}$ *with* $t > s$*, there exists an absolutely continuous path* $\gamma : [0, t-s] \to \mathbb{R}^N$ *such that*

$$\begin{cases} \dot\gamma(\tau) = & \displaystyle\sum_{k=1}^m \omega_k(\tau) X_k(\gamma(\tau)) + X_0(\gamma(\tau)) \\ \gamma(0) = & x, \quad \gamma(t-s) = y, \end{cases} \tag{1.3}$$

with $\omega_1, \dots, \omega_m \in L^\infty([0, t-s])$.

In the sequel, the solution of (1.3) will be denoted by $\gamma((x,t),(y,s),\omega)$.

Operators of the form (1.1), verifying hypotheses [C] and [H], have been considered by Kogoj and Lanconelli in [10] and [11]. An invariant Harnack inequality for the postive solutions of $Lu = 0$ is proved in [10], a general procedure for the construction of sequences of operator satisfying assumptions [C] and [H], is given in [11]. We next give some comments about these assumptions. We first compare the controllability property [C], with some properties of the commutators of $X_1, \dots, X_m, Y$. It is known that condition [H] implies that the coefficients a_j^k's of the X_k's are polynomial functions, hence we can rely on a classical results (see Derridj and Zuily [5] and Oleĭnik and Radkevič [16], Chap. II, Sec. 8) to see that [C] yields

$$\text{rank Lie}\{X_1, \dots, X_m, Y\}(z) = N + 1, \qquad \forall z \in \mathbb{R}^{N+1}. \tag{1.4}$$

Note that it is not true that [C] is a consequence of (1.4), nevertheless it is well known that the condition

$$\text{rank Lie}\big\{X_1, \dots, X_m\big\}(x) = N, \qquad \forall x \in \mathbb{R}^N, \tag{1.5}$$

(which is stronger than (1.4)) implies [C] (see for instance the books of Agrachev and Sachkov [1] and Jurdjevic [9]).

In the theory of the partial differential equations, the above properties are strongly related to the regularity problem for L. Specifically, condition (1.4) is the well known sufficient condition for the hypoellipticity of L introduced by Hörmander in [7]. In [10] it is proved that L has a fundamental solution Γ which is invariant with respect to the group operation, is smooth out of its poles and δ_λ-homogeneous of degree $2 - Q$:

$$\Gamma(z,\zeta) = \Gamma(\zeta^{-1} \circ z, 0), \qquad \Gamma(\delta_\lambda z, 0) = \lambda^{2-Q}\Gamma(z,0), \tag{1.6}$$

for every $z, \zeta \in \mathbb{R}^{N+1}$ and $\lambda > 0$ (here Q denotes the homogeneous dimension of the Lie group $\mathbb{G}$, see Section 2). Moreover, $\Gamma(x,t,\xi,\tau) > 0$ for $t > \tau$, and $\Gamma(x,t,\xi,\tau) = 0$ for $t \leq \tau$.

The main purpose of this paper is to adapt a method due to Moser [14] and used by Aronson and Serrin [2], [3], in order to prove a Gaussian lower bound of Γ. We recall that the method by Moser has been introduced in the study of uniformly parabolic operators and is based on the repeated use of an invariant Harnack inequality. In that framework, the Gaussian bound reads as follows: *let h be the fundamental solution of an uniformly parabolic operator. Then there exists a positive constant c such that*

$$h(x-y, t-s) \geq \frac{c}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{c\,(t-s)}}, \tag{1.7}$$

for every $(x,t), (y,s) \in \mathbb{R}^{N+1}$ with $t > s$. In order to adapt the method to operators of type (1.1), we rely on the following invariant Harnack inequality proved by Kogoj and Lanconelli. Consider the sets $H_r(z_0) = z_0 \circ \delta_r(H_1)$, and $S_r(z_0) = z_0 \circ \delta_r(S_1)$, where

$$\begin{aligned} H_1 &= \left\{(x,t) \in \mathbb{R}^{N+1} \;\mid\; \|(x,t)\|_{\mathbb{G}} \leq 1,\; t \leq 0\right\}, \\ S_1 &= \{(x,t) \;\in H_1 \mid 1/4 \leq -t \leq 3/4\}\,. \end{aligned}$$

Then the following result holds (see [10], Theorem 7.1). *Let Ω be an open subset of $\mathbb{R}^{N+1}$ containing $H_r(z_0)$ for some $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$. Then, there exist two positive constants θ and M, only depending on the operator L, such that*

$$\sup_{S_{\theta r}(z_0)} u \leq M\, u(z_0), \tag{1.8}$$

for every non-negative solution u of $Lu = 0$ in Ω. Our first result is a *non-local* lower bound for positive solutions to $Lu = 0$ obtained by the (local) Harnack inequality (1.8).

Proposition 1.1 *Let L be as defined in* (1.1)*, satisfying assumptions [C] and [H]. Then there exist three constants $\theta \in]0,1[, h > 0$ and $M > 1$, only depending on the operator L, such that the following statement is true. If $u : \mathbb{R}^N \times]T_0, T_1] \to \mathbb{R}$ is a positive solution to $Lu = 0$, $(x,t), (y,s) \in \mathbb{R}^N \times]T_0, T_1]$ are two points such that $T_1 - \theta^2(T_1 - T_0) \leq s < t \leq T_1$, and $\gamma((x,t),(y,s),\omega)$ is a solution to* (1.3)*, then*

$$u(y,s) \leq M^{1+\frac{\Phi(\omega)}{h}} u(x,t),$$

where

$$\Phi(\omega) = \int_0^{t-s} \left(\omega_1^2(\tau) + \cdots + \omega_m^2(\tau)\right) d\tau.$$

The above Proposition extends a previous result by Pascucci and Polidoro (Theorem 1.1 in [17]) and gives a bound for *any* solution γ of (1.3). In order to obtain the best exponent we formulate the natural optimal control problem: we consider the function $\omega_1, \dots, \omega_m$ as the *control* of the path γ in (1.3) and we look for the one minimizing the total cost Φ among the paths γ satisfying (1.3). We then define the value function

$$V(x,t,y,s) = \inf \left\{ \Phi(\omega) \mid \gamma((x,t),(y,s),\omega) \text{ is a solution to (1.3) } \right\}. \quad (1.9)$$

As a straightforward corollary of Proposition 1.1, we obtain that

$$u(y,s) \le M^{1+\frac{V(x,t,y,s)}{h}} u(x,t), \quad (1.10)$$

provided that u satisfies the assumptions of Proposition 1.1. A further direct consequence is the following lower bound for the fundamental solution Γ of L:

Theorem 1.2 *Let L be as defined in* (1.1)*, satisfying assumptions [C] and [H]. Then there exist two constants $C > 0$ and $\theta \in]0,1[$, only depending on the operator L, such that*

$$\Gamma(x,t,0,0) \ge \frac{1}{C\, t^{\frac{Q-2}{2}}} e^{-CV(x,\theta^2 t,0,0)} \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Thanks to (1.6), Theorem 1.2 provides a lower bound for $\Gamma(x,t,y,s)$ with $t > s$.

We next compare the above result with the known estimates of the fundamental solution due to Jerison and Sánchez-Calle [8], Kusuoka and Stroock [12], Varopoulos, Saloff-Coste and Coulhon [18], concerning operators in the form (1.1) without the drift term X_0. The main result in [8], [12], and in [18] is the bound

$$\frac{1}{C\sqrt{|\mathcal{B}_{t-s}(x)|}} e^{-\frac{C\, d^2(x,y)}{t-s}} \le \Gamma(x,t,y,s) \le \frac{C}{\sqrt{|\mathcal{B}_{t-s}(x)|}} e^{-\frac{d^2(x,y)}{C\,(t-s)}}, \quad (1.11)$$

for every $(x,t),(y,s) \in \mathbb{R}^N \times]T_0,T_1]$ with $t > s$, where $d(x,y)$ denotes the Carnot-Carateodory distance associated to the problem (1.3), in which the vector field X_0 is set to zero, (see [15]) and $|\mathcal{B}_r(x)|$ is the volume of the metric ball with center at x and radius r. The lower bound stated in Theorem 1.2 agrees with the one stated in (1.11), since

$$V(x,t,y,s) = \frac{d^2(x,y)}{t-s} \qquad \text{when } X_0 = 0. \tag{1.12}$$

The identity (1.12) fails when the drift term X_0 is needed to fulfill condition [C]. Consider for instance the Kolmogorov operators

$$Ku = \sum_{i,j=1}^{p_0} a_{i,j}\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u - \partial_t u,$$

where $A = (a_{ij})_{i,j=1,\dots,p_0}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ are real constant matrices, A is symmetric and positive. We recall that assumptions [C] and [H] are equivalent to some explicit conditions on the matrices A and B (see [13]). Moreover, the explicit expression of the value function for this class of operators is explicitly known (see [6]). In the simplest case, the Kolmogorov equation reads

$$\partial^2_{x_1} u + x_1 \partial_{x_2} u = \partial_t u$$

and the value function related to the Kolmogorov group is

$$\begin{aligned} V(x,t,y,s) = & \frac{(x_1-y_1)^2}{t-s} + \\ & 3\frac{(x_1-y_1)(x_2+(t-s)y_1-y_2)}{(t-s)^2} + 3\frac{(x_2+(t-s)y_1-y_2)^2}{(t-s)^3}, \end{aligned}$$

which clearly does not satisfy equation (1.12).

Aiming to show that the estimate given in Theorem 1.2 is sharp, we remark that one can prove an analogous upper bound for the fundamental solution. More specifically, under suitable conditions on the vector fields $X_0, \dots, X_m$, which guarantee the existence of global solutions of the problem (1.3), and assuming that there are no singular minimizers, then one has

$$\Gamma(x,t,0,0) \le \frac{C_\varepsilon}{t^{\frac{Q-2}{2}}} e^{-\frac{V((0,\varepsilon t)\circ(x,t)\circ(0,\varepsilon t),0,0)}{32}} \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

for every positive ε. The above inequality is obtained by a suitable adaptation of the method introduced by Aronson in [2] (details are given in [4]).

We recall that in the case of Kolmogorov equations, for every $\widetilde{c} > 1$ there exists a positive constant $\widetilde{C}$ such that $V(x,t,0,0) \leq \widetilde{C}\, V(x,\widetilde{c}t,0,0) \leq \widetilde{C}\, V(x,t,0,0)$, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ (see formula (6.13) in [6]). As a consequence, bounds analogous to (1.11) hold

$$\frac{1}{C(t-s)^{\frac{Q-2}{2}}} e^{-CV(x,t,y,s)} \leq \Gamma(x,t,y,s) \leq \frac{C}{(t-s)^{\frac{Q-2}{2}}} e^{-\frac{V(x,t,y,s)}{C}},$$

for every $(x,t),(y,s) \in \mathbb{R}^{N+1}$ with $t > s$.

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2 Proof of the main results

A Lie group $\mathbb{G} = \left(\mathbb{R}^{N+1}, \circ\right)$ is called *homogeneous* if a family of dilations $(\delta_\lambda)_{\lambda>0}$ exists on $\mathbb{G}$ and $\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta)$ for every $z, \zeta \in \mathbb{R}^{N+1}$ and for any $\lambda > 0$. In our setting, hypotheses [C] and [H] imply that $\mathbb{R}^N$ has a direct sum decomposition

$$\mathbb{R}^N = V_1 \oplus \cdots \oplus V_n$$

such that, if $x = x^{(1)} + \cdots + x^{(n)}$ with $x^{(k)} \in V_k$, then the dilations are

$$\delta_\lambda(x^{(1)} + \cdots + x^{(n)}, t) = (\lambda x^{(1)} + \cdots + \lambda^n x^{(n)}, \lambda^2 t), \tag{2.1}$$

for any $(x,t) \in \mathbb{R}^{N+1}$ and $\lambda > 0$. We may assume that

$$\begin{aligned} x^{(1)} &= (x_1, \ldots, x_{m_1}, 0, \ldots, 0) \in V_1, \\ x^{(k)} &= (0, \ldots, 0, x_1^{(k)}, \ldots, x_{m_k}^{(k)}, 0, \ldots, 0) \in V_k, \end{aligned}$$

for some basis of $\mathbb{R}^N$, where

$$x_i^{(k)} = x_{m_1+\cdots+m_{k-1}+i}, \qquad i = 1, \ldots, m_k \equiv \dim V_k.$$

The natural number

$$Q = \sum_{k=1}^{n} k\, m_k + 2$$

is usually called the *homogeneous dimension* of $\mathbb{G}$ with respect to (δ_λ). We also introduce the following δ_λ-homogeneous norms on $\mathbb{R}^{N+1}$ and $\mathbb{R}^N$:

$$|x|_{\mathbb{G}} = \max\left\{ \left|x_i^{(k)}\right|^{\frac{1}{k}} \ \mid \ k=1,\dots,n,\ i=1,\dots,m_k \right\},$$
$$\|(x,t)\|_{\mathbb{G}} = \max\left\{ |x|_{\mathbb{G}}, |t|^{\frac{1}{2}} \right\}.$$

Since $X_1,\dots,X_m$ and Y are smooth vector fields which are δ_λ-homogeneous respectively of degree one and two, then

$$\begin{aligned} X_k &= \sum_{j=1}^{n} a_{j-1}^k(x^{(1)},\dots,x^{(j-1)}) \cdot \nabla^{(j)}, \qquad k=1,\dots,m, \\ Y &= \sum_{j=2}^{n} b_{j-2}(x^{(1)},\dots,x^{(j-2)}) \cdot \nabla^{(j)} - \partial_t, \end{aligned} \tag{2.2}$$

where

$$\nabla^{(j)} = \big(0,\dots,0,\partial_{x_1^{(j)}},\dots,\partial_{x_{m_j}^{(j)}},0,\dots,0\big).$$

and a_j^k and b_j are $\delta_\lambda-$homogeneous polynomial functions of degree j with values in V_{j+1} and V_{j+2} respectively. Let us explicitly note that formula (2.2) says that $\mathrm{span}\big\{X_1(0),\dots,X_m(0)\big\} = V_1$; then we may assume $m = m_1$ and $X_j(0) = \mathbf{e}_j$ for $j = 1,\dots,m$ where $\{\mathbf{e}_i\}_{1\le i\le N}$ denotes the canonical basis of $\mathbb{R}^N$. Also note that from (2.2) it follows that $V_2 = \mathrm{span}\big\{X_0(0),[X_j,X_k](0), j,k=1,\dots,m\big\}$. Moreover, up to the linear change of variable $(x,t)\mapsto(x-t\,b_0,t)$, we may (and we do) assume that $b_0 = X_0(0) = 0$.

As said in the introduction, our argument mainly relies on the Harnack inequality (1.8) by Kogoj and Lanconelli ([10], Theorem 7.1). We first state a corollary of it, we refer to Proposition 3.2 in [17] for the proof.

Proposition 2.1 *Let Ω be an open set in $\mathbb{R}^{N+1}$ containing $H_r(z_0)$ for some $z_0\in\mathbb{R}^{N+1}$ and $r>0$. Then*

$$u(z_0\circ z) \le M\,u(z_0) \tag{2.3}$$

for every non-negative solution u of $Lu=0$ in Ω and for every z in the positive cone

$$\mathcal{P}_r = \left\{(x,-t)\in\mathbb{R}^{N+1} \ \mid\ |x|_{\mathbb{G}}^2 \le 2\,t,\ 0<t\le 2\,\theta^2 r^2\right\}. \tag{2.4}$$

In order to prove Proposition 1.1 we need a preliminary result.

Lemma 2.2 *Let $\gamma : [0,T] \to \mathbb{R}^N$ be a solution to (1.3), and let $r = \frac{\sqrt{2T}}{2\theta}$. There exists a positive constant h, only depending on the operator L, such that $(\gamma(s), t-s) \in (x,t) \circ \mathcal{P}_r$ for every $s \in [0,T]$ such that $\int_0^s |\omega(\tau)|^2 d\tau \le h$.*

PROOF. We first prove the claim in the case $(x,t) = (0,0)$, namely

$$\dot\gamma(\tau) = \sum_{j=1}^{m} \omega_j(\tau) X_j(\gamma(\tau)) + X_0(\gamma(\tau)), \qquad \gamma(0) = 0. \tag{2.5}$$

The result in the general case directly follows from the invariance of the vector fields $X_1, \dots, X_m$ and Y with respect to the operation "$\circ$". We then prove that, for sufficiently small s, $(\gamma(s), -s) \in \mathcal{P}_r$, that is

$$\left| \gamma^{(k)}(s) \right|_{\mathbb{G}}^2 = \max_{i=1,\dots,m_k} \left| \gamma_i^{(k)}(s) \right|^{\frac{2}{k}} \le 2s, \tag{2.6}$$

for any $k = 1, \dots, n$. To this aim, we consider the function

$$F(s) = \int_0^s |\omega(\tau)|^2 d\tau, \quad \text{for } 0 \le s \le T.$$

We claim that

$$\left| \gamma^{(k)}(s) \right|^2 \le c_k \left(F(s) + F(s)^k \right) s^k, \quad \text{for every } s \in [0,T], \tag{2.7}$$

for $k = 1, \dots, n$, and for some positive constants $c_1, \dots, c_n$ that only depend on the operator L. Since $F(0) = 0$ and F is a continuous increasing function, from (2.7) it follows that we can choose a positive h such that condition (2.6) holds whenever $F(s) \le h$. Hence we only need to prove (2.7).

We first consider $\gamma_j(\tau)$ for $j = 1, \dots, m$. Since $X_j(0) = \mathbf{e}_j$ for $j = 1, \dots, m$, we have

$$|\gamma_j(s)| = \left| \int_0^s \omega_j(\tau) d\,\tau \right| \le \int_0^s |\omega_j(\tau)|\, d\,\tau \le \left(\int_0^s |\omega(\tau)|^2\, d\,\tau \right)^{\frac{1}{2}} \sqrt{s}, \tag{2.8}$$

so that condition (2.7) is satisfied for $k = 1$ with $c_1 = \frac{1}{2}$.

Next, we have

$$\dot\gamma^{(2)}(\tau) = \sum_{j=1}^{m} \omega_j(\tau)\, a_1^j(\gamma^{(1)}(\tau))$$

where the $a_1^1, \dots, a_1^m$ are linear functions (recall that $b_0 = 0$). Then,

$$\left|\gamma^{(2)}(s)\right| \le c_2' \int_0^s |\omega(\tau)| \left|\gamma^{(1)}(\tau)\right| d\tau \le c_2' \left(\int_0^s |\omega(\tau)|^2 \, d\tau\right)^{\frac{1}{2}} s\sqrt{F(s)/2},$$

by (2.8), where the constant c_2' only depends on the coefficients a_1^j. Hence the components $\gamma^{(2)}(s)$ satisfy condition (2.7) with $c_2 = (c_2')^2/2$.

We also explicitly consider $k = 3$:

$$\dot\gamma^{(3)}(\tau) = \sum_{j=1}^m \omega_j(\tau)\, a_2^j(\gamma^{(1)}(\tau), \gamma^{(2)}(\tau)) + b_1(\gamma^{(1)}(\tau))$$

where the a_2^j's are δ_λ-homogeneous functions of degree 2 and b_1 is linear. Then,

$$\begin{aligned}\left|\gamma^{(3)}(s)\right| \le & c_3' \int_0^s \left(|\omega(\tau)| \left(\left|\gamma^{(1)}(\tau)\right|^2 + \left|\gamma^{(2)}(\tau)\right|\right) + \left|\gamma^{(1)}(\tau)\right|\right) d\tau \le \\ & c_3'' \left(\left(\int_0^s |\omega(\tau)|^2 \, d\tau\right)^{\frac{1}{2}} F(s) s^{\frac{3}{2}} + F(s)^{\frac{1}{2}} s^{\frac{3}{2}}\right),\end{aligned}$$

by the previous estimates of $\gamma^{(1)}$ and $\gamma^{(2)}$, where the constant c_3' only depends on the coefficients of a_2^j and b_1, while c_3'' depends on c_1 and c_2. Hence the components $\gamma^{(3)}(s)$ satisfy condition (2.7), for some c_3 that depends on L.

For $k = 4, \dots, n$, we have

$$\begin{aligned}\dot\gamma^{(k)}(\tau) = \sum_{j=1}^m \omega_j(\tau) a_{k-1}^j(\gamma^{(1)}(\tau), \dots, \gamma^{(k-1)}(\tau)) + \\ b_{k-2}(\gamma^{(1)}(\tau), \dots, \gamma^{(k-2)}(\tau)),\end{aligned}$$

and, since a_k^j and b_k are δ_λ-homogeneous functions of degree k, a straightforward inductive argument yields

$$\begin{aligned}\left|\gamma^{(k)}(s)\right| \le c_k' \int_0^s \left(|\omega(\tau)|\, \tau^{\frac{k-1}{2}} \left(F(\tau)^{\frac{1}{2}} + F(\tau)^{\frac{k-1}{2}}\right) + \right. \\ \left. \tau^{\frac{k-2}{2}} \left(F(\tau)^{\frac{1}{2}} + F(\tau)^{\frac{k-2}{2}}\right)\right) d\tau\end{aligned}$$

where the constant c'_k depends on $c_1, \dots, c_{k-1}$ and on the coefficients a^j_{k-1} and b_{k-2}. By the Hölder inequality we then find

$$\left|\gamma^{(k)}(s)\right| \le c''_k \Bigg(\left(\int_0^s |\omega(\tau)|^2 \, d\tau \right)^{\frac{1}{2}} \cdot \left(F(s)^{\frac{1}{2}} + F(s)^{\frac{k-1}{2}} \right) + \left(F(s)^{\frac{1}{2}} + F(s)^{\frac{k-2}{2}} \right) \Bigg) s^{\frac{k}{2}},$$

and the inequality (2.7) then follows for k. This concludes the proof. □

Proof of Proposition 1.1. Let h, θ and M be the constants of the Lemma 2.2, let $T = t - s$ and note that $H_r(x,t) \subset \mathbb{R}^N \times]T_0, T_1]$ for $r = \sqrt{t - T_0}$.

If $\int_0^{t-s} |\omega(\tau)|^2 d\tau \le h$, then the result is an immediate consequence of Lemma 2.2 and Proposition 2.1, since $t - s < \theta^2 r^2$, by our assumption.

If the above inequality is not satisfied, we set

$$k = \max \left\{ j \in \mathbb{N} : j\, h < \int_0^{t-s} |\omega(\tau)|^2 d\tau \right\}, \tag{2.9}$$

and define

$$\sigma_j = \inf_{\sigma > 0} \int_0^{\sigma} |\omega(\tau)|^2 d\tau > j\, h, \qquad t_j = t - \sigma_j, \qquad j = 1, \dots, k.$$

Note that $s < t_k < \cdots < t_1 < t$, so that

$$H_{r_j}(\gamma(\sigma_j), t_j)) \subset \mathbb{R}^N \times]T_0, T_1] \qquad \text{for } r_j = \sqrt{t_j - T_0} \; j = 0, \dots, k, \tag{2.10}$$

and $t_j - t_{j+1} < \theta^2 r_j^2$ for $j = 0, \dots, k$ (here $t_0 = t$), and $t_k - s < \theta^2 r_k^2$.

By Lemma 2.2 $(\gamma(\sigma_1), t_1) \in (x,t) \circ \mathcal{P}_{r_0}$, then we can use Proposition 2.1 to get $u(\gamma(\sigma_1), t_1) \le M\, u(x,t)$. We next repeat the above argument: Lemma 2.2 ensures that $(\gamma(\sigma_2), t_2) \in (\gamma(\sigma_1), t_1) \circ \mathcal{P}_{r_1}$. We then recall (2.10) and apply Proposition 2.1, that gives $u(\gamma(\sigma_2), t_2) \le M u(\gamma(\sigma_1), t_1) \le M^2\, u(x,t)$. We iterate the argument until, at the $(k+1)$-th step, we find

$$u(y,s) \le M u(\gamma(\sigma_k), t_k) \le M^{k+1}\, u(x,t).$$

The thesis then follows from (2.9). □

Proof of Theorem 1.2. Let $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$. Under the hypothesis of Proposition 1.1, applied with $T_0 = 0, T_1 = t$ and $(y,s) = \big(0, \left(1 - \theta^2\right) t\big)$, it follows from (1.10) that

$$\Gamma(x,t,0,0) \ge M^{-1-\frac{1}{h} V\left(x,t,0,(1-\theta^2)t\right)} \Gamma\left(0, \left(1-\theta^2\right) t, 0, 0\right).$$

The proof then follows from the fact that

$$\Gamma\left(0, \left(1-\theta^2\right) t, 0, 0\right) = \frac{\Gamma(0,1,0,0)}{\left(t(1-\theta^2)\right)^{\frac{Q-2}{2}}},$$

as a consequence of the second identity in (1.6), and that

$$V\left(x, t, 0, (1-\theta^2)t\right) = V\left(x, \theta^2 t, 0, 0\right).$$

□

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