Some estimates for the bilinear Schrödinger equation with discrete spectrum

Thomas Chambrion
(joint work with U. Boscain, M. Caponigro and M. Sigalotti)

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Quantum systems

The state of a quantum system evolving in a space (Ω, μ) can be represented by its *wave function* ψ. Under suitable hypotheses, the dynamics for ψ is given by the Schrödinger equation :

$$i\frac{\partial \psi}{\partial t}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t)$$

Ω : finite dimensional manifold, for instance a bounded domain of $\mathbf{R^d}$, or $\mathbf{R^d}$, or $SO(3)$,...
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The well-posedness is far from obvious. It may require to add boundary conditions ($\psi_{|\partial\Omega} = 0$ if Ω is a bounded subspace of $\mathbf{R^d}$) and hypotheses on V and W.

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with the assumptions

- H complex Hilbert space ;
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Under these assumptions

$$\forall u \in U, \exists\ e^{t(A+uB)} : H \to H \text{ group of unitary transformations}$$

Definition of solutions

$$i\frac{\partial\psi}{\partial t}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + u(t)W(x)\psi(x,t)$$

We choose piecewise constant controls

Definition

We call $\Upsilon^u_T(\psi_0) = e^{t_k(A+u_kB)} \circ \cdots \circ e^{t_1(A+u_1B)}(\psi_0)$ the solution of the system starting from ψ_0 associated to the piecewise constant control $u_1\chi_{[0,t_1]} + u_2\chi_{[t_1,t_1+t_2]} + \cdots$.

If B is bounded, it is possible to extend this definition for controls u that are only measurable bounded or locally integrable.

Controllability

Exact controllability

ψ_a, ψ_b given. Is it possible to find a control $u : [0, T] \to U$ such that $\Upsilon^u_T(\psi_a) = \psi_b$?

Approximate controllability

$\epsilon > 0$, ψ_a, ψ_b given. Is it possible to find a control $u : [0, T] \to U$ such that $\|\Upsilon^u_T(\psi_a) - \psi_b\| < \epsilon$?

Simultaneous approximate controllability

$\epsilon > 0$, $\psi_a^1, \psi_a^2, \ldots, \psi_a^p$, $\psi_b^1, \ldots, \psi_b^p$ given. Is it possible to find a control $u : [0, T] \to U$ such that $\|\Upsilon^u_T(\psi_a^j) - \psi_b^j\| < \epsilon$ for every j ?

A negative result

Theorem (Ball-Marsden-Slemrod, 1982 and Turinici, 2000)

If $\psi \mapsto W\psi$ is bounded, then the reachable set from any point (with L^{1+r} controls) of the control system :

$$i\frac{\partial \psi}{\partial t}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) + u(t)W(x)\psi(x,t)$$

has dense complement in the unit sphere.

Non controllability of the harmonic oscillator (I)

$\Omega = \mathbf{R}$

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}x^2\psi - u(t)x\psi$$

Theorem (Mirrahimi-Rouchon, 2004)

The quantum harmonic oscillator is not controllable.

(see also Illner-Lange-Teismann 2005 and Bloch-Brockett-Rangan 2006)

Non controllability of the harmonic oscillator (II)

The Galerkin approximation of order n is controllable (in $U(n)$) :

$$A = -\frac{i}{2}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n+1 \end{pmatrix}$$

$$B = -i\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \sqrt{2} & \ddots & & \vdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & \sqrt{n+1} \\ 0 & \cdots & \cdots & 0 & \sqrt{n+1} & 0 \end{pmatrix}$$

Exact controllability for the potential well

$\Omega = (-1/2, 1/2)$

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - u(t)x\psi$$

Theorem (Beauchard, 2005)

The system is exactly controllable in the intersection of the unit sphere of L^2 with $H^7_{(0)}$.

Generic controllability results via geometric methods

Theorem (Boscain-Chambrion-Mason-Sigalotti, 2009)

If $(\lambda_{n+1} - \lambda_n)_{n\in\mathbf{N}}$ is $\mathbf{Q}$-linearly independent and if B is connected w.r.t. A, then for every $\delta > 0$ $(A, B, (0, \delta))$ is approximately controllable on the unit sphere.

- The family $(\lambda_{n+1} - \lambda_n)_{n\in\mathbf{N}}$ is $\mathbf{Q}$-linearly independent if for every $N \in \mathbf{N}$ and $(q_1, \ldots, q_N) \in \mathbf{Q}^N \setminus \{0\}$ one has $\sum_{n=1}^{N} q_n(\lambda_{n+1} - \lambda_n) \neq 0$.
- B is connected w.r.t. A if for every $\{j, k\}$ in $\mathbf{N}^2$, $\exists p \in \mathbf{N}$, $\exists j = l_1, l_2, \ldots, l_p = k$ such that $b_{l_i,l_{i+1}} \neq 0$, for $1 \leq i \leq p$.

Lyapounov techniques

$$i\frac{\partial \psi}{\partial t}(x,t) = \underbrace{-\Delta\psi(x,t) + V(x)\psi(x,t)}_{A\psi} + u(t)\underbrace{W(x)\psi(x,t)}_{B\psi}$$

Ω is a bounded domain of $\mathbf{R^d}$, with smooth boundary.

Theorem (Nersesyan, 2009)

If

- $b_{1,j} \neq 0$ *for every* $j \geq 1$ *and*
- $|\lambda_1 - \lambda_j| \neq |\lambda_k - \lambda_l|$ *for every* $j > 1$, $\{1, j\} \neq \{k, l\}$

then the control system is approximately controllable on the unit sphere of L^2 *for* H^s *norms.*

Fixed point theorem

$\Omega = (0,1)$

$$i\frac{\partial \psi}{\partial t}(x,t) = \underbrace{-\Delta\psi(x,t)}_{A\psi} + u(t)\underbrace{W(x)\psi(x,t)}_{B\psi}$$

Theorem (Beauchard-Laurent, 2009)

If there exists $C > 0$ such that for every $j \in \mathbf{N}$,

$$|b_{1,j}| > \frac{C}{j^3}$$

then the system is exactly controllable in the intersection of the unit sphere with $H^3_{(0)}$.

A new result (simple statement)

Definition

$S \subset \mathbf{N}^2$ is a non resonant chain of connectedness of (A, B) if

- *for every $j \leq k$ in $\mathbf{N}$, there exists a sequence $(s_1^1, s_2^1), \ldots, (s_1^p, s_2^p)$ in $S \cap \{1, \ldots, k\}$ such that $s_1^1 = j, s_2^p = k,\ s_2^l = s_1^{l+1}$;*
- *$b_{s_1,s_2} \neq 0$ for every $(s_1, s_2) \in S$*
- *for every (j, k) in $\mathbf{N}^2$, $(s_1, s_2) \in S$, $\{s_1, s_2\} \neq \{j, k\} \Rightarrow |\lambda_{s_1} - \lambda_{s_2}| \neq |\lambda_j - \lambda_k|$ or $b_{j,k} = 0$*

Theorem (Boscain-Caponigro-Chambrion-Sigalotti)

If A has simple spectrum and (A, B) admits a non resonant chain of connectedness, then, for every $\delta > 0$, (A, B) is approximately simultaneously controllable by means of controls in $[0, \delta]$.

Idea of the geometric proof

Up to a time reparametrization, $e^{t(A+uB)} = e^{tu\left(\frac{1}{u}A+B\right)}$ the control system is

$$\dot{X} = \mathcal{P}uAX + BX, \qquad \mathcal{P}u > \frac{1}{\delta}.$$

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After the change of variable $Y = e^{-\int \mathcal{P}uA}X$, one finds

$$\dot{Y} = e^{-\int \mathcal{P}uA} B e^{\int \mathcal{P}uA} Y$$

For every k, $|\langle \phi_k, Y\rangle| = |\langle \phi_k, X\rangle|$

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Galerkin approximation :

$$\dot{Y} = \left[e^{i(\lambda_j - \lambda_k)\int \mathcal{P}u} b_{j,k} \right]_{j,k} Y.$$

Tracking

Non-resonant chain of connectedness : for every (j,k) in $\mathbf{N}^2$, $(s_1,s_2)\in S$, $\{s_1,s_2\}\neq\{j,k\}\Rightarrow|\lambda_{s_1}-\lambda_{s_2}|\neq|\lambda_j-\lambda_k|$ or $b_{j,k}=0$.

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For every $\epsilon > 0$, for every $\theta \in \mathbf{R}$, there exists a piecewise constant control u such that the system can track (in projection), up to ϵ, the finite dimensional system :

$$\dot{Y} = \rho \begin{pmatrix} 0 & \cdots & 0 & & \cdots & 0 \\ \vdots & & e^{i\theta} b_{j,k} & 0 & & \vdots \\ \vdots & & 0 & & \cdots & 0 \\ 0 & e^{-i\theta} b_{k,j} & 0 & & \cdots & 0 \\ 0 & \cdots & \cdots & & \cdots & 0 \end{pmatrix} Y$$

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$$\rho\geq\prod_{k=2}^{\infty}\cos\left(\frac{\pi}{2k}\right)\approx 0.4298156$$

Non-simple spectrum

The result extends to the case where A has finitely degenerated eigenvalues if (A, B, Φ) satisfies the extra condition

Hypothesis

$j \neq k$ *and* $\lambda_j = \lambda_k \Rightarrow b_{j,k} = 0.$

This is just a particular choice of the Hilbert basis Φ.

The result (non simple spectrum)

Theorem (Boscain-Caponigro-Chambrion-Sigalotti)

If (A, B, Φ) admits a non resonant chain of connectedness, then the control system is approximately simultaneously controllable on the sphere.

Example :

$$A = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad B = i \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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Slightly weaker hypotheses as for the finite result of controllability on the sphere for finite dimensional systems, obtained in 2000 by Turinici.

Estimates

Theorem (Boscain-Caponigro-Chambrion-Sigalotti)

If (A, B, Φ) *admits a non resonant chain of connectedness containing* $(1, 2)$*, then, for every* $\delta > 0$*, for every* $\epsilon > 0$*, there exist a piecewise constant control* $u : [0, T] \to [0, \delta]$ *such that*

$$\|\Upsilon^u_T(\phi_1) - \phi_2\| < \epsilon \text{ and } \|u\|_{L^1} \leq \frac{5\pi}{4|\langle \phi_1, B\phi_2 \rangle|}$$

The planar molecule

Let us consider a 2D-planar molecule submitted to a laser

$$i\frac{\partial\psi}{\partial t}(\theta,t) = -\frac{1}{2}\partial_\theta^2\psi(\theta,t) + u(t)\cos(\theta)\psi(\theta,t) \qquad \theta \in \mathbf{R}/2\pi$$

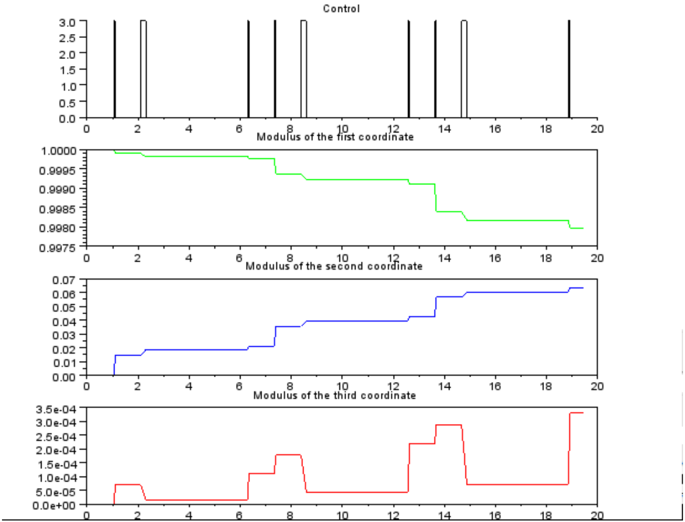
- The parity of ψ cannot change $\Rightarrow$ no global controllability
- We just look at the even part
- We try to steer the system from the first even eigenstate to the second even eigenstate

Galerkin approximation

$$A = i \begin{pmatrix} 0 & 0 & \dots & \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & 4 & \ddots \\ & \vdots & \ddots & 9 \end{pmatrix} \quad B = i \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \dots \\ 1/\sqrt{2} & 0 & 1/2 & \ddots \\ 0 & 1/2 & 0 & 1/2 \\ \vdots & \ddots & 1/2 & 0 \end{pmatrix}$$

$\{(k, k \pm 1); k \in \mathbf{N}\}$ is a non-resonant chain of connectedness.

Moduli of the first coordinates for $0 \leq t \leq 20$



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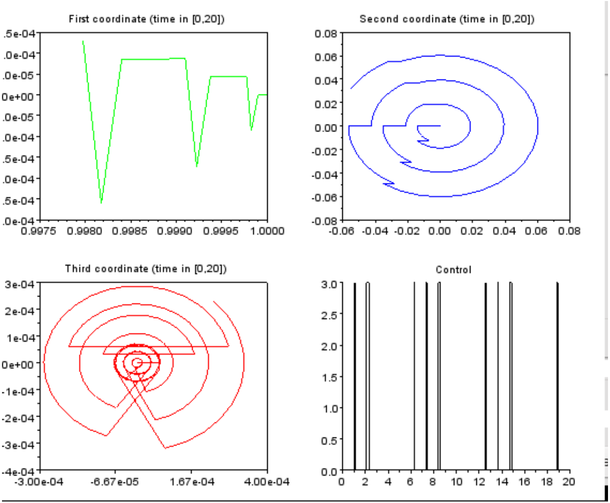
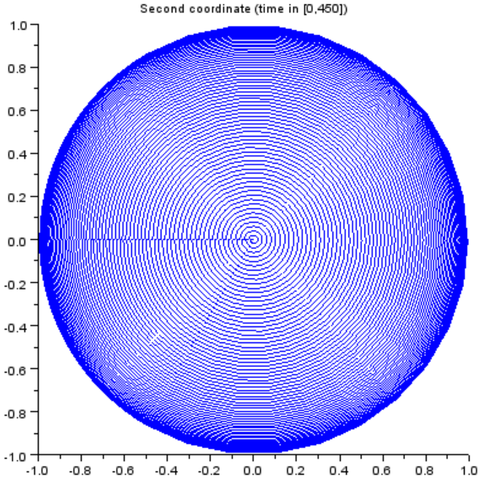
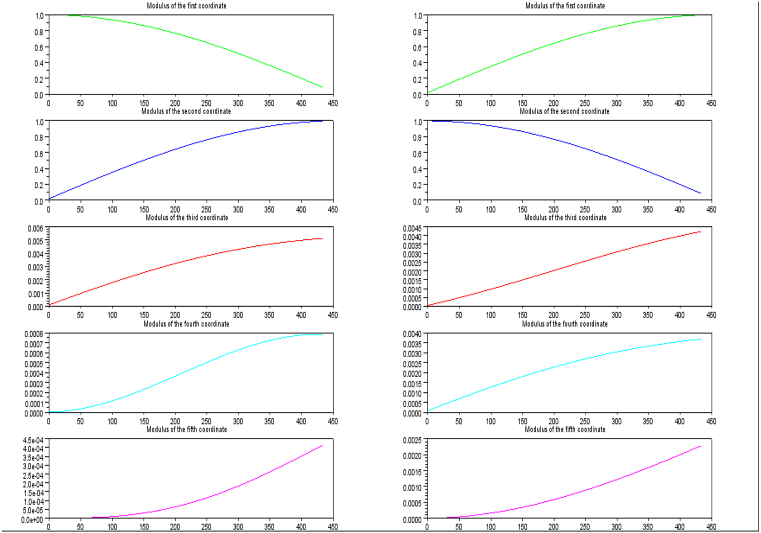


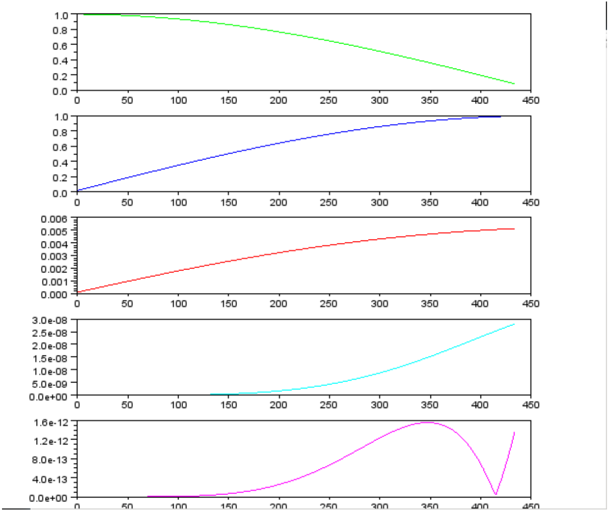
Second coordinate for $0 \leq t \leq 420$



Simultaneous control ($0 \leq t \leq 420$)



Moduli of coordinates 1, 2, 3, 8, 10 for $0 \leq t \leq 420$



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  - an explicit construction of the control (effective numerical computations) ;
  - easily computable estimates of the L^1 norm of the control.

Future works

- Simultaneous approximate controllability in higher norms

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- Implementation in the real life ?