Optimal parametrizations of adiabatic paths

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Outline

- Optimal parameterization problem
- Announcement of Results
- Dephasing Lindbladians
- Adiabatic theorem
- Results

Adiabatic control problem

Evolution:

$$\dot{\rho}(t) = -i[H(u(t)), \rho(t)]; \quad \rho(0) = P(0)$$

where

$$H(u) = \sum e_i P_i, \quad P(u) := P_0(u) \quad e_0 < e_i$$

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Adiabatic control: $\dot{u} \to 0 \quad T \to \infty$ (observe that $\int_0^T \dot{u} = 1$)

Examples

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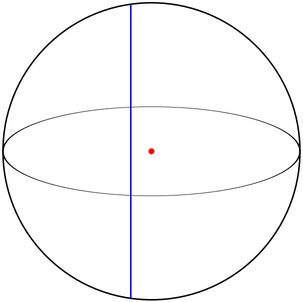

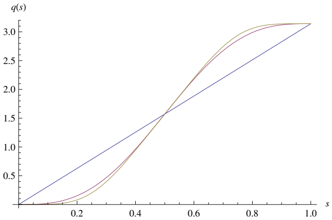

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Theorem (Adiabatic evolution)

Let $H(s) = \sum e_i(s)P_i(s)$ be smooth family of non-degenerate Hamiltonians. Then the above time-dependent Liouville equation has a solution

$$\rho(t) = P(u) + \dot{u}\mathcal{L}_u^{-1}P'(u) + O(\dot{u}^2).$$

Solutions are

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The idea: Add feedback to the model

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- No optimal parameterization for unitary evolutions
- Unique optimizer for dephasing Lindblad evolution
- The optimizer is "smooth"
- The optimizer is local
- Keep tunneling constant

Introduction to Lindblad

- Evolution of continuously observed system
- Observation is made at random times
- Distribution of observations is Poisson
- Entropy, purity, etc no more conserved

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- The rate of observation γ_α
- $\dot{\rho} = L\rho$ with

$$L = -i[H, \rho] - \frac{1}{2}\gamma_\alpha(\Gamma_\alpha^* \Gamma_\alpha \rho + \rho \Gamma_\alpha^* \Gamma_\alpha) + \gamma_\alpha \Gamma_\alpha \rho \Gamma_\alpha^*$$

Properties of Lindblad - Operational definition

Evolution generated by $\dot{\rho} = L\rho$

- Map states to states
- Preserve trace
- Is Markovian; semigroup; independent of history
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Reverse holds true.

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- $\Gamma = \Gamma(H)$, e.g. $\Gamma_i = P_i$
- $\mathrm{Ker}(L) = \mathrm{Ker}([H, \cdot])$

$$L\rho = -i[H, \rho] + \frac{\gamma}{2}(P_i\rho + \rho P_i) + \gamma P_i \rho P_i$$

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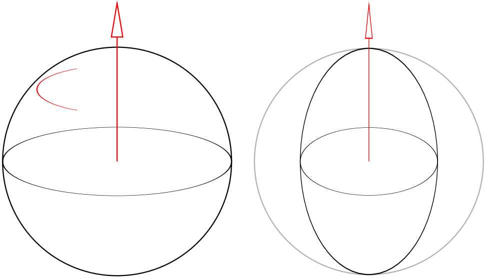
- Energy is conserved, Entropy increase without heat transfer, Phase information is lost

Example of dephasing Lindbladian

For a qubit

$$H = g \cdot \sigma = e_+ P_+ + e_- P_-$$
$$L\rho = -i[H, \rho] + \gamma(P_+ \rho P_- + P_- \rho P_+)$$



Adiabatic theorem revisited

Consider equation $\dot{\rho}(t) = \mathcal{L}(u)\rho,\ \rho(0) = P(0)$

$$\mathcal{L}\rho = -i[H,\ \rho] + \frac{\gamma}{2}(P_i\rho + \rho P_i) + \gamma P_i \rho P_i$$

Theorem (Adiabatic evolution for dephasing)

Let $H(s) = \sum e_i(s)P_i(s)$ be smooth family of non-degenerate Hamiltonians. Then the above time-dependent Lindblad equation has a solution

$$\rho(t) = P(u) + \dot{u}\mathcal{L}_u^{-1}P'(u) + \sum_{j\neq 0}(P_j(u) - P(u)) \int_0^T \mathrm{Tr}(P_j'(u)\mathcal{L}_u^{-1}P'(u))\dot{u}^2(\tau)\mathrm{d}\tau + O(\dot{u}^2).$$

Cost function

The cost function $J(u) := 1 - \mathrm{Tr}(\rho(T)P(1))$ is

$$J(u) = 2\gamma \sum_{i \neq 0} \int_0^T \frac{\mathrm{Tr}(P_i(P_0')^2)}{|e_i - e_0|^2 + \gamma^2} \dot{u}(\tau)^2 \mathrm{d}\tau + O(\dot{u}^2)$$

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- Irreversible
- Closed system $\gamma = 0$, $J(u) = O(\dot{u}^2)$
- $J(u)$ is quadratic in derivative
- Optimization is an Euler-Lagrange problem

Optimal speed

$$\dot{u} \sim \sum_{i \neq 0} \frac{\sqrt{g_i^2 + \gamma^2}}{\kappa_i}$$

where

$$\kappa_i^2 := \mathrm{Tr}(P_i \dot{P}_0^2),\ g_i := |e_i - e_0|$$

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- Local
- Gives algorithm for optimization
- Move slowly when gap is small
- Move inversely propotional to length on control - Fubini-Study metric

Conclusions

- Optimization of control is regularized by observing the system
- Framework for adiabatic control with feedback?