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CLASSIFICATION OF STABLE TIME-OPTIMAL CONTROLS ON 2-MANIFOLDS

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ABSTRACT. In this paper, we provide a topological classification via graphs of time-optimal flows for generic control systems of the form $\dot{x} = F(x) + uG(x)$, $x \in M$, $|u| \leq 1$, on two-dimensional orientable compact manifolds, also proving the structural stability of generic optimal flows. More precisely, adding some additional structure to topological graphs, more precisely, rotation systems, and owing to a theorem of Heffter, dating back to the 19th century, we prove that there is a one-to-one correspondence between graphs with rotation systems and couples formed by a system and the 2-D manifold of minimal genus in which the system can be embedded.

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1. Introduction

The aim of this paper is to provide a topological classification of optimal flows for the following geometric problem. Consider a smooth two-dimensional manifold M and a pair X, Y of smooth vector fields on M. Given a point x_0, we want to reach every point of M from this point at a minimum time, gluing together trajectories of these two vector fields X and Y.

It may happen that the minimum time is obtained only by a trajectory γ whose velocity $\dot{\gamma}(t)$ belongs to the segment joining $X(\gamma(t))$ and $Y(\gamma(t))$ (not being an extreme point). Thus, for existence purposes, we consider the set of velocities $\{vX(x) + (1-v)Y(x) : 0 \leq v \leq 1\}$ which does not change the value of the infimum time.

This natural geometric problem can be restated as the minimum-time problem for the following control system. Defining $F = \dfrac{Y+X}{2}$ and $G = \dfrac{Y-X}{2}$, we can write this control system

$$\dot{x} = F(x) + uG(x), \quad x \in M, \quad |u| \leq 1, \tag{1}$$

where F and G are $\mathcal{C}^\infty$ vector fields on M.

This problem (with the additional hypothesis $F(x_0) = 0$ guaranteeing the local controllability) was studied for $M = \mathbb{R}^2$ in [5, 6, 16, 17]. In these papers, under generic assumptions, was proved the existence of an optimal synthesis in the sense of Boltyanskii–Brunovsky (see [3, 7, 8]) i.e., a collection of optimal trajectories, one for each point of the plane, representing the "optimal dynamics." Moreover, their structural stability was proved and a complete classification via some topological graphs was given. The classification program consists in finding a open dense set Π in the space of pairs of vector fields Ξ, an equivalence relation $\sim$ on Ξ, and a class of topological graphs $\mathcal{G}$ such that the following holds. For

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every pair in Π, there exists an "optimal dynamics." Two systems are equivalent if the corresponding "optimal dynamics" are topologically equivalent and the following conditions hold:

1. There is a bijective correspondence between the set of equivalence classes of $\sim$ in Π and the set of graphs $\mathcal{G}$.
2. There are admissibility conditions that individualize the graphs corresponding to some pair.
3. The pairs of Π are structurally stable, i.e., a small perturbation does not change the equivalence class.

This research was inspired by the classical work of Peixoto on the classification of dynamical systems on two-dimensional manifolds. We recall that for two-dimensional smooth dynamical systems, the set $\mathcal{G}$ consists of topological graphs and the equivalence relation $\sim$ is an orbital equivalence (see [2]). Let us explain this in more detail.

The set $V^r(M)$ of C^r-smooth vector fields on a manifold M can be transformed into a metric space in which $X, Y \in V^r(M)$ are ε-close whenever X, Y and their derivatives up to the order r are ε-close. Recall that set $V_0 \subset V^r(M)$ is said to be "generic" if V_0 is open and dense in $V^r(M)$.

A dynamical system is structurally stable (or Morse–Smale) if every ε-small perturbation does not affect the (local and global) qualitative pattern of its trajectories. We identify $V^r(M)$ with continuous one-parameter dynamical systems (or flows) on M.

One of the most beautiful and deep results in low-dimensional topology (due to M. M. Peixoto) says that Morse–Smale flows are "generic" in $V^r(M)$, provided M is a closed oriented 2-manifold. Note that if M is a nonoriented surface, e.g., the Klein bottle, or $\dim\ M \geq 3$, then the above statement is no longer true. Peixoto's theorem illustrates how profoundly the topology and the dynamics depend on each other.

Historically, the structural stability in case of simply connected domains was introduced in 1937 by Andronov and Pontryagin. Influenced by Morse and advised by Lefschetz, M. M. Peixoto generalized the structural stability to the case of 2-manifolds [14]. Peixoto described in geometric terms the Morse–Smale flows as those having only "rough" limit cycles and fixed points.

Let $V_0 \subset V^r(M)$ be the set of Morse–Smale vector fields on M. In [15], Peixoto sets up a classification scheme for V_0 based on the concept of "distinguished graph." Peixoto's classification theorem establishes a bijection between equivalence classes of V_0 and $\mathcal{G}$, where $\mathcal{G}$ is the set of distinguished graphs. From now on, the classification of the Morse–Smale flows reduces to a problem of graph theory, precisely, find and classify all distinguished graphs. A simplification of Peixoto's graphs can be attained using the concept of "rotation systems" due to Heffter [10].

The aim of this paper is to extend the results of [5, 6, 16, 17] to a more general setting of a two-dimensional smooth manifold. Using the local analysis of [5, 16, 17] and some ideas coming from the topology of two-dimensional manifolds (see [12, 13]), we provide a complete classification of "optimal dynamics."

There are various concepts of solution for the minimum-time problem from x_0 for (1). The classical concept of solution is a feedback control. An optimal feedback control for the above problem is a function $u : M \to [-1, 1]$ such that the corresponding trajectories are time-optimal connecting x_0 to the points of the manifold M. In general, an optimal feedback for this problem is discontinuous but has enough regularity to give a good definition of weak solution and ensure forward existence and backward uniqueness of trajectories. Another way of finding a solution is to construct an optimal synthesis, i.e., a collection of trajectories for (1) of the form $\Gamma = \{\gamma_x : x \in M\}$ such that γ_x steers x_0 to x at a minimum time. The concept of synthesis was discussed in [18], where the main advantages are illustrated.

A first step for the global classification was the classification of local structurally stable singularities (see [17]). This local classification remains valid for the case of a general two-dimensional manifold. The equivalence relation used in [5, 6] is an orbital type equivalence preserving the topological structure of the optimal synthesis. We use the same definition of equivalence in this paper. The classification obtained in [6] is based on topological graphs with some additional structure. This additional structure is necessary for obtaining a minimal set of invariants to individualize an optimal synthesis.

A key ingredient to obtain a satisfactory classification is the structural stability. The structural stability was guaranteed by a detailed study of singularities and ensured under generic conditions. The analysis of systems of the form (1) can be pushed much further as explained in [4].

In this paper, we consider the whole set of systems on general two-dimensional orientable manifolds: pairs $(\mathcal{M}, \Sigma)$, where M is a manifold and Σ a control system of the form (1) on M. Therefore, we need to put more structure on graphs to individualize both the topology of the manifold and the system corresponding to a graph. This is obtained by using the concept of rotation system on graph and Heffter's theorem (see [10]). Moreover, we replace the additional information on the topological graphs introduced in [6] by some properties of the graph itself and of the dual graph that is naturally defined once one has the rotation system. The obtained structure is called *optgraph* (see Definition 8) and is denoted by a pair $(\mathcal{G}, R)$, where $\mathcal{G}$ is a topological graph and R is a rotation system on $\mathcal{G}$. Also, we are able to provide a complete description of the possible faces of a graph with the rotation system corresponding to a pair manifold–system; see Sec. 4.2.

Assuming that the vector fields are complete, fixing a positive time τ, and restricting to trajectories whose total time is less than or equal to τ, we can consider the case of a compact manifold. The corresponding generic set of systems admitting an optimal synthesis is denoted by Π_τ. The first main result is the following.

Theorem 1. *Let Σ_1 and Σ_2 be two control systems on a compact 2-dimensional orientable manifold M. Assume that $\Sigma_1, \Sigma_2 \in \Pi_\tau$; then $\Sigma_1 \sim \Sigma_2$ if and only if the associated optgraphs $(\mathcal{G}_1, R_1)$ and $(\mathcal{G}_2, R_2)$ are equivalent.*

Moreover, the structural stability is established as for the planar case.

To complete the classification program, we introduce the definition of an admissible graph and prove the following assertion.

Theorem 2. *Let $(\mathcal{G}, R)$ be an arbitrary admissible optgraph. Then there exist a compact 2-dimensional orientable manifold M and a system Σ on M such that there exists a cellular embedding of $(\mathcal{G}, R)$ in M and the optgraph associated to Σ is equivalent to $(\mathcal{G}, R)$.*

The paper is organized as follows. In Sec. 2, we introduce the optimal control problem associated to our original problem and recall the notation and main results from [4–6, 16, 17]. Section 3 contains an introduction to the topology of two-dimensional manifolds. In Sec. 4, we introduce the concept of optgraph and prove our main results.

2. Classification of Time-Optimal Syntheses for Generic Planar Systems

In this section, we recall the results obtained in [5, 6, 16, 17] (see also [4]) for the problem of reaching every point of $\mathbb{R}^2$ starting from the origin for the control system

$$\dot{x} = F(x) + uG(x), \quad x \in \mathbb{R}^2, \quad |u| \leq 1, \tag{2}$$

where $F(0) = 0$. In what follows, we will assume that the vector fields F and G are $\mathcal{C}^\infty$ in the following sense.

Definition 1 ($\mathcal{C}^\infty$ vector fields)**.** We say that a vector field is $\mathcal{C}^\infty$ if its components admit partial derivatives of any order that are bounded on the whole plane.

Let Ξ be the set of all couples of $\mathcal{C}^\infty$ vector fields $\Sigma = (F, G)$ such that $F(0) = 0$. From now on, we endow Ξ with the $\mathcal{C}^3$ *topology*, i.e., the topology induced by the following norm:

$$\|(F,G)\| = \sup\left\{ \left|\frac{\partial^{\alpha_1+\alpha_2} F_i(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}\right|, \left|\frac{\partial^{\alpha_1+\alpha_2} G_i(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}\right| : x \in \mathbb{R}^2;\, i = 1,2;\, \alpha_1, \alpha_2 \in \mathbf{N} \cup \{0\};\, \alpha_1 + \alpha_2 \leq 3 \right\}. \tag{3}$$

We say that a subset of Ξ is generic if it contains an open and dense set. Analogously, a property P is said to be generic if the set satisfying P is generic.

Given a measurable function $u : [a, b] \to [-1, 1]$, a trajectory of (2) corresponding to u is an absolutely continuous mapping $\gamma : [a, b] \to \mathbb{R}^2$ such that $\dot{\gamma}(t) = F(\gamma(t)) + u(t)G(\gamma(t))$ for almost all $t \in [a, b]$. Since the system is autonomous, we can always assume that $\mathrm{Dom}(\gamma) = [0, a]$ for some $a \in \mathbb{R}$, $a > 0$, where Dom denotes the domain. Moreover, we denote by $\mathrm{Supp}(\gamma)$ the set $\gamma([a, b])$. A trajectory $\gamma : [0, a] \to \mathbb{R}^2$ is (time) optimal if for every trajectory $\gamma' : [0, b] \to \mathbb{R}^2$ with $\gamma(a) = \gamma'(b)$, we have $a \le b$. A trajectory γ corresponding to a constant control ± 1 is called a *bang arc*. A *bang-bang* trajectory is a finite concatenation of bang arcs, and a time at which the control changes sign is called a *switching time* and the corresponding point is called a *switching point*. The symbols $\gamma^{\pm}$ indicate the curves starting at the origin and corresponding to the constant control $u = \pm 1$.

We are interested in the *reachable set* within some fixed time $\tau > 0$ from the origin, i.e.,

$$\begin{aligned}\mathcal{R}(\tau) :=\{x \in\ \mathbb{R}^2 :\ &\text{there exist}\, t \in [0, \tau] \text{ and a trajectory}\\ &\gamma : [0, t] \to \mathbb{R}^2 \text{ of (2) such that} \gamma(0) = 0,\ \gamma(t) = x\}.\end{aligned} \tag{4}$$

The most important and powerful tool for the study of optimal trajectories is the well-known Pontryagin Maximum Principle (in the following PMP; see, e.g., [1, 11, 19]). It is a first-order necessary condition for optimality and generalizes the Euler–Lagrange equations and the Weierstrass conditions of the calculus of variations to problems with nonholonomic constraints. For each optimal trajectory, the PMP provides a lift to the cotangent bundle, i.e., a solution of a suitable pseudo-Hamiltonian system.

2.1. Properties of optimal trajectories and existence of an optimal synthesis.

We first introduce some definitions.

Definition 2 (stratification)**.** A stratification of $\mathcal{R}(\tau)$, $\tau > 0$, is a finite collection $\{M_i\}$ of connected embedded $\mathcal{C}^1$ submanifolds of M, called strata, such that the following holds. If $M_j \cap \mathrm{Clos}(M_k) \neq \emptyset$ with $j \neq k$ then $M_j \subset \mathrm{Clos}(M_k)$ and $\dim(M_j) < \dim(M_k)$.

Definition 3 (regular optimal synthesis)**.** A regular optimal synthesis for $\Sigma \in \Xi$ on $\mathcal{R}(\tau)$ is a collection of trajectory-control pairs $\{(\gamma_x, u_x) : x \in \mathcal{R}(\tau)\}$ satisfying the following properties:

1. For every $x \in \mathcal{R}(\tau)$, $\gamma_x : [0, t_x] \to \mathbb{R}^2$ steers the origin to x at a minimum time.
2. If $y = \gamma_x(t)$ for some $t \in \mathrm{Dom}(\gamma_x)$, then γ_y is the restriction of γ_x to $[0, t]$.
3. There exists a stratification of $\mathcal{R}(\tau)$ such that $u(x) = u_x(t_x)$ is smooth on each stratum (assuming each u_x is left continuous).

Finally, we define the following two functions:

$$\Delta_A(x); = \mathrm{Det}\big(F(x), G(x)\big) = F_1(x)G_2(x) - F_2(x)G_1(x),$$

$$\Delta_B(x); = \mathrm{Det}\big(G(x), [F, G](x)\big) = G_1(x)[F, G]_2(x) - G_2(x)[F, G]_1(x)$$

that determines the structure of optimal trajectories as explained below. Using PMP and suitable second order conditions one can prove the following theorem (see [16, 17] or [4]).

Theorem 3. *Under generic conditions*

- *every optimal trajectory is a finite concatenation of bang arcs (corresponding to constant controls ± 1) or singular arcs, i.e., running on the set $\Delta_B^{-1}(0)$ and corresponding to the singular feedback*

$$\varphi(x) = -\frac{\nabla\Delta_B(x) \cdot F(x)}{\nabla\Delta_B(x) \cdot G(x)}; \tag{5}$$

- *there exists a regular optimal synthesis.*

The optimal synthesis is constructed by an algorithm $\mathcal{A}$ that is defined by induction. At the step k, it constructs all extremal trajectories (i.e., satisfying PMP) formed by k arcs each of which is either bang or singular and then deletes those arcs that are not optimal.

To describe the algorithm $\mathcal{A}$, we need the key definition of *optimal strip.* An optimal strip is essentially a one-parameter continuous family of optimal trajectories formed by the same sequence of arcs.

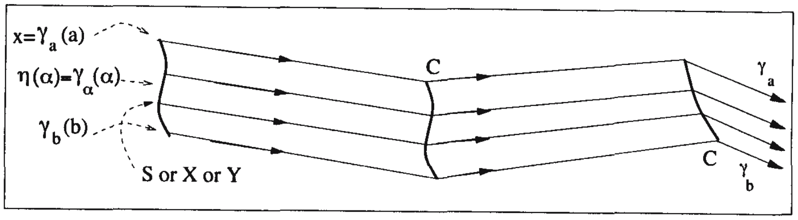

Fig. 1. Optimal strip.

Definition 4. Let a, b be two real numbers such that $0 \leq a < b \leq \tau$, $x \in \mathcal{R}(\tau)$, and $f : [a,b] \to \mathbb{R}$ be a function such that $f(\alpha) \geq \alpha$ for every $\alpha \in [a,b]$. A set of trajectories $\mathcal{S}^{a,b,x,f} = \{\gamma_\alpha : [0, f(\alpha)] \to \mathbb{R}^2, \alpha \in [a,b], \gamma_a(a) = x\}$ is called an optimal strip if the following conditions hold:

(i) $\forall\, \alpha \in [a,b]$, $\gamma_\alpha : [0, f(\alpha)] \to \mathbb{R}^2$ is an optimal trajectory for the control problem (2). Moreover, there exists $\varepsilon > 0$ such that $\gamma|_{[\alpha,\alpha+\varepsilon]}$ corresponds to a constant control ± 1.

(ii) $\forall\, \alpha \in]a,b[$, γ_α does not switch on $\Delta_A^{-1}(0) \cup \Delta_B^{-1}(0)$ after time α.

(iii) The set $\mathcal{B}^{a,b,x,f} = \{y \in \mathbb{R}^2 : \exists \alpha \in]a,b[$ and $t \in]\alpha, f(\alpha)[$ such that $y = \gamma_\alpha(t)$, t is a switching time for $\gamma_\alpha\}$ is never tangent to X or Y.

(iv) The mapping $\eta : \alpha \in [a,b] \mapsto \gamma_\alpha(\alpha) \in M$ is a bang or singular arc, and for $a \leq \alpha' \leq \alpha \leq b$, $\gamma_\alpha(t) = \gamma_{\alpha'}(t)$ holds for $t \in [0, \alpha']$.

The function η is called the *base* of the optimal strip. The concept of strip is illustrated in Fig. 1

The construction of the optimal synthesis is carried out in the following way. First, one constructs all optimal strips bifurcating from $\gamma^\pm$, i.e., having a part of $\mathrm{Supp}(\gamma^\pm)$ as base. Then one studies the evolution of each strip. It may happen that an optimal strip is divided into two strips when some trajectory of the strip enters a singular arc. (This happens at frame points of kind $(C,S)_1$ and $(S,K)_1$ (see definitions below); moreover, two strips with the singular arc as base start at these points.) A strip can terminate on some curve reached optimally by different trajectories, called overlap curve, or on the boundary of the reachable set. Finally, a strip can glue together with another strip. (This happens at frame points of kind $(Y,K)_3$, $(C,K)_1$, and $(S,K)_2$ (see definitions below); more precisely the strip is divided into two parts one of which ends on the overlap curve and the other glues together with another strip.)

Let $\{S^i_M\}_{i\in I}$ be the set of all the maximal strips, i.e., the strips with maximal base and maximal time $f(\alpha)$. Clearly, we may split the synthesis Γ as

$$\Gamma = \bigcup_{i\in I} S^i_M.$$

Note that if $S^{a,b,x,f}$ is a maximal strip, then $f(\alpha) < \tau$ for some $\alpha \in]a,b[$ if and only if there exists an overlap curve K such that $\gamma_\alpha(f(\alpha)) \in K$. We can split the maximal strips in such a way that they satisfy the following condition:

(v) either $\{\gamma_\alpha(f(\alpha)), \alpha \in [a,b]\}$ is an overlap curve or $f(\alpha) = \tau$ for every $\alpha \in [a,b]$.

The optimal synthesis constructed by the algorithm corresponds to a feedback $u(x)$ satisfying the following condition:

- on strata of dimension 2, we have $u(x) = \pm 1$,
- on strata of dimension 1, called frame curves (in the following FCs), $u(x) = \pm 1$ or $u(x) = \varphi(x)$.

The strata of dimension 0 are called frame points (in the following FPs). A complete classification of generic types of FPs and FCs is found in [17]. The generic FCs are the following:

- FCs of kind Y (resp. X) correspond to subsets of $\mathrm{Supp}(\gamma^+)$ (resp. subsets of $\mathrm{Supp}(\gamma^-)$),
- FCs of kind C, called switching curves, are curves made of switching points,
- FCs of kind S are singular arcs,

- FCs of kind K, called overlaps, reached optimally by two different trajectories.
- FCs of kind F are subsets of the topological frontier of $\mathcal{R}(\tau)$.
- FCs of kind γ_0 are arcs of optimal trajectories (not of type X, Y) that pass through FPs. These trajectories "transport" a special information.

Every FP can be determined as the intersection of exactly two FCs of type Y, C, S, K, or F (with possibly also some γ_0, a curve passing through the point). A FP x that is the intersection of two frame curves D_1 and D_2 is called a (D_1, D_2) frame point. [1] There are 24 topological equivalence classes of FPs: (X,Y), $(Y,C)_{1,2,3}$, (Y,S), $(Y,K)_{1,2,3}$, $(C,C)_{1,2}$, $(C,S)_{1,2}$, $(C,K)_{1,2}$, $(S,K)_{1,2,3}$, (K,K), $(Y,F)_1$, $(Y,F)_2$, (S,F), (C,F), (K,F), (F,F). The optimal synthesis near each Frame Point is showed in Fig. 2.

2.2. Structural Stability. A key ingredient to obtaining a satisfactory classification is the structural stability that guarantees persistence of syntheses of main features under small perturbations.

Denote by $u_{F,G}$ the optimal feedback corresponding to $(F,G) \in \Xi$. Introduce an equivalence relation between couples of vector fields: $(F,G) \sim (F',G')$ determined by the topological equivalence of the corresponding flows

$$\dot{x} = F(x) + u_{F,G}(x)G(x), \tag{6}$$

and

$$\dot{x} = F'(x) + u_{F',G'}(x)G'(x). \tag{7}$$

Roughly speaking, this equivalence requires the existence of a homeomorphism defined on a suitable subset of the plane which maps oriented arcs of trajectories of (6) onto oriented arcs of trajectories of (7). We need to exclude overlap curves from the domain of the homeomorphism to avoid the creation of very small equivalence classes. This is well illustrated in Remark 4.

Consider a system $\Sigma \in \Xi$; let $\mathcal{R}$ be the reachable set for Σ at time τ, and let Γ be the corresponding optimal synthesis. Now, if K_1 and K_2 are two frame curves of K type of Γ, then we set $K_1 \sim K_2$ if they have a point in common. Given an overlap curve K, the union of the elements of an equivalence class is a connected curve denoted by $[K]$. Define

$$\mathcal{K} = \big\{x \mid x \in [K] \setminus \partial[K],\ K \text{ is an overlap curve of } \Gamma\big\};$$

let $\mathcal{R}' = \mathcal{R} \setminus \mathcal{K}$. Recall that, for each $x \in \mathcal{R}$, we denote by $t \mapsto \gamma_x(t)$ the trajectory of Γ which reaches x from the origin at a minimum time.

Definition 5 (equivalence of feedback flows)**.** We say that the time-optimal feedback flows for Σ_1 on $\mathcal{R}_1$ and Σ_2 on $\mathcal{R}_2$ are *equivalent*, or simply that $\Sigma_1 \sim \Sigma_2$, if there exists a homeomorphism $\Psi : \mathcal{R}'_1 \mapsto \mathcal{R}'_2$ such that the following conditions hold:

(E1) Ψ maps arcs of optimal trajectories for Σ_1 onto arcs of optimal trajectories for Σ_2. More precisely, for every $x \in \mathcal{R}'_1$, one has $\{\Psi(\gamma^1_x(t)) : t \in \mathrm{Dom}(\gamma^1_x)\} = \{\gamma^2_{\Psi(x)}(t) : t \in \mathrm{Dom}(\gamma^2_{\Psi(x)})\}$.

(E2) Ψ induces a bijection on frame curves that are not overlap curves, i.e., for each frame curve D_1, which occurs in the construction of the optimal feedback for Σ_1 and is not a K-curve, we have that $\Psi(D_1)$ is a frame curve of the same type corresponding to Σ_2, and vice versa.

(E3) If A is an open region of $\mathcal{R}'_1$ enclosed by frame curves and entirely covered by constant control $+1$ trajectories (constant control -1 trajectories), then $\Psi(A)$ is enclosed by the corresponding frame curves and is covered constant control $+1$ trajectories (constant control -1 trajectories).

[1]Here we need to distinguish three kind of these curves:

- curves of kind γ_A that are abnormal extremals,
- curves of kind γ_k that are curves starting at the terminal point of an overlap (i.e., that start at the FPs of kind $(Y,K)_3$, $(S,K)_2$, $(C,K)_1$ (see below)),
- curves of kind γ_0 that are the other arcs of optimal trajectories that start at FPs.

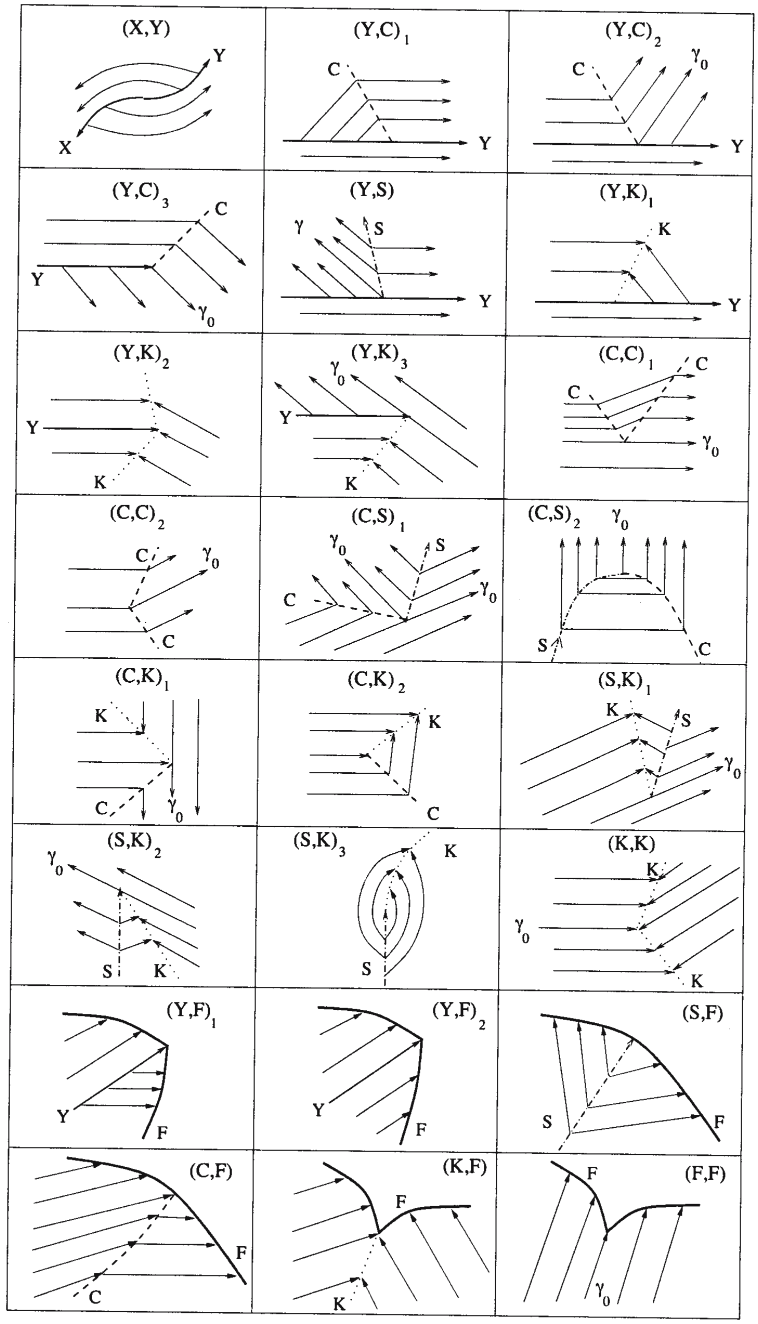

Fig. 2. FPs of the optimal synthesis.

Remark 1. The optimal synthesis is essentially unique in the following sense. Generically, there exists finitely many embedded connected one-dimensional manifolds such that on the complement, the optimal trajectories are uniquely determined. The nonuniqueness happens exactly on overlap curves and on γ_0 curves starting at the terminal point of an overlap (i.e., that start at the FPs of kind $(Y,K)_3$, $(S,K)_2$, $(C,K)_1$; see above). The exclusion of overlap curves is performed so as not to have too small equivalence classes and hence not too many of them (see [4, Remark 31, p. 89]).

Definition 6 (structural stability)**.** We say that a system $\Sigma \in \Xi$ is *structurally stable* if there exists a neighborhood $\mathcal{N}$ of Σ in the space Ξ (endowed with the $\mathcal{C}^3$ norm; see formula (3)) such that for every $\Sigma' \in \mathcal{N}$, the feedback flows for Σ and Σ' are equivalent.

To ensure the structural stability, we need to impose some conditions on the synthesis. First, we introduce the following definition.

Definition 7. If x_1, x_2 are two frame points that are not of type $(C,C)_2$, (K,K), (F,F), we let $x_1 \sim x_2$ if and only if there exist some points $y_0 = x_1, y_2, \dots, y_n = x_2$ such that the following holds. Each y_i belongs to a frame curve D_i. If y_i is a frame point, then it is not of (K,K) type; if y_i is not a frame point, then D_i is not of K type. For every y_i, $i = 1, \dots, n-1$, there exist a constructed trajectory γ_i and $a_i, b_i \in \mathrm{Dom}(\gamma_i)$ verifying $\gamma_i(a_i) = y_i$, $\gamma_i(b_i) = y_{i+1}$, $\gamma_i\,|_{[a_i,b_i]}$ is an X or Y trajectory, and $\gamma_i(]a_i,b_i[) \cap \mathrm{Supp}(\gamma^{\pm}) = \emptyset$. That is, there exists a curve connecting x_1 with x_2, formed by X and Y arcs of the constructed trajectories, that does not intersect the frame curves $\gamma^{\pm}$, the relative interior of overlap curves, and (K,K) frame points.

A FC or FP is stable if, perturbing the system, it persists. This can be directly verified from the nonsingularity of the functions defining them; see [4].

Theorem 4. *Given $\tau > 0$, the set $\Pi_\tau \subset \Xi$ of the system satisfying the conditions*

($\mathcal{A}1$) *all frame curves and points satisfy the stability conditions*;

($\mathcal{A}2$) *if x_1, x_2 are two frame points and $x_1 \sim x_2$, then $x_1 = x_2$,*

is generic. Moreover, every $\Sigma \in \Pi_\tau$ is structurally stable.

Since all definitions and proof techniques are local, the same result remains valid for a general two-dimensional manifold M.

3. Topology of 2-Manifolds

In this section we recall the basics of graph theory and of topology of two dimensional manifolds; see [9, 12, 13].

A *CW-complex* is an ascending sequence

$$X^0 \subset X^1 \subset X^2 \subset \cdots \tag{8}$$

of closed subspaces of a Hausdorff space X such that X^0 is a discrete space and each X^n is obtained from X^{n-1} by an adjunction of cells of dimension $n > 0$. It is supposed that $X = \cup_{n=0}^{\infty} X^n$ and X, together with the subspaces X^n, have a weak topology: a subset $A \subset X$ is closed if and only if $A \cap \bar{e}^q$ is closed for each q-cell e^q. An *n-dimensional* CW-complex is a CW-complex that admits no cells of dimension $>n$.

A 1-dimensional CW-complex $K^1(X)$ is called a *graph*. The topological structure on $K^1(X)$ consists of a Hausdorff space X and a closed discrete subspace X^0; a point of X^0 is called a *vertex* of X. The complementary set $X \backslash X^0$ is a disjoint union of open subsets e_i; every e_i is homeomorphic to an open interval $I \subset \mathbb{R}$ and is called an *edge* of X. For each edge e_i, its boundary ∂e_i is a subset of X^0 consisting either of one or two points; in the case, where ∂e_i consists of two points, the set $\overline{e}_i$ is homeomorphic to a closed interval $\overline{I} = [0,1] \subset \mathbb{R}$; in the case where ∂e_i consists of one point, the set $\overline{e}_i$ is homeomorphic to the unit circle S^1.

Let X and X' be two finite graphs. A graph mapping $f : X \to X'$ consists of a vertex function $f_V : V_X \to V_{X'}$ and an edge function $f_E : E_X \to E_{X'}$ such that the incidence structure is preserved. If X is a directed graph (an *orgraph*), we also require that f preserves orientation at every edge. A graph mapping $f : X \to X'$ between two graphs X and X' is called an *isomorphism* if both its vertex function f_V and edge function f_E are one-to-one and onto (surjective). Two graphs X and X' are said to be isomorphic if there exists an isomorphism $f : X \to X'$.

Let $u, v \in V$ be the vertices of a graph X. A *walk* w on X from u to v of length n means an alternating sequence of vertices and directed edges such that the initial vertex $v_0 = u$ and the final vertex $v_n = v$ for $i = 1, \dots, n$. If $u \neq v$, then the walk is said to be open, otherwise it is closed. An open walk is called a *path* if all its vertices are distinct. A path on X such that its initial vertex coincides with its final vertex is called a *cycle* on X.

By a *surface* one usually understands a 2-dimensional (real) compact manifold. Further we mostly deal with *orientable* surfaces. Orientable surfaces are described by a *genus* $g \geq 0$ which counts a number of 'handles' glued to the sphere S^2.

An *embedding* $i : X \to M$ of a graph X to a surface M is a 1-1 continuous map of the topological space X, taken as a 1-complex $K^1(X)$, into the topological space M.[1] Two embeddings i_1 and i_2 of X in a surface M are equivalent if there exists a homeomorphism $h : M \to M$ such that $h \circ i_1 = i_2$ (in other words, h brings the image $i_1(X)$ to the image $i_2(X)$).

If one takes an embedding $i : X \to M$ of a connected graph X in M, then the set $M \backslash i(X)$ is a union of open regions V_m. Clearly, gluing up handles to each V_m, it is possible to obtain embeddings of X in the surfaces of an arbitrary high genus. An embedding $i : X \to M$ is said to be 2*-cell* (*or cellular*) if all open regions V_m are homeomorphic to an open disk. Further, we consider both cellular and noncellular embeddings.

Denote by $i : X \to M$ a 2-cell embedding of an orgraph X in a surface M. Let $N \backslash i(X) = F_1 \cup \cdots \cup F_m$ be a union of open disk regions in M. A *dual graph* $X^{\#}$ associated to i is a graph with the vertex set $V_{X^{\#}} = \{F_1, \ldots, F_m\}$. An edge $e^{\#} \in E_{X^{\#}}$ between F_i and F_j can be drawn (the case $i = j$ is not excluded) if and only if there is an edge $e \in E_X$ between F_i and F_j, i.e., $e \subseteq \overline{F}_i \cap \overline{F}_j$.

A *local rotation* of a vertex v is an oriented cyclic order (defined up to a cyclic permutations) of all edges incident to v. (The local rotation of 1-valent vertices is uniquely defined and is said to be trivial.) A *rotation system* R (or, simply, a *rotation*) of a graph X is a union of all local rotations over all vertices of X. Rotations give rise to a certain system of faces ($\equiv$ cycles) on X.

The following *face tracing algorithm* allows us to determine all faces of a graph X corresponding to the rotation R. Take an arbitrary vertex $v_1 \in V(X)$ and an edge a_{v_1} incident to v_1. Let v_2 be a vertex of X connected with v_1 by the edge a_{v_2}, and let b_{v_2} be an edge of the vertex v_2, which lies *to the right*[2] in the cyclic order from a_{v_1}. Moving along the edge b_{v_2} to a vertex v_3, we define an edge c_{v_3}, which lies *to the right* from b_{v_2}. Proceeding inductively, we stop the process at an edge z_{v_n} if thenext two edges will be again a_{v_1} and b_{v_2}. Thereby a cycle $a_{v_1}, b_{v_2}, \ldots, z_{v_n}$ of length n, which defines a face F_1 on X, will be traced. For tracing the next face F_2, one must start with an edge which lies *to the right* of any edge of the face F_1 and such that a corner between them does not occur in F_1 and then apply the above construction. All faces $F_1, F_2, \ldots, F_m$ on X will be traced when no unused corners remain.

Theorem 5 (see [10])**.** *Let X be a finite graph endowed with a rotation system R. Then there exists a* 2*-cell embedding of X in an orientable surface M such that one of the two rotations induced by this embedding coincides with R. Moreover, two embeddings are equivalent if and only if they have equivalent rotation systems.*

Each embedding $i : X \to M$ induces a pair of rotation systems R and R^*, where R^* is a mirror image of R (i.e, can be obtained from R by reversing the cyclic order of all local rotations). The corresponding embeddings $i(X)$ and $i^*(X)$ are conjugate by a homeomorphism $h : M \to M$, which is not close to id_M.

4. Topological Classification of Optimal Syntheses

In this section, we describe a procedure to associate a topological graph with a rotation system (and some additional structure) to every system in Π_τ. The points and edges of this topological graph correspond to frame points and curves of the synthesis. Then we provide a classification of optimal syntheses via these topological graphs.

We treat the case of a general smooth two-dimensional oriented manifold M replacing the origin by a fixed initial point x_0 such that $F(x_0) = 0$. All the geometric techniques used to prove results of Sec. 2 are

[1] That is, the CW-topology of $K^1(X)$ coincides with the induced topology on $i(X)$.

[2] For the 1-valent vertices v_i with the edge e_i, the edge lying to the right of e_i will be again e_i. In other words, such a rotation is trivial.

local and, thus, it is possible to establish analogous results for a control system defined on a general smooth two-dimensional manifold by a pair of complete vector fields. Since $\mathcal{R}(\tau)$ is compact, for simplicity, we restrict ourselves to an orientable compact smooth 2-D manifold.

We consider connected finite graphs $\mathcal{G}$ with oriented and nonoriented edges endowed by a rotation system R; moreover, we introduce some additional structure as follows.

Definition 8. An *optgraph* is a pair $(\mathcal{G}, R)$, where $\mathcal{G}$ is a connected finite graph and R is a rotation system on $\mathcal{G}$ satisfying the following. $\mathcal{G}$ presents edges of seven types: X, Y, C, S, K, F, and γ_0. The edges of types X, Y, S, and γ_0 have an orientation. As explained in the previous section, with $(\mathcal{G}, R)$ we can associate a finite number of faces $A_1, \dots, A_m$. We let the faces A_i that are not enclosed by edges of type F have a sign $\pm$ (corresponding to the fact that the optimal feedback is equal to ± 1). In other words, we put a sign ± 1 on some vertices of the dual graph $\mathcal{G}^{\#}$.

We now describe a canonical way of associating an optgraph to a system $\Sigma \in \Pi_\tau$. For every FP, we construct a vertex of $\mathcal{G}$. For every FC D, $\partial D = \{x_1, x_2\}$, we construct an edge E of $\mathcal{G}$ of the same type connecting the points of $\mathcal{G}$ corresponding to x_1, x_2. If D is an X, Y, S or γ_0 FC, then D has the orientation of increasing time, and we endow E with the corresponding orientation. At each FP, there is an oriented cyclic order inherited by the manifold M, and we put the same local rotation at the corresponding vertex. This defines a rotation system R. For every region $A \subset \mathcal{R}(\tau)$ enclosed by frame curves, there is a corresponding face A' defined by the rotation system R. If A is covered by constant control $+1$ trajectories, to A' we assign the positive sign, otherwise we assign to A' the negative sign.

We now give some examples that motivate the definition of optgraph.

4.1. Example of optgraphs.

Example A. Consider the system

$$\begin{cases} \dot{x}_1 = 3x_1 + u, \\ \dot{x}_2 = x_1^2 + x_1. \end{cases}$$

For every time $\tau > \ln(4)/3$, the reachable set at time τ contains two switching curves starting from γ^-. There are two frame points of type (X, C) that are not topologically equivalent. See [4, Sec. 2.6.4, Example 3] for an accurate description of this system. Figure 3, shows the reachable set of this example, and Fig. 4, its associated optgraph $\mathcal{G}_A$.

Remark 2. If we do not specify a sign for every region of $\mathcal{G}_A$, then the two (X, C) frame points are not distinguishable. Hence, for some system Σ with a frame point of type $(X, C)_1$ or $(X, C)_2$, we can construct a system with the same graph, except for the signs of the regions, but not equivalent to Σ. This show the necessity of specifying a sign for every region.

Example B. Consider ε, $0 < \varepsilon < 1$, $\tau > \dfrac{\pi}{\sqrt{1-\varepsilon}}$ and the system Σ:

$$\begin{cases} \dot{x}_1 = \varepsilon x_2 + u x_2, \\ \dot{x}_2 = u(1 - x_1). \end{cases} \tag{9}$$

We obtain

$$\Delta_B(x_1, x_2) = -\varepsilon x_2^2 + \varepsilon(1 - x_1)^2. \tag{10}$$

Hence every singular arc runs on $S = \{(x_1, x_2) : x_2 = \pm(1 - x_1)\}$. It is easy to verify that the trajectory γ^+ intersects the set S at a point (x_1^+, x_2^+) of the first quadrant. The algorithm $\mathcal{A}$ constructs the FC $S_1 = \{(x_1, x_2) : x_2 = 1 - x_1, x_1^+ \le x_1 < 1\}$. The singular control φ_S^1 on S_1 (see (5)) is

$$\varphi_S^1(x_1, x_2) = -\frac{\varepsilon x_2}{1 - x_1 + x_2} > -1. \tag{11}$$

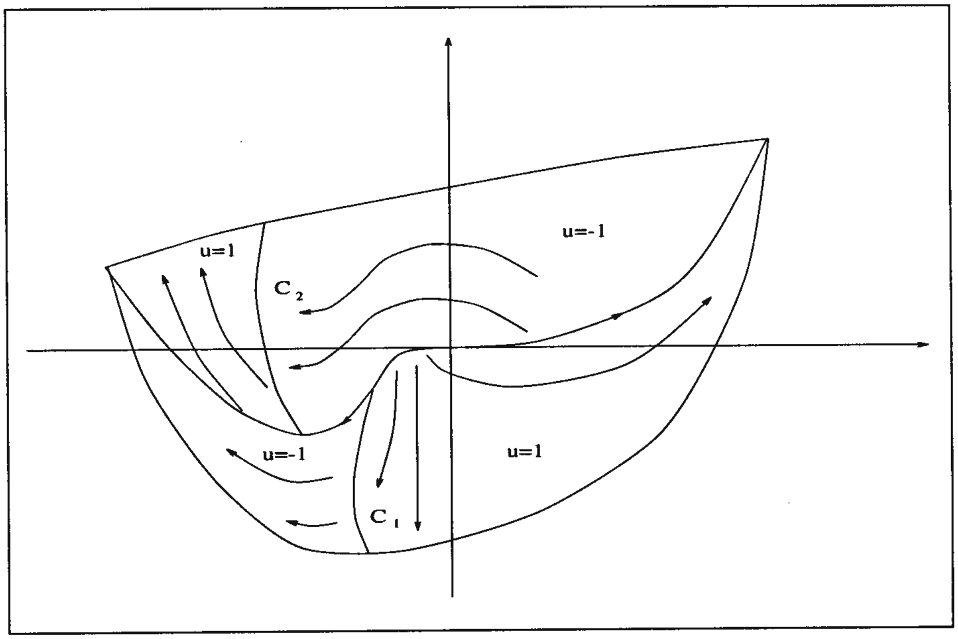

Fig. 3. Reachable set of Example A.

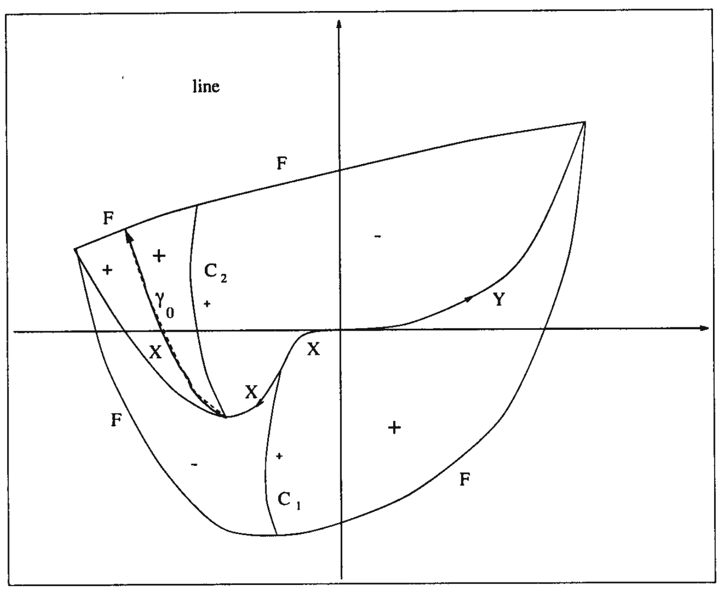

Fig. 4. Graph corresponding to Example A.

From (11), we have

$$\dot{x}_1(\varphi_S^1) = \frac{\varepsilon}{2}(1 - x_1);$$

hence the point $(1, 0)$ is not reached at a finite time by a singular trajectory.

Similarly one can verify the presence of another FC $S_2 = \{(x_1, x_2) : x_2 = x_1 - 1, x_1 \leq x_1^-\}$.

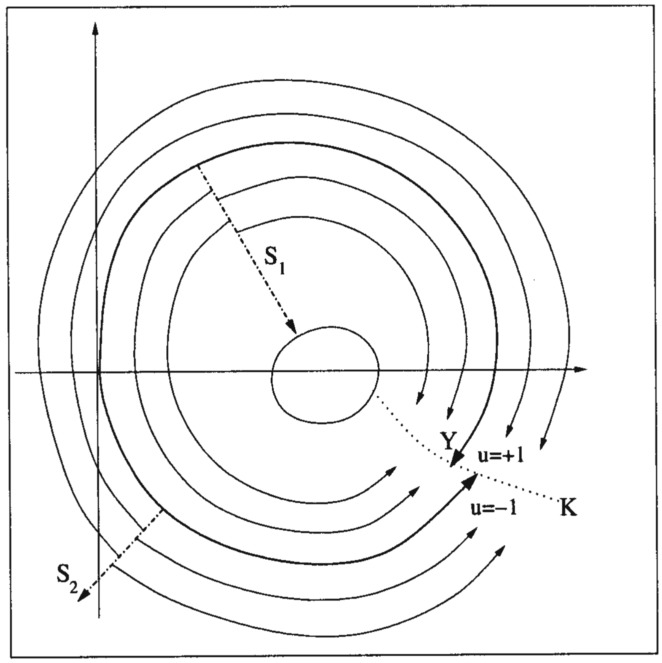

Fig. 5. $(Y, K)_2$ frame point.

The trajectories $\gamma^{\pm}$ are very close to the circle A of radius 1 centered at $(1,0)$; γ^{+} runs clockwise and γ^{-} counterclockwise. We have that γ^{+} lies inside A, γ^{-} outside, and

$$\gamma^{+} \cap \gamma^{-} \cap A = \{(0,0),(2,0)\}.$$

However, the two trajectories $\gamma^{\pm}$ do not reach $(2,0)$ at the same time. Indeed,

$$(2,0) = \gamma^{+}\left(\frac{\pi}{\sqrt{1+\varepsilon}}\right) = \gamma^{-}\left(\frac{\pi}{\sqrt{1-\varepsilon}}\right).$$

An overlap curve K is generated, and $\gamma^{\pm}$ end on it. $\mathcal{R}(\tau)$ is represented iIn Fig. 5.

Remark 3. In Example B, there is a region A that is a connected component of the complement of the reachable set and is bounded. In the corresponding graph, we cannot give a sign to the region corresponding to A. Otherwise, we would have equivalent systems corresponding to different graphs. The regions enclosed by edges all of F type correspond exactly to the complement or to the holes of the reachable set.

Remark 4. The exclusion of overlap curves is performed in order to have not too small equivalence classes and hence not too many of them. Consider Example B; let γ_1 and γ_2 be the trajectory corresponding to control -1, resp. $+1$, starting from the (X,S) point of the first, resp. fourth, orthant. If we include the overlap curves in the definition of equivalence, then the relative position of the endpoints of γ_1 and γ_2 on the K curve is an invariant of the equivalence class and the same for all relative positions of points that can be obtained similarly from them concatenating arcs of X and Y trajectories (see the Definition 7).

4.2. Admissible faces. There is some more information we can get on an optgraph that is associated to a system. More precisely, not every type of faces can occur, but only a finite number with 3 or 4 edges of specified type. We first give the following definition.

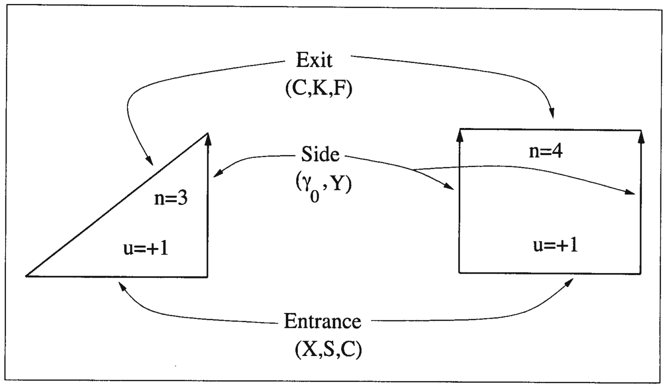

Fig. 6. Admissible faces in the case $u = +1$.

Definition 9. Let $(\mathcal{G}, R)$ be an optgraph, let A_i be a face, and let $\partial A_i = \{e_1, \ldots, e_n\}$. An edge e is called a *side* if e is of type γ_0 and of type X if the region has sign $-$ and of type Y if the region has sign $+$. The face A_i is said to be admissible if:

(i) $n = 3$ or 4;
(ii) if $n = 3$, then there is only one side; otherwise, there are two nonincident sides and there is no orientation of ∂A_i compatible with both orientations of the sides;
(iii) the edge containing the initial point(s) of the side(s) is said to be the *entrance* and is of type X (if the sign is $+$), Y (if the sign is $-$), S, or C;
(iv) the edge containing the terminal point(s) of the side(s) is said to be the *exit* and is of type C, K, or F.

In Fig. 6, we represent all admissible faces.

Proposition 1. *If $(\mathcal{G}, R)$ is an optgraph associated to a system, then every face not enclosed by F edges is admissible.*

Proof. The proof is worked out by induction following the algorithm $\mathcal{A}$; see Sec. 2.1.

At the first step, the algorithm constructs the trajectories $\gamma^{\pm}$. Then it constructs the strips having subsets of $\mathrm{Supp}(\gamma^{\pm})$ as a base. New strips may be generated with only singular curves as a base. On the other hand, all strips end on K or F curves.

As consequence of all these operations, we obtain optimal strips with Y or γ_0 curves enclosing them. Now it is sufficient to note that the optimal strips have only C curves in the interior. Finally, cutting optimal strips along C curves one obtains only admissible faces. $\square$

4.3. Correspondence between systems and optgraphs.

To ensure that the canonical way of associating an optgraph to a system is well defined, we have to prove that two systems are equivalent if and only if the associated optgraphs are equivalent.

Since we have defined the equivalence between systems in a *weak* form excluding overlap curves, equivalent systems may correspond to optgraphs having a different number of K-edges. Hence we need to define an equivalence relation excluding K-edges.

Given two optgraphs $(\mathcal{G}_1, R_1)$, $(\mathcal{G}_2, R_2)$, we say that they are equivalent and we write $(\mathcal{G}_1, R_1) \sim (\mathcal{G}_2, R_2)$ if there is a (possibly multivalued) graph mapping f (see definition in Sec. 3) between $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the following holds.

(H0) f_V is a bijection between the sets of vertices not of type $(Y, K)_2$, (K, K), and (K, F). f_E is the mapping defined on edges, probably multivalued and not injective on the set of K-edges, but it is bijective restricted to the edges not of K-type.
(H1) For every edge E_1 of $\mathcal{G}_1$ not of K-type, $f_E(E_1)$ is an edge of the same type.

(H2) If $\mathcal{K}_1$ and $\mathcal{K}_2$ are the sets of equivalence classes of K-edges (for the incidence relation) of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, then f_E induces a bijective correspondence between $\mathcal{K}_1$ and $\mathcal{K}_2$.

(H3) f preserves rotation systems.

(H4) If A' is a region enclosed by edges $E_1, \dots, E_n$ of $\mathcal{G}_1$, then the region enclosed by $f_E(E_1), \dots, f_E(E_n)$ has the same sign.

Proof of Theorem 1. First, assume that $\Sigma_1 \sim \Sigma_2$; let Ψ be as in Definition 5, p. 3114. We use the symbols Γ_1 and Γ_2 for the corresponding optimal syntheses. To prove the equivalence, we construct the required graph mapping f in the following way. Given a frame curve D of Γ_1 that is not a K-curve, let E_1, and E_2 be the edges corresponding to D and $\Psi(D)$, respectively. We define $f_E(E_1) = E_2$. If we extend Ψ by continuity, then, for every K_1, the overlap curve $\Psi([K_1])$ of Γ_1 is the union of elements of an equivalence class of K-curves of Γ_2. Therefore, we can define f_E on $\mathcal{K}_1$. In the same way, we define f_V.

From (E1) and (E2), it follows that (H0), (H1), (H2), and (H3) hold, while from (E3), it follows (H4).

Now, assume that $(\mathcal{G}_1, R_1) \sim (\mathcal{G}_2, R_2)$. We note that condition (H3) is not necessary to prove the conclusion.

Let E_1 be an X, Y or S-edge of $\mathcal{G}_1$, $E_2 = f_E(E_1)$, and let D_1, D_2 be the frame curves corresponding to E_1, E_2, respectively. From (H1), we have that D_1, D_2 are of the same type. We define Ψ on D_1 in such a way that Ψ is an homeomorphism and $\Psi(D_1) = D_2$.

For every $y \in D_1$ consider the constructed trajectories $\gamma_y \in \Gamma_1$ for which $y = \gamma_y(b_y)$ is a switching point. If D_1 is of X or Y type, then there is at most one such trajectory; if D_1 is of S type, then there are two such trajectories. If D_1 is of type X or Y and there exists γ_y, then from (H4), there exists a trajectory $\gamma_{\Psi(y)} \in \Gamma_2$ having the same property. Let $c_y > b_y$ be the first time in which γ_y reaches another frame curve; define $b_{\Psi(y)}$, $c_{\Psi(y)}$ similarly. We set

$$\Psi\big(\gamma_y(t)\big) \doteq \gamma_{\Psi(y)} \left(b_{\Psi(y)} + \frac{c_{\Psi(y)} - b_{\Psi(y)}}{c_y - b_y}(t - b_y) \right) \quad \forall t \in [b_y, c_y].$$

In this way, we have also defined Ψ on the frame curves that are reached by the trajectories γ_y. We proceed in the same way defining Ψ on the images of the constructed trajectories that switch at the point of these new frame curves. After a finite number of steps, we define Ψ on the whole reachable set $\mathcal{R}_1$ of the system Σ_1. Note that we can have two different definitions of Ψ on the K frame curves, but thanks to (H2) Ψ restricted to $\mathcal{R}'_1$ (see the Definition 5, p. 3114) is well defined. The condition (E1) follows by construction.

Conditions (H0) and (H1) ensure that the corresponding trajectories have the same history, i.e., they cross the same type of frame curves in the same order and are composed by the same elementary arcs. Finally, from conditions (H0), (H1), (H2), and (H4), we have that Ψ satisfies (E1)–(E3). $\square$

4.4. Admissible graphs. To complete the classification program, we give some admissibility conditions to characterize the class of optgraphs corresponding to the systems.

From the previous analysis, it follows that there exists a set of graphs $\mathcal{E}$ whose elements are defined locally, and each of which corresponds to a type of frame point. A point x' of an optgraph $(\mathcal{G}, R)$ is said to be *admissible* if there exist a graph $\mathcal{G}' \in \mathcal{E}$ such that $\mathcal{G}$ contains a copy of $\mathcal{G}'$ to which x' belongs. We use the same terminology for the points of $\mathcal{G}$, e.g., the (X, Y) point. The first condition is as follows:

($\mathcal{G}1$) All points of $(\mathcal{G}, R)$ are admissible.

We consider optgraphs that contain exactly one point of the type (X, Y) and we call this point the *origin* of the optgraph. Assume that ($\mathcal{G}1$) holds. Let E be a Y-edge, and let x be the initial point of E. If x is not the origin, then there exists a Y-edge E_1 for which x is the terminal point. We consider the initial point x_1 of E_1 and do the same considerations. Since $\mathcal{G}$ is finite, proceeding by induction, we find a finite collection $E_1, \dots, E_n$ of Y-edges such that E_i is incident to E_{i+1}, $i = 1, \dots, n-1$, and the initial point of E_n is the origin. Then, since there is only one origin, the Y-edges form a set $\{E_1, \dots, E_m\}$ such that the initial point of E_1 is the origin and for each $i = 1, \dots, m-1$, the terminal point of E_i is the initial point of E_{i+1}. We call η^+ the union of these edges. Analogously, we define η^- for the X-edges.

All the possibilities for the sequence of frame points on a curve γ^+ of a system Σ are described in [4]. We say that η^+ is *admissible* if there exists a system Σ such that the curve γ^+ correspond to η^+ canonically. That is, there is a correspondence defined for points, edges of η^+, and for the regions to which η^+ belongs that follows the rules of canonical correspondence. This happens exactly when η^+ and γ^+ have an ordered sequence of corresponding points. The second condition is as follows:

($\mathcal{G}2$) $\mathcal{G}$ has exactly one (X, Y) point called the origin. The collections of edges $\eta^\pm$ are admissible.

The incidence relation partitions the set of F-edges into a finite number of equivalence classes. For every frame curve F, we indicate by $[F]$ the closed path consisting of the union of elements of corresponding equivalence class.

($\mathcal{G}3$) All $[F]$ enclose a face of $(\mathcal{G}, R)$.

Finally,

($\mathcal{G}4$) Every face of $(\mathcal{G}, R)$ not enclosed by F-edges is admissible.

Remark 5. $[F]$ enclosing a face correspond to the complement and the holes of the reachable set, e.g., the system in Example B. Therefore, the conditions in $(\mathcal{G}_4)$ is given only for regions not enclosed by F edges.

Definition 10. If an optgraph $(\mathcal{G}, R)$ satisfies conditions $(\mathcal{G}1), \ldots, (\mathcal{G}4)$, then we say that $(\mathcal{G}, R)$ is *admissible*.

Remark 6. It is easy to verify that if $\mathcal{G}$ corresponds to a system Σ then $\mathcal{G}$ is admissible.

Now assume that $(\mathcal{G}, R)$ is an admissible optgraph. We want to find a manifold M and a system Σ on M such that $\mathcal{G}$ is associated to Σ in the canonical way, i.e., prove Theorem 2. This and Theorem 1 show that the correspondence $(M, \Sigma) \leftrightarrow (\mathcal{G}, R)$ is a bijection between the set of equivalence classes of manifold-system couples and the set of equivalence classes of admissible optgraphs.

Remark 7. Note that the manifold M of Theorem 2 is unique up to diffeomorphism because of the cellular embedding. The same system Σ can be put on a manifold with higher genus by adding an arbitrary number of handles.

Proof of Theorem 2. From Theorem 5, it follows that there exists a compact 2-dimensional orientable manifold M and a cellular embedding of $(\mathcal{G}, R)$ in M such that one of the two rotation systems induced by the embedding coincides with R.

From condition $(\mathcal{G}1)$, it follows that the set of $[F]$s partition M in a finite number of connected components. By $(\mathcal{G}3)$, only one such component contain edges of the image of $\mathcal{G}$. Indeed assume on the contrary that there are two such components M_1 and M_2. If $\mathrm{Clos}(M_1) \cap \mathrm{Clos}(M_2) \neq \emptyset$, then the $[F]$ separating them violates $(\mathcal{G}3)$. Otherwise, since $\mathcal{G}$ is connected, there are edges connecting ∂M_1 and ∂M_2 and we obtain again a contradiction.

Due to $(\mathcal{G}1)$ and $(\mathcal{G}2)$, we can construct locally some systems around frame points and frame curves using the explicit examples of [4] such that they correspond to $\mathcal{G}$.

It remains to prove that we can glue these system in a smooth fashion, but this can be done, using $(\mathcal{G}4)$, exactly as in [4, Appendix A.5] or [6]. $\square$

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