Control and stabilization of the Bloch equation

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Plan

The Bloch equation

An ensemble of non interacting spins, in a magnetic field $B(t) := (u(t), v(t), B_0)$, with dispersion in the Larmor frequency $\omega = \gamma B_0 \in (\omega_*, \omega^*)$ (=rotation speed around z).

$$\text{one spin :} \quad M(t,\omega) \in S^2$$

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad \omega \in (\omega_*, \omega^*)$$

State : M **Controls :** u, v

controllability of an ODE, simultaneously w.r.t. $\omega \in (\omega_*, \omega^*)$

Li-Khaneja(06)

Application : Nuclear Magnetic Resonance

Controllability question for the Bloch equation

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty)\times(\omega_*,\omega^*)$$

Ex : $M_0(\omega) \equiv -e_3, \quad M_f(\omega) \equiv +e_3,$
But spins with different ω have different dynamics !

Goal : Use the control to compensate for the dispersion in ω.

Rk : If ω is fixed, the controllability of one ODE on S^2 is trivial.

A prototype for infinite dimensional bilinear systems with continuous spectrum

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad \omega \in (\omega_*, \omega^*)$$

$$\mathcal{A}M := \omega e_3 \wedge M(\omega) \qquad \to \qquad \mathrm{Sp}(\mathcal{A}) = -i(\omega_*, \omega^*) \bigcup i(\omega_*, \omega^*)$$

$$\lambda = \pm i\widetilde{\omega} \qquad \to \qquad M_\lambda(\omega) = \begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix} \delta_{\widetilde{\omega}}(\omega)$$

$\Rightarrow$ Toy model $\qquad i\partial_t \psi = (-\Delta + V)\psi - u(t)\mu(x)\psi$

State of the art : bilinear control for Schrödinger PDEs

Quite well understood :
exact controllability 1D

- negative results : Ball-Marsden-Slemrod(82), Turinici(00), Ilner-Lange-Teismann(06), Mirrahimi-Rouchon(04) Nersesyan(10).
- positive local results with discrete spectrum + gap (1D) : KB(05), KB-Laurent(09).
- positive global results : KB-Coron(06), Nersesyan(09).

approximate controllability with discrete spectrum
Chambrion-Mason-Sigalotti-Boscain(09), Nersesyan(09), Ervedoza-Puel(09).

Not well understood : with continuous spectrum : Mirrahimi(09)

Linearized system around ($M \equiv e_3, u \equiv v \equiv 0$) : non exact controllability, approximate controllability

$$M = (x, y, z), \quad \mathcal{Z}(t,\omega) := (x + iy)(t,\omega), \quad w(t) := (v - iu)(t)$$

$$\mathcal{Z}(T,\omega) = \Big(\mathcal{Z}_0(\omega) + \int_0^T w(t) e^{-i\omega t} dt\Big) e^{i\omega T}$$

- $T > 0$, the reachable set from $\mathcal{Z}_0 = 0$ is $\mathcal{F}[L^1(-T,0)]$
- the $\mathcal{Z}_0$ asymptotically zero controllable are $\mathcal{F}[L^1(0,+\infty)]$
- $\forall \mathcal{Z}_0$ in that space, the control is unique
- $\forall T > 0$, approximate controllability in $C^0[\omega_*, \omega^*]$ with $C_c^\infty(0,T)$-controls.

We will see that the NL syst has better controllability properties.

Whole space : structure of the reachable set

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T) \times \mathbb{R}$$

Theorem : Let $T > 0$ and $R := 1/(8\sqrt{3T})$.

- $\forall u, v \in B_R[L^2(0,T)], \exists! M = (x,y,z)$ solution with $\mathcal{Z} := x + iy \in C^0([0,T], L^2(\mathbb{R})) \cap C^0_b([0,T] \times \mathbb{R})$,
- the image of

$$\begin{array}{rccc} F_T : & B_R[L^2(0,T)]^2 & \to & L^2 \cap C^0_b(\mathbb{R}) \\ & (u,v) & \mapsto & \mathcal{Z}(T,.) \end{array}$$

is a non flat **submanifold** of $L^2 \cap C^0_b(\mathbb{R})$, with ∞ codim.

Proof : Inverse mapping $dF_T(0,0).(U,V) \sim \mathcal{F}(U+iV) \; + \; 2^{nd}$ order

On a bounded interval : analyticity argument

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T) \times (\omega_*, \omega^*)$$

- $T > 0, u, v \in L^2(0,T) \Rightarrow \mathcal{Z}(T,.)$ analytic
- $T > 0$, $R := 1/(8\sqrt{3T})$.
  There exists arbitrarily small **analytic** targets that cannot be reached exactly in time T with controls in $B_R[L^2(0,T)]$.

The non controllability is not a question of regularity.

Solutions associated to Dirac controls

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T)\times(\omega_*,\omega^*)$$

Classical solution for $u, v \in L^1_{loc}(\mathbb{R})$.

If $u = \alpha\delta_a$ and $v = 0$ then

$$M(a^+,\omega) = \exp(\alpha\Omega_x)M(a^-,\omega)$$

$\rightarrow$ instantaneous rotation of angle α around the x-axis, $\forall\omega$

Rk : limit $[\epsilon \rightarrow 0]$ of solutions associated to $u = \frac{\alpha}{\epsilon}1_{[a,a+\epsilon]}$.

Approximate controllability result $-\infty < \omega_* < \omega^* < +\infty$

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*)$$

Theorem : Let $M_0 \in H^1((\omega_*, \omega^*), S^2)$. There exist $(t_n)_{n\in\mathbb{N}} \in [0,+\infty)^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ finite sums of Dirac masses such that

$$U[t_n^+; u_n, v_n, M_0] \to e_3 \text{ weakly in } H^1.$$

Rk : Same result with $u, v \in L^\infty_{loc}[0,+\infty)$: $\quad \alpha\delta_a \leftarrow \frac{\alpha}{\epsilon} 1_{[a,a+\epsilon]}$
Approximate controllability in H^s, $\forall s < 1$, in L^∞...

First step : Li-Khaneja 's non commutativity result

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty)\times(\omega_*,\omega^*)$$

Theorem : Let $P, Q \in \mathbb{R}[X]$. $\forall \epsilon > 0, \exists \tau^* > 0$ such that $\forall \tau \in (0,\tau^*)$, $\exists T > 0, u, v \sim$ Dirac such that

$$\Big\| U[T^+; u, v, .] - \Big(I + \tau [P(\omega)\Omega_x + Q(\omega)\Omega_y] \Big) \Big\|_{H^1(\omega_*,\omega^*)} \leqslant \epsilon\tau.$$

Proof : Explicit controls $\to$ cancel the drift term, Lie brackets.

Rk : It is not sufficient for the global approximate controllability.
$\tau\omega^N$ needs $T_N \sim 2^N \tau^{\frac{1}{N}}$ and more than 2^N N-S.

Second step : Variationnal method

Let $M_0 \in H^1((\omega_*, \omega^*), S^2)$ be such that $M_0 \neq e_3$.

Goal : Find $U[t_n^+; u_n, v_n, M_0] \rightharpoonup e_3$ in H^1 when $n \to +\infty$

$$K := \left\{ \widetilde{M} \,;\, \exists U[t_n^+; u_n, v_n, M_0] \rightharpoonup \widetilde{M} \text{ in } H^1 \right\}$$

$$m := \inf \left\{ \|\widetilde{M}'\|_{L^2}; \widetilde{M} \in K \right\}$$

1) $\exists e \in K$ such that $m = \|e'\|_{L^2}$
2) $m = 0$. Otherwise, one may decrease more : $\exists P, Q \in \mathbb{R}[X]$ st

$$\left\| \frac{d}{d\omega} \left[\Big(I + \tau [P(\omega)\Omega_x + Q(\omega)\Omega_y] \Big) e \right] \right\|_{L^2} < \|e'\|_{L^2}$$

3) $e_3 \in K \cap S^2$

Conclusion

Theorem : Let $M_0 \in H^1((\omega_*, \omega^*), S^2)$. There exist $(t_n)_{n\in\mathbb{N}} \in [0, +\infty)^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ finite sums of Dirac masses such that

$$U[t_n^+; u_n, v_n, M_0] \to e_3 \text{ weakly in } H^1.$$

Advantages :

- global result
- strong cv in H^s, $\forall s < 1$, L^∞

Flaws : How to do ? The strategy of the proof may

- not work,
- take a long time,
- cost a lot (N-S).

Explicit controls for the asymptotic exact controllability

Notations : - $(\omega_*, \omega^*) = (0, \pi)$, $f : (0, \pi) \to \mathbb{C}$ identified with $\tilde{f} : \mathbb{R} \to \mathbb{R}$, 2π periodic symmetric, $N(f) := \sum_{n \in \mathbb{Z}} |c_n(f)|$.
- $M = (x, y, z)$, $\quad \mathcal{Z} := x + iy$

Theorem : $\exists \delta > 0$ / $\forall M_0 : (0, \pi) \to S^2$ with $N[\mathcal{Z}_0] < \delta$ and $z_0 > 1/2$, the solution of the Bloch equation with

$$u(t) := \pi \delta_k(t) - \sum_{p=1}^{2k-1} \Im\Big(c_{-k+p}(\mathcal{Z}_0)\Big) \delta_{k+p}(t) + \pi \delta_{3k}(t),$$
$$v(t) := -\sum_{p=1}^{2k-1} \Re\Big(c_{-k+p}(\mathcal{Z}_0)\Big) \delta_{k+p}(t),$$

where $k = k(\mathcal{Z}_0)$ / $\sum_{|n|>k} |c_n(\mathcal{Z}_0)| < N(\mathcal{Z}_0)/4$ satisfies

$$N[\mathcal{Z}(3k^+)] < \frac{N(\mathcal{Z}_0)}{2} \quad \text{and} \quad z(3k^+) > 1/2.$$

Ideas of the proof

1) 'cancel' $c_n(\mathcal{Z}_0)$ **for** $n \leqslant 0$ with $w(t) = \sum_{k=0}^{N} c_{-k}\delta_k(t)$

$$\begin{aligned}\mathcal{Z}(N^+,\omega) &\sim \left(\mathcal{Z}_0(\omega) - \int_0^N w(t)e^{-i\omega t}dt\right)e^{i\omega N}\\ &\sim \left(\sum_{n\in\mathbb{Z}} c_n e^{in\omega} - \sum_{k=0}^{N} c_{-k}e^{-ik\omega}\right)e^{i\omega N}\end{aligned}$$

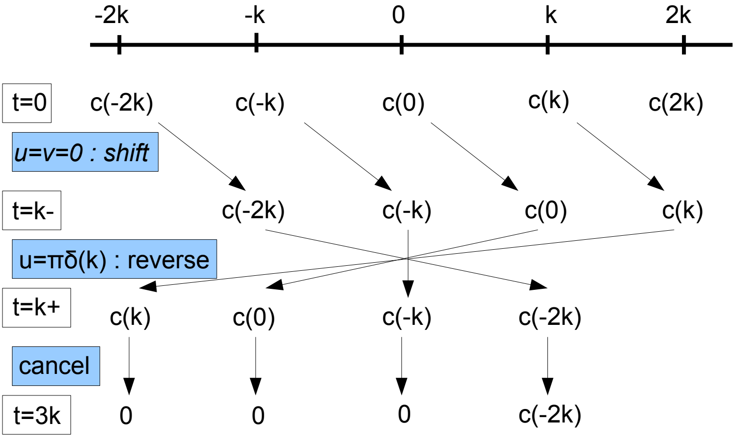
2) shift to the right with $u \equiv v \equiv 0$,

$$\mathcal{Z}(N,\omega) = \mathcal{Z}_0(\omega)e^{iN\omega} = \sum_{n\in\mathbb{Z}} c_n e^{i(n+N)\omega}$$

3) reverse with $u(t) = \pi\delta_0(t)$, $M(0^+) = \exp(\pi\Omega_x)M_0$

$$\mathcal{Z}(0^+,\omega) = \overline{\mathcal{Z}_0(\omega)} = \sum_{n\in\mathbb{Z}} \overline{c_n}e^{-in\omega}$$

Proof


-2k
-k
0
k
2k
t=0
c(-2k)
c(-k)
c(0)
c(k)
c(2k)
u=v=0 : shift
t=k-
c(-2k)
c(-k)
c(0)
c(k)
u=πδ(k) : reverse
t=k+
c(k)
c(0)
c(-k)
c(-2k)
cancel
t=3k
0
0
0
c(-2k)

Goal

Propose **explicit feedback laws** that stabilize the Bloch equation around a uniform state of spin $+1/2$ or $-1/2$.

$$M(t,\omega) \underset{t\to+\infty}{\longrightarrow} e_3 \quad \text{uniformly wrt} \quad \omega \in (\omega_*, \omega^*)$$

Interest : less sensible to random perturbations than open loop controls

Strategy

Feedback design tool : control Lyapunov function

Convergence for ODEs : LaSalle invariance principle

Convergence for PDEs : several adaptions
- approximate stabilization : with discrete [KB-Mirrahimi(09)] or continuous spectrum [Mirrahimi(09)]
- weak stabilization :
under a strong compactness assumption [Ball-Slemrod(79)]
without [this work, KB-Nersesyan(10)]
- strong stabilization :
with compact trajectories [d'Andréa-Novel-Coron(98)]
strict Lyapunov function [Coron-d'Andréa-Novel-Bastin(07)]

The impulse train structure control

In view of the previous results, it is natural to consider

$$u = u_{smooth} + \sum_{k=1}^{\infty} \pi \delta(t - kT)$$

$$x(kT^+) = x(kT^-) \qquad y(kT^+) = -y(kT^-) \qquad z(kT^+) = -z(kT^-)$$

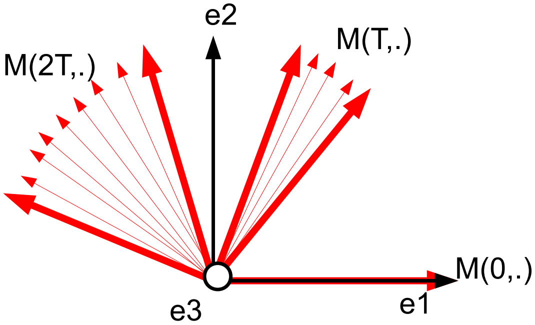
With $\epsilon(t) = (-1)^{E(t/T)}$, the change of variables

$$(x, y, z) \leftarrow (x, \epsilon(t) y, \epsilon(t) z), \qquad u \leftarrow u + \sum_{k=1}^{\infty} \pi \delta(t - kT), \qquad v \leftarrow \epsilon(t) v$$

transforms the Bloch equation into

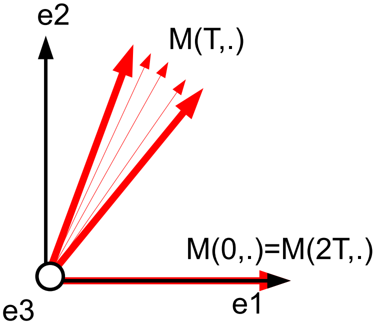
$$\frac{\partial M}{\partial t}(t, \omega) = \Big[u(t) e_1 + v(t) e_2 + \epsilon(t) \omega e_3 \Big] \wedge M(t, \omega)$$

The impulse train structure reduces the dispersion



Initial free system

The impulse train structure reduces the dispersion



New free system

Driftless form

$$M=(x,y,z) \qquad \mathcal{Z}:=x+iy \qquad \Omega:=v-iu$$

$$\begin{cases} \frac{\partial \mathcal{Z}}{\partial t}(t,\omega)=i\epsilon(t)\omega\mathcal{Z}(t,\omega)+\Omega(t)z(t,\omega) \\ \frac{\partial z}{\partial t}(t,\omega)=-\Re[\Omega(t)\overline{\mathcal{Z}(t,\omega)}] \end{cases}$$

$$\mathcal{Z}(t,\omega)\leftarrow \mathcal{Z}(t,\omega)e^{-i\omega\zeta(t)} \qquad \text{where} \qquad \zeta(t):=\int_0^t \epsilon(s)ds$$

$$\begin{cases} \frac{\partial \mathcal{Z}}{\partial t}(t,\omega)=\Omega(t)z(t,\omega)e^{-i\omega\zeta(t)} \\ \frac{\partial z}{\partial t}(t,\omega)=-\Re[\Omega(t)\overline{\mathcal{Z}(t,\omega)}e^{-i\omega\zeta(t)}] \end{cases}$$

Control design : control Lyapunov function

$$\mathcal{L}(t) := \int_{\omega_*}^{\omega^*} \Big[|\mathcal{Z}'(t,\omega)|^2 + z'(t,\omega)^2 + z(t,\omega) \Big] d\omega$$

$$\frac{d\mathcal{L}}{dt}(t) = \Re\left[\Omega(t)\mathcal{H}(t)\right]$$

where

$$\mathcal{H}(t) := \int_{\omega_*}^{\omega^*} \Big[i\zeta(t)[\overline{\mathcal{Z}}z' - \overline{\mathcal{Z}}'z] - \overline{\mathcal{Z}(t,\omega)} \Big] e^{-i\omega\zeta(t)} d\omega$$

So we take

$$\Omega(t) := -\overline{\mathcal{H}(t)} \qquad \text{then} \qquad \frac{d\mathcal{L}}{dt}(t) = -|\Omega(t)|^2$$

Local stabilization

Theorem : There exists $\delta > 0$ such that, for every $M_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\|M_0 + e_3\|_{H^1} < \delta$, the solution of the closed loop system satisfies

$$M(t) \rightharpoonup -e_3 \text{ in } H^1(\omega_*, \omega^*) \text{ when } t \to +\infty.$$

Rk : $M(t, \omega) \to -e_3$ uniformly with respect to $\omega \in (\omega_*, \omega^*)$.

Proof : 1. Invariant set $= \{-e_3\}$ locally.
2. $\Omega(t) \to 0$ a.e.
3. $-e_3$ is the only possible weak H^1-limit :
If $M(t_n) \to M_\infty^0$ weakly in H^1 and strongly in $H^{1/2}$ then $M(t_n + \tau) \to M_\infty(\tau)$ strongly in $H^{1/2}$, $\forall \tau > 0$, thus $\Omega[M(t_n + \tau)] \to \Omega[M_\infty(\tau)]$. Therefore $\Omega[M_\infty] \equiv 0$.

Key point : $\Omega(M)$ is well defined for M only in $H^{1/2}$

No global stabilization

Topological obstructions : $H^1((\omega_*, \omega^*), S^2)$ cannot be continuously deformed to one point.

Actually, there is an infinite number of invariant solutions, that may be expressed explicitly.

Numerical simulations

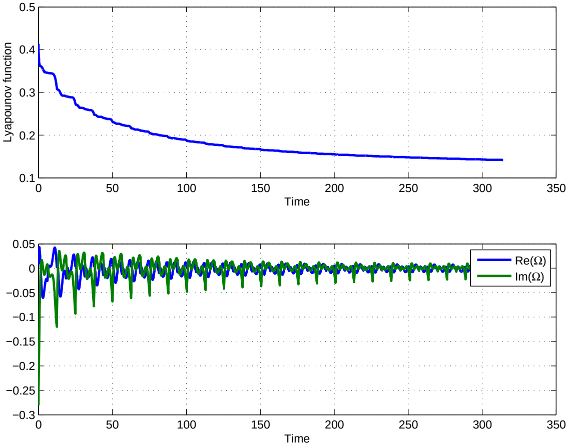
Parameters : $(\omega_*, \omega^*) = (0, 1)$, $T = 2\pi$, $G := 1/(2T^2)$

$$\begin{pmatrix} x_0(\omega) \\ y_0(\omega) \\ z_0(\omega) \end{pmatrix} := \begin{pmatrix} \cos(\pi, \omega)\sqrt{1 - z_0(\omega)^2} \\ \sin(\pi, \omega)\sqrt{1 - z_0(\omega)^2} \\ 0.8 - 0.1\sin(4\pi\omega) \end{pmatrix}.$$

Simulation until $T_f = 50T$

Conclusion : The convergence speed is rapid at the beginning but decreases at the end.

Numerical simulations



Conclusion of the talk : Controllability

Linearized system :

- non exact controllability, L^1 controls : $\mathcal{F}[L^1(-T,0)]$
- non asymptotic zero controllability
- uniqueness of the control
- approximate controllability, unbounded controls

Nonlinear system :

- non exact controllability, $B_R[L^2(0,T)]$-controls : manifold
- approximate controllability in H^s, $s<1$, unbounded controls : *non commutativity + variationnal method*
- explicit controls for the (local) asymptotic exact controllability to e_3 : *Fourier method,* many controls work

The nonlinearity allows to recover controllability.

Conclusion of the talk : Stabilization

- impulse train control
- driftless form
- control Lyapunov function : H^1-distance to the target
- explicit damping feedback laws
- weak H^1 local stabilization

Open problems, perspectives

- **exact** controllability in **finite time** with unbounded controls ?
- **strong** stabilization with the same feedback laws ?
- explicit feedbacks for the **semi-global** stabilization
- convergence rates ? arbitrarily fast stabilization ?