Approximate controllability for a system of Schrödinger equations modelling a single trapped ion

Sylvain Ervedoza    Jean-Pierre Puel

Institut de Mathématiques de Toulouse & CNRS

December 9th, 2010
Outline of the talk

1. Introduction
2. The Cauchy problem
3. A physical approach
   - The Law-Eberly system
   - Formal derivation of the approximate system
4. Our approach
   - In low-frequency invariant spaces
   - In $(L^2)^2$
   - Estimates in higher order norms
5. Conclusion
Let $A$ be the harmonic operator on $\mathbb{R}$

$$A = \frac{1}{2} \left( -\partial_{xx}^2 + x^2 \right).$$

Properties of $A$

This operator is **self-adjoint, with compact resolvent.**

The spectrum of $A$ is explicit

$$A\Phi_j = \lambda_j \Phi_j, \quad (\Phi_j) = \text{Hermite functions}, \quad \lambda_j = j + 1/2.$$
Model of a single trapped ion:

\[
\begin{align*}
    i\partial_t \psi_e &= \omega A \psi_e + \frac{\Omega}{2} \psi_e + (u + u^*) \psi_g, \quad (t, x) \in (0, T) \times \mathbb{R}, \\
    i\partial_t \psi_g &= \omega A \psi_g - \frac{\Omega}{2} \psi_g + (u + u^*) \psi_e, \quad (t, x) \in (0, T) \times \mathbb{R}, \\
    \psi_e(0, x) &= \psi_e^0(x), \quad \psi_g(0, x) = \psi_g^0(x), \quad x \in \mathbb{R}. \tag{1}
\end{align*}
\]

- \(\omega, \Omega\) are large real numbers! \(\Omega \gg \omega \gg 1\).
- \(u\) is the control function, superposition of 3 lasers:

\[
u(t, x) = u_0 e^{i(\Omega t - \sqrt{2} \eta_0 x)} + u_r e^{i((\Omega - \omega) t - \sqrt{2} \eta_r x)} + u_b e^{i((\Omega + \omega) t - \sqrt{2} \eta_b x)}.
\]
The model

Physical constraints on the control function:

\[ u(t, x) = u_0 e^{i(\Omega t - \sqrt{2} \eta_0 x)} + u_r e^{i((\Omega - \omega) t - \sqrt{2} \eta_r x)} + u_b e^{i((\Omega + \omega) t - \sqrt{2} \eta_b x)}, \]

- \((u_0, u_b, u_r) \in \mathbb{C}^3\).
- \(t \mapsto (u_0(t), u_b(t), u_r(t))\) is piecewise constant.
- At each time \(t\), there is at most one control “on”.
- \(\eta\) are the Lamb-Dicke parameters, assumed small

\[ \eta \ll 1. \]
The model, main assumptions

\[
\begin{align*}
&i \partial_t \psi_e = \omega A \psi_e + \frac{\Omega}{2} \psi_e + (u + u^*) \psi_g, \quad (t, x) \in (0, T) \times \mathbb{R}, \\
&i \partial_t \psi_g = \omega A \psi_g - \frac{\Omega}{2} \psi_g + (u + u^*) \psi_e, \quad (t, x) \in (0, T) \times \mathbb{R}, \\
&\psi_e(0, x) = \psi_e^0(x), \quad \psi_g(0, x) = \psi_g^0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

\(\Omega \gg \omega \gg 1, \quad \eta \ll 1.\)

\[
u(t, x) = u_0 e^{i(\Omega t - \sqrt{2} \eta_0 x)} + u_r e^{i((\Omega - \omega) t - \sqrt{2} \eta r x)} + u_b e^{i((\Omega + \omega) t - \sqrt{2} \eta b x)}.
\]

**Problem**

Can we control this systems with such controls?
Our result: Approximate controllability

Theorem

Let \((\psi^0_e, \psi^0_g)\) and \((\psi^1_e, \psi^1_g)\) of unit \((L^2)^2\) norm. Then \(\forall \delta > 0, \exists (\mathcal{K}, \eta_0, \rho_0)\), such that for all \((\omega, \Omega)\) with \(2\Omega \geq 3\omega\) and

\[ \eta \leq \eta_0, \quad KT = \mathcal{K}/\eta, \quad \omega\eta \geq \rho_0, \]

there exists a control \(u(t, x)\) as above, furthermore satisfying

\[ \sup\{|u_0(t)|, |u_r(t)|, |u_b(t)|\} \leq K, \]

such that the solution \((\psi_e, \psi_g)\) of (1) with initial data \((\psi^0_e, \psi^0_g)\) satisfies, for some \(\beta\) of modulus 1,

\[ \left\| (\psi_e(T), \psi_g(T)) - \beta(\psi^1_e, \psi^1_g) \right\|_{0 \times 0} \leq \delta. \]
• Many different interpretations of the conditions

\[ \eta \leq \eta_0, \quad KT = \aleph / \eta, \quad \frac{\omega \eta}{K} \geq \rho_0: \]

  - \textit{K fixed}, then \( \eta \ll 1, \ T = T^*/\eta, \) and \( \omega \geq \omega^*/\eta. \)
  - \textit{T fixed}: \( \eta \ll 1, \ K = K^*/\eta, \ \omega \geq \omega^*/\eta^2. \)
  - \( \omega, \Omega \) fixed: \( \eta \ll 1, \ \eta/K \gg 1, \ T = T^*/(K\eta). \)
  - \( K = \eta: \eta \ll 1, \ \omega \gg 1, \ T = T^*/\eta^2. \)

• We always have \( \omega \gg K. \)

• If, at time \( T, \) the control is turned off, the solution stays in a \( \delta \) neighborhood of the target trajectory.

• Can be generalized for all norms \( \| (\cdot, \cdot) \|_{k \times k}, \) see later.
• **Approximate controllability:**
  - Through a stabilization approach: Beauchard, Coron, Mirrahimi, Rouchon, Turinici, Nersesyan...
  - Through optimal control: Baudouin, Puel, Salomon, Ito, Kunisch, ...
  - Via a geometric approach: Bloch, Brockett, Rangan, Agrachev...
  - Via a finite dimensional point of view: Chambion, Mason, Sigalotti, Boscain. $\rightsquigarrow$ idea close to the one we shall use below.

• **Exact controllability:**
  - Negative results: Ball, Marsden, Slemrod, Turinici, Mirrahimi, Rouchon.
  - Positive results: Beauchard, Coron, Laurent.
The Cauchy problem

Notations

\[ \|\psi\|_k = \left\| A^{k/2} \psi \right\|_{L^2(\mathbb{R})}, \quad \forall \psi \in D(A^{k/2}) \]

\[ \|(\psi_1, \psi_2)\|_{k \times k} = \left( \|\psi_1\|_k^2 + \|\psi_2\|_k^2 \right)^{1/2}, \quad \forall (\psi_1, \psi_2) \in D(A^{k/2})^2 \]

\[
\begin{cases}
    i \partial_t \psi_e = \omega A \psi_e + \frac{\Omega}{2} \psi_e + f \psi_g, & (t, x) \in (0, T) \times \mathbb{R}, \\
    i \partial_t \psi_g = \omega A \psi_g - \frac{\Omega}{2} \psi_g + f \psi_e, & (t, x) \in (0, T) \times \mathbb{R},
\end{cases}
\]

\[ \psi_e(0, x) = \psi_e^0(x), \quad \psi_g(0, x) = \psi_g^0(x), \quad x \in \mathbb{R}. \]  

Here \( f = f(t, x) \) is real valued.
The Cauchy problem

Theorem

Let $T > 0$. Let $f : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$, $f \in L^\infty((0, T); C^0_b(\mathbb{R}))$. Then, for all initial data $(\psi_e^0, \psi_g^0) \in L^2(\mathbb{R})^2$, there exists a unique solution $(\psi_e, \psi_g)$ of (2) in $C([0, T]; L^2(\mathbb{R})^2)$, and $\forall t > 0$,

$$\| (\psi_e(t), \psi_g(t)) \|_{0 \times 0} = \| (\psi_e^0, \psi_g^0) \|_{0 \times 0}.$$

Moreover, if $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and $f \in L^\infty((0, T); C^k_b(\mathbb{R}))$, then $(\psi_e, \psi_g) \in C([0, T]; \mathcal{D}(A^{k/2})^2)$. 
Sketch of the proof

- **Step 1:** Prove that the map

\[
\psi_e(\psi_e, \psi_g)(t) = S(t)e^{-i\Omega t/2}\psi^0_e + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2}f(s)\psi_g(s) \, ds,
\]

\[
\psi_g(\psi_e, \psi_g)(t) = S(t)e^{i\Omega t/2}\psi^0_g + i\int_0^t S(t-s)e^{i\Omega(t-s)/2}f(s)\psi_e(s) \, ds,
\]

on \(Y = C([0, T]; \mathcal{D}(A^{k/2})^2)\) endowed with the norm

\[
\|(\psi_e, \psi_g)\|_Y = \sup_{t \in [0, T]} \left\{ e^{-\lambda t} \|(\psi_e(t), \psi_g(t))\|_{k \times k} \right\},
\]

is a contraction, for a good choice of \(\lambda\). \((S(t) = \exp(-it\omega A)\)
is the free Schrödinger semigroup).

- **Step 2.** **A priori estimates** for smooth solutions.
- **Step 3.** Limit for **low-regularity data**.
An approximate system: Law-Eberly

Let us consider the approximate system

\[
\begin{align*}
    i \partial_t \phi_e &= \left( u_0^* + v_r^* a + v_b^* a^\dagger \right) \phi_g, \quad (t, x) \in (0, T) \times \mathbb{R}, \\
    i \partial_t \phi_g &= \left( u_0 + v_r a^\dagger + v_b a \right) \phi_e, \quad (t, x) \in (0, T) \times \mathbb{R},
\end{align*}
\]

with

\[
a = \frac{1}{\sqrt{2}} \left( x + \partial_x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( x - \partial_x \right).
\]

Here \( v_r \) and \( v_b \) respectively correspond to \( -i \eta u_r \) et \( -i \eta u_b \).

Advantage: This system is exactly controllable!
Let us consider the approximate system

\[
\begin{align*}
\forall t, x \in (0, T) \times \mathbb{R}, \\
\frac{d}{dt} \phi_e &= \left( u_0^* + v_r^* a + v_b^* a^\dagger \right) \phi_g, \\
\frac{d}{dt} \phi_g &= \left( u_0 + v_r a^\dagger + v_b a \right) \phi_e,
\end{align*}
\]

with

\[
a = \frac{1}{\sqrt{2}} \left( x + \partial_x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( x - \partial_x \right).
\]

Here \( v_r \) and \( v_b \) respectively correspond to \(-i\eta u_r \) et \(-i\eta u_b \).

Advantage: This system is exactly controllable!
Some spectral theory

The operators \( a \) and \( a^\dagger \) respectively are the **annihilation and creation** operators.

\[
A = a^\dagger a + \frac{1}{2} = aa^\dagger - \frac{1}{2},
\]

\[
a\Phi_0 = 0, \quad \begin{cases} a\Phi_{n+1} = \sqrt{n+1} \Phi_n, \\ a^\dagger \Phi_n = \sqrt{n+1} \Phi_{n+1}, \end{cases} \quad \forall n \in \mathbb{N}.
\]

Notations: For \( M \in \mathbb{N} \), we define

\[
V_M = \text{span} \left\{ \Phi_j ; \ 0 \leq j \leq M \right\}.
\]
Law-Eberly, a refined version

Theorem (Law Eberly revisited)

For all \((\phi_e^0, \phi_g^0), (\phi_e^1, \phi_g^1) \in V_M^2\) of same \((L^2)^2\)-norm, there exist \(T > 0\) and a control \(t \mapsto (u_0(t), v_r(t), v_b(t))\) such that

- \((\phi_e, \phi_g)\) solution of (3) with initial data \((\phi_e^0, \phi_g^0)\) satisfies \((\phi_e(T), \phi_g(T)) = \beta(\phi_e^1, \phi_g^1)\) for some \(\beta \in \mathbb{C}\) of unit modulus.
- \(\forall t \in [0, T], (\phi_e(t), \phi_g(t)) \in V_M^2\).
- There is at most one control “ON”
- There are at most \(2M\) switching times.
- Imposing \(|u_0| \leq K_0\) and \(|v_r|, |v_b| \leq K_1\), then one can take any \(T\) s. t. \(T \geq T^*\), with

\[
T^* = \frac{(M + 1)\pi}{K_0} + \frac{\pi}{K_1}\sum_{j=1}^{M} \frac{1}{\sqrt{j}}
\]
Sketch of the proof

- If $u_0$ is the only active control:

\[ i \partial_t \phi_e = u_0^* \phi_g, \quad i \partial_t \phi_g = u_0 \phi_e, \quad (t, x) \in (0, T) \times \mathbb{R}. \]

The ratio of populations $\langle \phi_e, \Phi_n \rangle$ and $\langle \phi_g, \Phi_n \rangle$ oscillate at frequency $|u_0|$.

- If $v_r$ is the only active control:

\[ i \partial_t \phi_e = v_r^* a \phi_g, \quad i \partial_t \phi_g = v_r a^\dagger \phi_e, \quad (t, x) \in (0, T) \times \mathbb{R}. \]

The ratio of populations $\langle \phi_e, \Phi_n \rangle$ and $\langle \phi_g, \Phi_{n+1} \rangle$ oscillate at frequency $|v_r| \sqrt{n}$.

\[ \rightsquigarrow \textbf{Idea:} \quad \text{Put everything on } (\phi_e, \phi_g) = (0, \Phi_0), \text{ and use reversibility to conclude.} \]
Formal derivation: Lamb-Dicke approximation

\[ u_{LD}(t, x) = \left( u_0 e^{i\Omega t} + u_r e^{i(\Omega - \omega)t} + u_b e^{i(\Omega + \omega)t} \right) (1 - i\sqrt{2}\eta x). \]

\[ \tilde{\phi}_e(0, x) = \psi^0_e(x), \quad \tilde{\phi}_g(0, x) = \psi^0_g(x), \quad x \in \mathbb{R}. \]
Formal derivation: the averaging approximation

\[ e^{it\omega A}(a + a^\dagger)e^{-it\omega A} = e^{-i\omega t}a + e^{i\omega t}a^\dagger. \]

Hence

\[
e^{i\Omega t}S(-t)(u_{LD} + u_{LD}^*)S(t)
= u_0e^{2i\Omega t}(1 - i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) + u_0^*(1 + i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) \\
+ u_re^{i(2\Omega - \omega)t}(1 - i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) \\
+ u_re^{i(2\Omega + \omega)t}(1 - i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) \\
+ u_be^{i(2\Omega + \omega)t}(1 - i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)) \\
+ u_be^{-i\omega t}(1 + i\eta(e^{-i\omega t}a + e^{i\omega t}a^\dagger)).
\]

\[ \rightsquigarrow \text{ Averaging: Cancel all oscillatory terms!} \]

Yields Law-Eberly equations by setting \( v_b = -i\eta u_b, \) \( v_r = -i\eta u_r \) as announced.
Our approach is as follows:

1. To precisely measure the error done in the previous approximations for initial and target data in $V_M^2$.
2. To truncate initial and target data to go back to the previous item.
Let $M \in \mathbb{N}$.

From Law-Eberly’s theorem, the time should be

$$T \geq T^* = \frac{(M + 1)\pi}{K_0} + 2 \frac{\pi}{K_1} \sum_{j=1}^{M} \frac{1}{\sqrt{j}}$$

under the constraints $|u_0| \leq K_0$ et $|v_b|, |v_r| \leq K_1$.

We want to consider the constraints $|u_0|, |u_r|, |u_b| \leq K$. Since $v_b = -i\eta u_b$ and $v_r = -i\eta u_r$, we take $K_0 = K$ and $K_1 = \eta K$ : for $\eta \leq 1/2\sqrt{M}$,

$$TK = \frac{3\pi \sqrt{M}}{\eta}.$$

Fix $T$ as above.
Let \((\psi^0_e, \psi^0_g)\) and \((\psi^1_e, \psi^1_g)\) in \(V^2_M\) of unit \((L^2)^2\)-norm.

Define \((\phi^0_e, \phi^0_g) = (\psi^0_e, \psi^0_g)\) and

\[
(\phi^1_e, \phi^1_g) = (S(-T) \exp(i\Omega T/2)\psi^1_e, S(-T) \exp(-i\Omega T/2)\psi^1_g)
\]

Then Law-Eberly’s theorem provides a control \(u\) that steers solutions of Law-Eberly approximate equations (3) from \((\phi^0_e, \phi^0_g)\) to \(\beta(\phi^1_e, \phi^1_g)\), for \(\beta \in \mathbb{C}\) of unit modulus.

In the sequel, we always consider this control!
Approximate controllability in $V^2_M$

**Theorem**

Let $(\psi^0_e, \psi^0_g)$ and $(\psi^1_e, \psi^1_g)$ in $V^2_M$ and the above control:

$\forall \delta > 0, \exists \eta_0 = \eta_0(\delta, M), \exists \rho_0 = \rho_0(\delta, M)$, s. t. $\forall K, \forall (\omega, \Omega)$ with $2\Omega \geq 3\omega$ and

$$\eta \leq \eta_0, \quad TK = \frac{3\pi \sqrt{M}}{\eta}, \quad \frac{\omega \eta}{K} \geq \rho_0,$$

the solution of the complete system (1) with initial data $(\psi^0_e, \psi^0_g)$ satisfies

$$\| (\psi_e(T), \psi_g(T)) - \beta(\psi^1_e, \psi^1_g) \|_{0 \times 0} \leq \delta.$$
Sketch of the proof

In the interaction frame:

\[ \xi_e = S(-t)e^{i\Omega t/2}\psi_e, \quad \xi_g = S(-t)e^{-i\Omega t/2}\psi_g. \]

Compare these with the functions \((\phi_e, \phi_g)\).

In the interaction frame, the equations read as follows:

\[
\begin{align*}
    i\partial_t \xi_e &= e^{i\Omega t}S(-t)(u + u^*)S(t)\xi_g, \\
    i\partial_t \xi_g &= e^{-i\Omega t}S(-t)(u + u^*)S(t)\xi_e,
\end{align*}
\]
Sketch of the proof

We set $\epsilon_e(t, x) = \xi_e - \phi_e, \epsilon_g(t, x) = \xi_g - \phi_g$, and we have to study

(with $f = u + u^*, f_{LD} = u_{LD} + u_{LD}^*$ with Lamb-Dicke approximation, and $f_{LE}$ Law-Eberly approximation)

\[
\begin{align*}
    i\partial_t \epsilon_e &= e^{i\Omega t} S(-t)f(t, x)S(t)\epsilon_g + e^{i\Omega t} S(-t)(f - f_{LD})S(t)\phi_g \\
    &\quad + \left( e^{i\Omega t} S(-t)f_{LD}S(t) - f_{LE}(t, x) \right)\phi_g, \quad (t, x) \in (0, T) \times \mathbb{R},
\end{align*}
\]

\[
\begin{align*}
    i\partial_t \epsilon_g &= e^{-i\Omega t} S(-t)f(t, x)S(t)\epsilon_e + e^{-i\Omega t} S(-t)(f - f_{LD})S(t)\phi_e \\
    &\quad + \left( e^{-i\Omega t} S(-t)f_{LD}S(t) - f_{LE}(t, x) \right)\phi_g, \quad (t, x) \in (0, T) \times \mathbb{R},
\end{align*}
\]

$\epsilon_e(0) = 0, \quad \epsilon_g(0) = 0.$

- blue: Error coming from the **Lamb-Dicke** approximation.
- red: Error coming from the **averaging** approximation.
Sketch of the proof

Can be put under the form

\[
\begin{align*}
  i\partial_t \epsilon_e &= e^{i\Omega t} S(-t)f(t, x)S(t)\epsilon_g + h_{LDe}(t, x) + \partial_t h_{me}(t, x), \\
  i\partial_t \epsilon_g &= e^{-i\Omega t} S(-t)f(t, x)S(t)\epsilon_e + h_{LDg}(t, x) + \partial_t h_{mg}(t, x).
\end{align*}
\]

with

\[
\begin{align*}
  h_{LDe}(t, x) &= e^{i\Omega t} S(-t)(f(t) - f_{LD}(t))S(t)\phi_g(t), \\
  h_{me}(t, x) &= \int_0^t \left( e^{i\Omega s} S(-s)f_{LD}(s, x)S(s) - f_{LE}(s, x)^* \right) \phi_g(s) \, ds, \ldots
\end{align*}
\]

We prove \( \|h_{LD}\|_0 \leq C\eta^2 K(M+1) \) and
\[
\|h_{me}\|_0 \leq C \frac{K}{\omega - K} (M+1)^{3/2}.
\]

Then, by energy techniques, \( \sup_{t \in [0, T]} \|(\epsilon_e(t), \epsilon_g(t))\|_{0 \times 0} \) is small for \( \eta \) small and \( \omega\eta/K \) large.
For $\delta > 0$, we set $M$ large enough so that $\text{dist}((\psi_0, \psi_0), V^2_M)$ and $\text{dist}((\psi_1, \psi_1), V^2_M)$ are small. We then look at $(\tilde{\psi}_0, \tilde{\psi}_0), (\tilde{\psi}_1, \tilde{\psi}_1)$ in $(V^2_M)^2$ of unit norm s.t.

$$\| (\psi_0, \psi_0) - (\tilde{\psi}_0, \tilde{\psi}_0) \|_{0 \times 0} \leq \frac{\delta}{3}, \quad \| (\psi_1, \psi_1) - (\tilde{\psi}_1, \tilde{\psi}_1) \|_{0 \times 0} \leq \frac{\delta}{3}.$$ 

We then apply the previous theorem to $(\tilde{\psi}_0, \tilde{\psi}_0), (\tilde{\psi}_1, \tilde{\psi}_1)$, with the parameters as in the previous theorem

$$\| (\psi_e(T), \psi_g(T)) - \beta(\psi_1, \psi_1) \|_{0 \times 0} \leq \delta.$$ 

Indeed, the truncature error stays constant.
Again, two main steps:

- In $V_M^2$ for $\| (\cdot, \cdot) \|_{k \times k}$
- For data in $\mathcal{D}(A^{k/2})$, a truncation argument with $M$ large enough.
Theorem

For \((\psi_e^0, \psi_g^0)\) and \((\psi_e^1, \psi_g^1)\) in \(V_M^2\), and the control constructed above:

\[
\forall \delta > 0, \exists \eta_k = \eta_k(\delta, M), \exists \rho_k = \rho_k(\delta, M), \text{ s.t. } \forall K, \forall (\omega, \Omega) \text{ with } 2\Omega \geq 3\omega \text{ and }
\]

\[
\eta \leq \eta_k, \quad TK = \frac{3\pi \sqrt{M}}{\eta}, \quad \frac{\omega \eta}{K} \geq \rho_k,
\]

the solution of the exact system (1) with initial data \((\psi_e^0, \psi_g^0)\) satisfies

\[
\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{k \times k} \leq \delta.
\]
Sketch of the proof

We do as before, except that we need stronger estimates on the error terms:

$$\|h_{LD}(t, x)\|_k \leq C_1(k) \eta^2 K (M + 1)^{(k+2)/2},$$

$$\|h_m(t, x)\|_k \leq C_2(k) \frac{K}{(\omega - K)} (M + 1)^{(k+3)/2}.$$

Yields the proof similarly by standard energy estimates. (induction on $k$).
Theorem

Let \((\psi_0^e, \psi_0^g)\) and \((\psi_1^e, \psi_1^g)\) in \(\mathcal{D}(A^{k/2})^2\) of unit \((L^2)^2\)-norm. Then \(\forall \delta > 0, \exists (\kappa, \eta_k, \rho_k)\), such that for \((\omega, \Omega)\) with \(2\Omega \geq 3\omega\) and

\[
\eta \leq \eta_k, \quad KT = \kappa / \eta, \quad \frac{\omega \eta}{K} \geq \rho_k,
\]

there exists a control \(u(t, x)\) as above, satisfying the additional constraints

\[
\sup\{|u_0(t)|, |u_r(t)|, |u_b(t)|\} \leq K.
\]

such that the solution \((\psi_e, \psi_g)\) of (1) with initial data \((\psi_0^e, \psi_0^g)\) satisfies, for some \(\beta \in \mathbb{C}\) of modulus 1,

\[
\left\| (\psi_e(T), \psi_g(T)) - \beta (\psi_1^e, \psi_1^g) \right\|_{k \times k} \leq \delta.
\]
Again, we do a truncation argument, but this is more subtle!

\[ i \partial_t \epsilon_e = e^{i \Omega t} S(-t) f(t, x) S(t) \epsilon_g, \quad i \partial_t \epsilon_g = e^{-i \Omega t} S(-t) f(t, x) S(t) \epsilon_e. \]

But the equation is not isometric on \( \mathcal{D}(A^{k/2})^2 \)!

One shall do a commutator estimate

\[ \langle f(t, x) \psi_1, A^k \psi_2 \rangle = \langle f(t, x) A^{k/2} \psi_1, A^{k/2} \psi_2 \rangle + \text{a reminder}. \]
To bound the reminder, we use

\[
\sup \left\{ \left\| \partial_x^i f \right\|_{L^\infty((0,T) \times \mathbb{R})} : i = 1, \ldots, k \right\} \leq C K \eta,
\]

But the time is given \( T \simeq \frac{\sqrt{M}}{K \eta} \).

We then obtain

\[
\sup_{t \in [0,T]} \left\| (\epsilon_e(t), \epsilon_g(t)) \right\|_{\ell \times \ell} \leq C \sqrt{M} \sup_{t \in [0,T]} \left\| (\epsilon_e(t), \epsilon_g(t)) \right\|_{(\ell-1) \times (\ell-1)}
\]

\[
+ \left\| (\epsilon_e(0), \epsilon_g(0)) \right\|_{\ell \times \ell}.
\]

Miracle!

\[
\sqrt{M} \sup_{t \in [0,T]} \left\| (\epsilon_e(t), \epsilon_g(t)) \right\|_{(\ell-1) \times (\ell-1)} \simeq \frac{\delta}{M(k-\ell)/2},
\]

\[
\left\| (\epsilon_e(0), \epsilon_g(0)) \right\|_{\ell \times \ell} \simeq \frac{\delta}{M(k-\ell)/2}.
\]
Conclusion

Constructive method in the limits

\[ \eta \ll 1, \quad KT = \aleph / \eta, \quad \frac{\omega \eta}{K} \gg 1, \]

based on the finite dimension.

Remark: We have estimates on \( \aleph \) when the initial and target states are in \( \mathcal{D}(A^{k/2})^2 \) and we control approximately in \( \mathcal{D}(A^{\ell/2})^2 \) pour \( \ell < k \).
Can we do better than Law Eberly on the simplified model? *For instance, we used only two controls.*

Problem in combinatorics and graph theory.

*Cf Brockett’s talk*

More intricate models? Two ions coupled through oscillations...
Can we do **local exact controllability**?

**Preliminary question**: Consider

\[ i\partial_t \psi = (-\partial_{xx} + x^2)\psi + f(t)\eta(x)\psi, \quad (t, x) \in (0, T) \times \mathbb{R}. \]

**Local exact controllability** of this equation around the ground state trajectory \( \exp(-i\lambda_0 t)\Psi_0 \)?

Can we choose a profile function \( \eta = \eta(x) \) such that any initial data \( \psi_0 \) near \( \Psi_0 \), there exists a real valued control function \( f = f(t) \), such that the solution \( \psi \) satisfies \( \psi(T) = \exp(-i\lambda_0 T)\psi_0 \)?
Thank you for the attention!

Based on