New Estimates and Bounds on the Reachable Sets of Controlled Lindblad-Kossakowski Equations

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Preliminaries: The Lindblad-Kossakowski Equation (LKE)

Let $\rho \in \mathcal{D}_n$ be the density matrix of a quantum system with underlying state space $\mathbb{C}^n$, i.e. $\rho \in \mathbb{C}^{n\times n}$, $\rho = \rho^\dagger \geq 0$, and $\operatorname{tr}\rho = 1$. Its dynamics are given by

$$(\Sigma) \qquad \dot{\rho} = -\mathrm{i}[H,\rho] + \mathcal{L}(\rho). \qquad \text{(Lindblad-Kossakowski-Equation)}$$

Here, H denotes the system's Hamiltonian and $\mathcal{L}$ allows for interactions with the environment (= open quantum system).

Two equivalent representations of the interaction term $\mathcal{L}$:

$$\mathcal{L}(\rho) := \sum_{k=1}^{N} 2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k, \qquad \text{(Lindblad-Form)}$$

wherer $L_1,\ldots,L_N \in \mathbb{C}^{n\times n}$ and $N \in \mathbb{N}$ are arbitrary or, equivalently, ...

The Lindblad-Kossakowski Equation (cont'd)

Two equivalent representations of the interaction term $\mathcal{L}$:

$$\mathcal{L}(\rho) := \sum_{k=1}^{N} 2L_k\rho L_k^\dagger - L_k^\dagger L_k\rho - \rho L_k^\dagger L_k, \qquad \text{(Lindblad-Form)}$$

where $L_1, \dots, L_N \in \mathbb{C}^{n\times n}$ and $N \leq n^2 - 1$, or equivalently,

$$\mathcal{L}(\rho) := \sum_{j,k,=1}^{n^2-1} a_{jk}\Big([G_j, \rho G_k^\dagger] + [G_j\rho, G_k^\dagger]\Big), \qquad \text{(GKS-Form)}$$

where $G_1, \dots, G_{n^2-1}$ form an orthonormal basis of $\mathfrak{sl}_{\mathbb{C}}(n)$ and the GKS-matrix $A := (a_{jk})_{j,k=1,\dots,n^2-1}$ is positive semi-definite.

The vector of coherence representation which transfers (Σ) to a (bilinear control) system in $\mathbb{R}^{n^2-1}$ will not be used in this talk.

Preliminaries: Completely positive maps (CP-maps)

- A linear map $\mathcal{P} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is positive if and only if
$$\rho = \rho^\dagger \geq 0 \quad \Longrightarrow \quad \mathcal{P}(\rho) = \mathcal{P}(\rho)^\dagger \geq 0,$$
i.e. the set of all positive semidefinite matrices is invariant under $\mathcal{P}$.
- A linear map $\mathcal{P} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is completely positive if and only if
$$\mathrm{I}_p \otimes \mathcal{P} : \mathbb{C}^{p\times p} \otimes \mathbb{C}^{n\times n} \to \mathbb{C}^{p\times p} \otimes \mathbb{C}^{n\times n} \quad \text{is positive for all } p \in \mathbb{N}.$$
- A linear map $\mathcal{P} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is unital if and only if $\mathcal{P}(\mathrm{I}_n) = \mathrm{I}_n$.
- A linear map $\mathcal{P} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is trace-preserving if and only if
$$\operatorname{tr} \mathcal{P}(\rho) = \operatorname{tr} \rho \quad \text{for all } \rho \in \mathbb{C}^{n\times n}.$$

Remark

Positive maps which are unital and trace-preserving are often called doubly stochastic.

Completely positive maps (cont'd)

Facts:

- Any completely positive map $\mathcal{P} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is given by

$$\mathcal{P}(\rho) := \sum_{k=1}^{N} V_k \rho V_k^\dagger, \qquad \text{(diagonal Kraus-Form)}$$

  where $V_1, \dots, V_N \in \mathbb{C}^{n\times n}$ and $N \leq n^2$

- The forward flow of the LKE yields a one-parameter semigroup of completely positive, trace-preserving maps.

Preliminaries: Controlled LKEs

Introducing Hamiltonian controls to the LKE (in a semiclassical way) yields

$$(\Sigma_c) \qquad \dot{\rho} = -\mathrm{i}\Big[H_0 + \sum_{k=1}^{m} u_k(t) H_k, \rho\Big] + \mathcal{L}(\rho), \qquad \text{(controlled LKE)}$$

where $t \mapsto u_k(t) \in \mathcal{U} \subset \mathbb{R}^m$ are arbitrary measurable, locally bounded control functions.

General assumption: $\mathcal{U} = \mathbb{R}^m$

Further notation:

- $\mathcal{R}_{\leq T}(\rho_0) :=$ reachable set of ρ_0 up to time $T > 0$.
- $\mathcal{R}(\rho_0) :=$ entire reachable set of ρ_0

Preliminaries: The majorization preorder

Let $\hat{x}$ and $\hat{y}$ be decreasing rearrangements of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, respectively. Then, x is majorized by y (denoted by $x \prec y$) if and only if

$$\hat{x}_1 \leq \hat{y}_1, \quad \hat{x}_1 + \hat{x}_2 \leq \hat{y}_1 + \hat{y}_2, \quad \dots, \quad \sum_{k=1}^{n} \hat{x}_k = \sum_{k=1}^{n} \hat{y}_k.$$

For density matrices we define

$$\rho \prec \rho' \quad :\Longleftrightarrow \quad (\lambda_1, \dots, \lambda_n) \prec (\lambda'_1, \dots, \lambda'_n),$$

where $\lambda_1, \dots, \lambda_n$ and $\lambda'_1, \dots, \lambda'_n$ are the eigenvalues of ρ and ρ', respectively.

Result I: The Majorization Theorem

Theorem I [H. Yuan, IEEE 2010]

Let (Σ) be any LKE. Then the following statements are equivalent:

(a) (Σ) is unital, i.e. $\mathcal{L}(\mathrm{I}_n) = 0$.

(b) The flow Φ_t of (Σ) is monotonically majorization-decreasing, i.e. $\Phi_t(\rho_0) \prec \Phi_{t'}(\rho_0)$ for all $t \geq t'$ and all $\rho_0 \in \mathcal{D}_n$.

Note: $\mathcal{L}(\mathrm{I}_n) = 0 \iff \Phi_t(\mathrm{I}_n) = \mathrm{I}_n$ for all $t \in \mathbb{R}$.

Corollary

Let (Σ_c) be any controlled unital LKE. Then one has

$$\mathrm{cl}\Big(\mathcal{R}(\rho_0)\Big) \subset \{\rho \in \mathcal{D}_n \mid \rho \prec \rho_0\} = \mathrm{conv}\, \mathcal{O}(\rho_0)$$

for all $\rho_0 \in \mathcal{D}_n$.

Result II: Reachable sets of 2-level systems

Theorem II[H. Yuan, IEEE 2010[1]]

Let $n = 2$ and consider a controlled unital LKE of the form

$$(\Sigma_c) \qquad \dot{\rho} = -\mathrm{i}\Big[\sigma_z + u_x(t)\sigma_x + u_y(t)\sigma_y, \rho\Big] + \mathcal{L}(\rho),$$

where σ_x, σ_y and σ_z denote the Pauli matrices. Then, for generic $\mathcal{L}$ one has

$$\mathrm{cl}\,\Big(\mathcal{R}(\rho_0)\Big) = \{\rho \in \mathcal{D}_2 \mid \rho \prec \rho_0\} = \mathrm{conv}\,\mathcal{O}(\rho_0)$$

for all $\rho_0 \in \mathcal{D}_2$.

[1] This result is not explicitly stated, but implicitly contained in the cited paper.

Refined Majorization Theorem: Unitary invariant norms and the von Neumann entropy

A norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$ is unitarily invariant if and only if

$$\|UXW\| = \|X\| \quad \text{for all } X \in \mathbb{C}^{n\times n} \text{ and all } U, W \in U(n)$$

Standard examples:

- $\|X\|_2 := \left(\operatorname{tr} X^\dagger X\right)^{1/2} = \left(\operatorname{tr} |X|^2\right)^{1/2}$ (Hilbert/Schmidt norm)
- More general, let $p \in [1, \infty)$ and set

$$\|X\|_p := \left(\operatorname{tr} |X|^p\right)^{1/2}, \quad |X| := \sqrt{X^\dagger X} \qquad \text{(p-norm)}$$

Let ρ be a density matrix. The von Neumann entropy of ρ is defined by

$$N(\rho) := -\operatorname{tr}(\rho \log \rho).$$

Refined Majorization Theorem: Statement

Theorem A

Let (Σ) be any LKE. Then the following statements are equivalent:

(a) (Σ) is unital, i.e. $\mathcal{L}(\mathrm{I}_n) = 0$.

(b) The flow Φ_t of (Σ) is monotonically majorization-decreasing, i.e. $\Phi_t(\rho_0) \prec \Phi_{t'}(\rho_0)$ for all $t \geq t'$ and all $\rho_0 \in \mathcal{D}_n$.

(c) The flow Φ_t of (Σ) is monotonically norm-decreasing for all unitarily invariant norms $\|\cdot\|$, i.e. $\|\Phi_t(\rho_0)\| \leq \|\Phi_{t'}(\rho_0)\|$ for all $t \geq t'$ and all $\rho_0 \in \mathcal{D}_n$.

(d) The flow Φ_t of (Σ) is monotonically norm-decreasing for at least one strictly convex, unitarily invariant norm, i.e. $\|\Phi_t(\rho_0)\| \leq \|\Phi_{t'}(\rho_0)\|$ for all $t \geq t'$ and all $\rho_0 \in \mathcal{D}_n$.

(e) The flow Φ_t of (Σ) is monotonically increasing with respect to the von Neumann entropy, i.e. $N\big(\Phi_t(\rho_0)\big) \geq N\big(\Phi_{t'}(\rho_0)\big)$ for all $t \geq t'$ and all $\rho_0 \in \mathcal{D}_n$.

Refined Majorization Theorem: Consequence

Corollary

The semi-flow $\Phi_{t\geq 0}$ of (Σ) is a (weak) contraction semi-group with respect to all unitarily invariant norm if and only if (Σ) is unital.

Remark

Theorem A generalizes several known results from the literature, e.g.

- Φ_t is purity decreasing if and only if (Σ) is unital. [Lidar, et al.]
- The semi-flow $\Phi_{t\geq 0}$ of (Σ) is a (weak) contraction semi-group with respect to all p-norm if and only if (Σ) is unital. [Wolf, et al.]

Refined Majorization Theorem: Proof

The essential ingredient to the proof of Theorem A is the following result:

Theorem (Uhlmann 1971, Ando 1989)

The following statements are equivalent:

(a) $\rho \prec \rho'$.

(b) $\rho \in \operatorname{conv} \mathcal{O}(\rho')$, *where* $\mathcal{O}(\rho')$ *denotes the unitray orbit of* ρ'.

(c) *There exists a unital, trace-preserving, CP-map* $\mathcal{P}$ *with* $\mathcal{P}(\rho') = \rho$.

(d) *There exists a unital, trace-preserving, positive map* $\mathcal{P}$ *(= doubly stochastic map) with* $\mathcal{P}(\rho') = \rho$.

Refined Majorization Theorem: Proof

"(a) $\Longrightarrow$ (b):" Ando (Thm. 7.1)

"(b) $\Longrightarrow$ (a):"

- "Monotonicity" implies $\Phi_t(\mathrm{I}_n/n) \prec \mathrm{I}_n/n$ for all $t \geq 0$.
- However, $\mathrm{I}_n/n \prec \rho$ for all density operators ρ.
- Hence, $\Phi_t(\mathrm{I}_n/n) = \mathrm{I}_n/n$ for all $t \geq 0$ and thus $\mathcal{L}(\mathrm{I}_n/n) = 0$.

"(a) $\Longrightarrow$ (c):" Ando (Corollary 7.8)

"(c) $\Longrightarrow$ (d):" ✓

"(a) $\Longrightarrow$ (e):" Ando (Thm. 7.4), Yuan (Prop. 5)

"(e) $\Longrightarrow$ (a):"

- "Monotonicity" implies $N\big(\Phi_t(\mathrm{I}_n/n)\big) \geq N(\mathrm{I}_n/n)$ for all $t \geq 0$.
- However, $N(\mathrm{I}_n/n) \geq N(\rho)$ for all $\rho \in \mathcal{D}_n$ and $N(\mathrm{I}_n/n) > N(\rho)$ if $\rho \neq \mathrm{I}_n/n$.
- Hence, $\Phi_t(\mathrm{I}_n/n) = \mathrm{I}_n/n$ for all $t \geq 0$ and thus $\mathcal{L}$ is unital.

Refined Majorization Theorem: Proof

"(d) $\Longrightarrow$ (a):"

- Let $\|\cdot\|$ be any strictly convex, unitarily invariant norm. Then, "Monotonicity" implies $\|\Phi_t(\mathrm{I}_n/n)\| \leq \|\mathrm{I}_n/n\|$ for all $t \geq 0$.
- Moreover, $\mathrm{I}_n/n \prec \rho$ for all $\rho \in \mathcal{D}_n$ and thus $\|\mathrm{I}_n/n\| \leq \|\Phi_t(\mathrm{I}_n/n)\|$ for all $t \geq 0$.
- Hence, $\|\mathrm{I}_n/n\| = \|\Phi_t(\mathrm{I}_n/n)\|$ for all $t \geq 0$.
- Assume w.l.o.g. $\Phi_{t_*}(\mathrm{I}_n/n) =: \rho_* = \mathrm{diag}(r_1, \dots, r_n) \neq \mathrm{I}_n/n$ for some $t_* > 0$. Then

$$\begin{aligned}\|\mathrm{I}_n/n\| &= \Big\| \sum_{\pi \in S_n} \alpha_\pi \,\mathrm{diag}(r_{\pi(1)}, \dots, r_{\pi(n)}) \Big\| \\ &\leq \sum_{\pi \in S_n} \alpha_\pi \big\| \mathrm{diag}(r_{\pi(1)}, \dots, r_{\pi(n)}) \big\| = \|\mathrm{I}_n/n\|.\end{aligned}$$

  with $\sum_{\pi \in S_n} \alpha_\pi = 1$ and $\alpha_\pi \geq 0$.
- This contradicts the strict convexity of $\|\cdot\|$. Therefore $\Phi_t(\mathrm{I}_n/n) = \mathrm{I}_n/n$ for all $t \geq 0$ and thus $\mathcal{L}$ is unital.

Generalized Reachability Result: Comments on Yuan's proof

Theorem II [H. Yuan, IEEE 2010[2]]

Let $n = 2$ and consider a controlled unital LKE of the form

$$(\Sigma_c) \qquad \dot{\rho} = -\mathrm{i}\Big[\sigma_z + u_x(t)\sigma_x + u_y(t)\sigma_y, \rho\Big] + \mathcal{L}(\rho),$$

where σ_x, σ_y and σ_z denote the Pauli matrices. Then, for generic $\mathcal{L}$ one has

$$\mathrm{cl}\,\Big(\mathcal{R}(\rho_0)\Big) = \{\rho \in \mathcal{D}_2 \mid \rho \prec \rho_0\} = \mathrm{conv}\,\mathcal{O}(\rho_0)$$

for all $\rho_0 \in \mathcal{D}_2$.

- Result II in Yuan's paper is even slightly more general, as he actually charaterized the closure of $\mathcal{R}_{\leq T}(\rho_0)$ for any $T > 0$.

[2] see also footnote on page 10

Generalized Reachability Result: Comments on Yuan's proof

- One essential ingredient of the proof is the fact that the Lie algebra generated by the control terms σ_x and σ_y coincides with the entire Lie algebra $\mathfrak{su}(2)$ and thus any $\mathrm{SU}(2)$-matrix can be reached approximately in time $T = 0$.

Such systems will be called fast Hamiltonian controllable.

conjecture

If the controlled LKE (Σ_c) is generic, fast Hamiltonian controllable and unital, then the closure of the reachable set of any $\rho_0 \in \mathcal{D}_n$ is given by $\operatorname{conv} \mathcal{O}(\rho_0)$.

- Yuan's proof heavily exploits the fact that the adjoint action of $\mathrm{SU}(2)$ on $\mathfrak{h}_0(2) := \mathrm{i}\,\mathfrak{su}(2) \simeq \mathbb{R}^3$ can be identified with the action of $SO(3)$ on $\mathbb{R}^3$.

Problem: " $\mathrm{Ad}_{SU(n)} \subsetneq SO(n^2 - 1)$ " for $n \geq 3$.

A controlled LKE with Hamiltonian controls $H_1, \dots, H_m$ is called fast Hamiltonian controllable if and only if

$$\langle \mathrm{i}H_1, \dots, \mathrm{i}H_m \rangle_{Lie} = \mathfrak{su}(n).$$

Theorem B

Let (Σ_c) be a generic, fast Hamiltonian controllable, unital LKE and let $\mathcal{L}$ be of simple double commutator form, i.e.

$$\mathcal{L}(\rho) = -\big[L, [L, \rho]\big]$$

for some Hermitian $(n \times n)$-matrix L. Then the closure of the reachable set of any $\rho_0 \in \mathcal{D}_n$ is given by

$$\mathrm{cl}\Big(\mathcal{R}(\rho_0)\Big) = \mathrm{conv}\,\mathcal{O}(\rho_0) = \{\rho \in \mathcal{D}_n \mid \rho \prec \rho_0\}.$$

Simple double commutator case: Proof

- Since (Σ_c) is unital we obtain from the Majorization Theorem

$$\mathrm{cl}\Big(\mathcal{R}(\rho_0)\Big) \subset \mathrm{conv}\,\mathcal{O}(\rho_0)$$

- Since (Σ_c) is also assumed to be fast Hamiltonian controllable, it suffices to show that all diagonal density matrices $\rho = \mathrm{diag}(s_1, \ldots, s_n)$, $s_1 \geq \cdots \geq s_n$ with $\rho \prec \rho_0$ are contained in $\mathrm{cl}\Big(\mathcal{R}(\rho_0)\Big)$
- Again due to the fast Hamiltonian controllability we assume w.l.o.g. $\rho_0 = \mathrm{diag}(r_1, \ldots, r_n)$, $r_1 \geq \cdots \geq r_n$.

Single double commutator case: Proof

- Now, let ρ_0 and ρ be represented as follows:

$$\rho_0 = \mathrm{I}_n/n + \sum_{j=1}^{n-1} \alpha_j D_j \quad \text{and} \quad \rho = \mathrm{I}_n/n + \sum_{j=1}^{n-1} \beta_j D_j,$$

with $0 \leq \beta_j \leq \alpha_j$ and

$$D_j := \mathrm{diag}(\underbrace{0,\dots,0}_{(j-1)-\text{times}},1,-1,0,\dots,0) = \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{bmatrix}.$$

- Using the identity

$$\mathrm{Ad}_{U^\dagger}\,\mathrm{ad}_L^2\,\mathrm{Ad}_U = \mathrm{ad}_{U^\dagger L U}$$

we can "turn" L to the form $\begin{bmatrix} c & x & & & \\ x & c & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \ddots \end{bmatrix}$, $x \geq 0$ in approx. time $T = 0$.

Single double commutator case: Proof

- Then, one has

$$[L,[L,D_j]] = \begin{cases} 4x^2 D_1 & \text{for } j = 1 \\ -2xD_1 & \text{for } j = 2 \\ 0 & \text{for } j \geq 3 \end{cases}$$

It follows

$$\mathrm{e}^{-t\,\mathrm{ad}_L^2}(D_j) = \begin{cases} \mathrm{e}^{-4x^2 t} D_1 & \text{for } j = 1 \\ \frac{\mathrm{e}^{-4x^2 t}-1}{2x} D_1 + D_2 & \text{for } j = 2 \\ D_j & \text{for } j \geq 3 \end{cases}$$

and thus

$$\mathrm{e}^{-t\,\mathrm{ad}_L^2}(\rho_0) = \mathrm{e}^{-t\,\mathrm{ad}_L^2}\Big(\mathrm{I}_n/n + \sum_{j=1}^{n-1} \alpha_j D_j\Big) = g(t)\alpha_1 D_1 + \mathrm{I}_n/n + \sum_{j=2}^{n-1} \alpha_j D_j,$$

where $g(t) := \mathrm{e}^{-4x^2 t} + \frac{\mathrm{e}^{-4x^2 t}-1}{2x}$.

Single double commutator case: Proof

- Since $t \mapsto g(t)$ is a monotonically decreasing function with $\lim_{t\to\infty} g(t) = 0$, there is some t^* such that $g(t^*)\alpha_1 = \beta_1$.
- Then continue with "α_2" and $L' := \begin{bmatrix} \lambda_1 & & & & \\ & c' & x' & & \\ & x' & c' & & \\ & & & \lambda_4 & \\ & & & & \ddots \end{bmatrix}$ with $x' \geq 0$.
- Recall that $0 \leq \beta_j \leq \alpha_j$ for all $j = 1, \ldots, n-1$.

Generalized Reachability Result: general double commutator case

Theorem C

Let $n \geq 3$ and (Σ_c) be any controlled unital LKE with $\mathcal{L}$ is of general double commutator form, i.e.

$$\mathcal{L}(\rho) := -\sum_{j=1}^{N} [L_j, [L_j, \rho]],$$

where $N \geq 2$ and $L_1, \ldots, L_N$ are generic Hermitian $(n \times n)$-matrices. Then there exist $\rho_0 \in \mathcal{D}_n$ such that the closure of their reachable set is in general a proper subset of $\operatorname{conv} \mathcal{O}(\rho_0)$, i.e.

$$\operatorname{cl}\Big(\mathcal{R}(\rho_0)\Big) \subsetneqq \operatorname{conv} \mathcal{O}(\rho_0) = \{\rho \in \mathcal{D}_n \mid \rho \prec \rho_0\}$$

even if (Σ_c) is fast Hamiltonian controllable.

Auxiliary results: A useful inequality

Lemma A

Let H be any Hermitian $(n \times n)$-matrix and let ρ be any positive semidefinite matrix. Then one has

$$\|H\|_2^2 \det \rho \leq \operatorname{tr}(\rho^\sharp H \rho H), \tag{1}$$

where $\rho^\sharp$ denotes the adjugate matrix of ρ.

Proof: One can assume that ρ is diagonal and positive definite, i.e. $\rho = \operatorname{diag}(r_1, \ldots, r_n)$ with $r_1 \geq r_2 \geq \cdots \geq r_n > 0$. Then, the identity $\rho^\sharp = \rho^{-1} \det \rho$ implies that (1) is equivalent to

$$\|H\|_2^2 \leq \operatorname{tr}(\rho^{-1} H \rho H).$$

Now, the standard estimate

$$2 \leq \left(\frac{r_i}{r_j} + \frac{r_j}{r_i}\right)$$

yields the desired result.

Auxiliary results: A neat estimate

Lemma B

Let (Σ_c) be any controlled unital LKE. Moreover, let $u(t)$ be any admissible control and $\rho(t)$ the respective solution of (Σ_c). Then, the function $t \mapsto \det\big(\rho(t)\big)$ is monotonically increasing.

Corollary

The set $\{\rho \in \mathcal{D}_n \mid \det \rho \neq 0\}$ *is forward-invariant under* (Σ_c).

Remark

Lemma B straightforwardly follows from a more general convexity result (Ando Thm. 7.4) which yields

$$\rho \prec \rho' \quad \Longrightarrow \quad \det \rho \geq \det \rho'.$$

However, the following proof provides additionally helpful information on the "degree" of monotonicity.

Generalized Reachability Result: A neat estimate

Proof: For simplicity, we assume that $\mathcal{L}$ is in double commutator form, i.e.

$$\mathcal{L}(\rho) := -\sum_{j=1}^{N}[L_j,[L_j,\rho]].$$

Thus we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\det\big(\rho(t)\big) &= \mathrm{D}\det\big(\rho(t)\big)\cdot\dot\rho(t) = \operatorname{tr}\Big(\rho^\sharp(t)\big(-\mathrm{i}[H,\rho(t)] + \mathcal{L}(\rho(t))\big)\Big) \\
&= \sum_{j=1}^{N}\operatorname{tr}\Big(\rho^\sharp(t)\big(2L_j\rho(t)L_j - \rho(t)L_jL_j - L_jL_j\rho(t)\big)\Big) \\
&\overset{\text{LemA}}{\geq} \det\big(\rho(t)\big)\Big(\sum_{j=1}^{N}2\|L_j\|_2^2 - 2\operatorname{tr}L_jL_j\Big) = 0.
\end{aligned}$$

General double commutator case: Proof

- For simplicity, we restrict to the case $n = 3$ and $N = 2$. Consider

$$\rho_0 := \begin{bmatrix} 3/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \rho_* := \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

  Note that $\rho_* \prec \rho_0$, $\det \rho_* = \det \rho_0$, $\rho_* \notin \mathcal{O}(\rho_0)$ and $\|\rho_*\| < \|\rho_0\|$.
- For generic L_1 and L_2 one has

$$\sum_{j=1}^{2} \operatorname{tr}\left(\rho^\sharp(2L_j\rho L_j - \rho L_j L_j - L_j L_j \rho)\right) > 0 \tag{2}$$

  for all $\rho \in \mathcal{O}(\rho_0)$.
- Assume that $\rho_* \in \mathcal{R}(\rho_0)$. Then $\|\rho_*\| < \|\rho_0\|$ implies that the "minimal" $T > 0$ to reach ρ_* is bounded away from zero.

General double commutator case: Proof

- By inequality (2) and the compactness of $\rho \in \mathcal{O}(\rho_0)$ one can find a neighborhood $\mathcal{W}$ of $\rho \in \mathcal{O}(\rho_0)$ such that $\mathcal{W} \cup \mathcal{O}(\rho_*) = \emptyset$ and

$$\sum_{j=1}^{2} \operatorname{tr}\left(\rho^{\sharp}(2L_j\rho L_j - \rho L_j L_j - L_j L_j \rho)\right) \geq c > 0.$$

- A similar argument as before shows that $\rho(t)$ cannot leave $\mathcal{W}$ in arbitrarily small time $T' > 0$.
- Thus one can guarantee that $\det \rho(T') \geq c' := cT'$, when leaving $\mathcal{W}$, and therefore by the monotonicity of $t \mapsto \det \rho(t)$ the solution $\rho(t)$ is bounded away from ρ_*.

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Thanks for paying attention!

Merci de votre attention!

Problems:

- What means "dissipation" in open quantum systems?
- In what sense does a LKE model dissipation?

Answer: No idea!

Naive approach: In correspondence to the concept of dissipation in classical systems, one would like to call an open quantum systems dissipative if and only if the expectation value of the "total energy" decreases in time, i.e. $t \mapsto \operatorname{tr}\big(H\rho(t)\big)$ is monotonically decreasing for all initial values $\rho(0) \in \mathcal{D}_n$.

Problem: Let (Σ) be a LKE. For simplicity, we assume that $\mathcal{L}$ is of simple double commutator form, i.e.

$$\mathcal{L}(\rho) := -\big[L, [L, \rho]\big] \qquad L \in \mathfrak{h}_0(n) := \mathrm{i}\,\mathfrak{su}(n).$$

Then, one has ...

Appendix: Some remarks on "dissipation" in open quantum systems

Then, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\big(H\rho(t)\big) &= \operatorname{tr}\big(H\dot{\rho}(t)\big) = \operatorname{tr}\Big(H\big(\mathrm{i}[H,\rho] - \big[L,[L,\rho]\big]\big)\Big)\\ &= -\operatorname{tr}\big(H\big[L,[L,\rho(t)]\big]\big) = -\operatorname{tr}\big(\big[L,[L,H]\big]\rho(t)\big)\end{aligned}$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\big(H\rho_0\big) \leq 0 \quad \text{for all } \rho(0) \in \mathcal{D}_n \iff \big[L,[L,H]\big] \geq 0 \iff \big[L,[L,H]\big] = 0.$$

Therefore, the total energy is decreasing if and only if L and H commute. But then the total energy is actually conserved and no "real" dissipation occures.

Better Concept: Taking the previous results into account. We suggest to call an open quantum systems dissipative if and only if the function $t \mapsto N\big(\rho(t)\big)$ is monotonically increasing for all initial values $\rho(0) \in \mathcal{D}_n$.

Consequence: A LKE is dissipative in the above sense if and only if it is unital.