Bilinear control of nonlinear Schrödinger and wave equation

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Quantum Control

Bilinear control

Model system

$$i\partial_t \psi(t,x) + \partial_x^2 \psi(t,x) = -u(t)\mu(x)\psi(t,x). \quad (1)$$

u (the control) and μ the real valued potential .
So, at each time t, the available control $u(t)$ is only the amplitude and not a distributed fonction.

Bilinear control

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Aim : local control by perturbation

Other results : nonlinear Schrödinger and nonlinear wave equation

Bibliography

Exact controllability

- **Negative result** : Ball-Marsden-Slemrod (82)
- **Positive result** : Local exact controllability in 1D : in H^7, in large time Beauchard (05), Coron(06) : $T_{min} > 0$,
  controllability in 1D between eigenstates : Beauchard and Coron (06)

Bibliography

Approximate controllability

- **By Gallerkin approximation and finite dimensional methods** Chambrion-Mason-Sigalotti-Boscain(09)
- **By stabilization** Nersesyan (09)
- **Exact controllability "at $T = \infty$"** Nersesyan-Nersisyan (10)

First obstruction Ball-Marsden-Slemrod

Theorem (Ball-Marsden-Slemrod 82)

If the multiplication by μ is bounded on the functional space X, then the set of reachable states is a countable union of compact sets of X
$\Rightarrow$ *no controllability in X.*

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Once the functional space X is chosen, we must chose a potentiel μ enough regular to be able to do a perturbation theory, but not too much otherwise Ball-Marsden-Slemrod applies.

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First solution given by K. Beauchard : use of Nash-Moser theorem.
Improved method (with K. Beauchard) : prove directly that the system can be well posed even if the potential is "bad" $\Rightarrow$ optimal with respect to regularity and time of control ; easier proof that can be extended to other cases.

Main results

Denote φ_k the eigenfunctions of the Dirichlet Laplacian operator.
We control near the ground eigenstate φ_1 with solution $\psi_1(t) = e^{-i\lambda_1 t}\varphi_1$.
$\mathcal{S}$ is the unit sphere of $L^2(]0,1[_x)$.

Theorem (with K. Beauchard)

Let $T > 0$ and $\mu \in H^3(]0,1[,\mathbb{R})$ be such that

$$\exists c > 0 \text{ such that } \frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|, \forall k \in \mathbb{N}^*. \tag{2}$$

There exists $\delta > 0$ such that for any $\psi_f \in \mathcal{S} \cap H^3_{(0)}(]0,1[,\mathbb{C})$ with $\|\psi_f - \psi_1(T)\|_{H^3} < \delta$ there exists a control $u \in L^2(]0,T[,\mathbb{R})$ s.t. the solution of (1) with initial condition

$$\psi(0) = \varphi_1$$

and control u satisfies $\psi(T) = \psi_f$.

Remarks about assumption (2)

$$\langle \mu\varphi_1, \varphi_k \rangle_{L^2_x} = \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu\varphi_1)'''(x)\cos(k\pi x)$$
$$= \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} + \frac{\ell^2 sequence}{k^3}.$$

and we can prove that assumption (2) is generic in $H^3(]0,1[)$.

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&= \frac{4[(-1)^{k+1}\mu'(1) - \mu'(0)]}{k^3\pi^2} + \frac{\ell^2 sequence}{k^3}.
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Such assumption implies that multiplication by μ does not map $H^3_{(0)}$ into itself.

Rk : there are some cases where assumption (2) is not fufilled but Beauchard and Coron manage to prove the controllability with additional techniques : return method or power series expansions.

"Regularizing" effect

$$
\begin{aligned}
H^3_{(0)} &= D\left((-\Delta_{Dirichlet})^{3/2}\right) \\
&= \left\{ u \in H^3 \,\middle|\, u(0) = u(1) = 0 = u''(0) = u''(1) \right\}
\end{aligned}
$$

Proposition (with K. Beauchard)

Let $f \in L^2((0,T), H^3 \cap H^1_0)$ (not necessarily $H^3_{(0)}$). Then, the solution ψ of

$$
\begin{cases}
i\partial_t \psi(t,x) + \partial_x^2 \psi(t,x) &= f \\
\Psi(0) &= 0
\end{cases}
$$

belongs to $C^0([0,T], H^3_{(0)})$

Method of proof

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Rk : In certain cases treated by Beauchard and Coron, we can get controllability even if the linearized system is not controllable (use return method and quasi-static transformation or expansion to higher order). Our result should improve the regularity in these results.

Controllability of the linearized system

We linearize around the trajectory $\psi_1(t,x) = e^{-i\lambda_1 t}\varphi_1$.

$$\begin{cases} i\partial_t \Psi(t,x) + \partial_x^2 \Psi(t,x) & = & -v(t)\mu(x)\psi_1(t,x) \\ \psi(0,x) & = & 0. \end{cases}$$

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$$\Psi(T) = \sum_{k=1}^{\infty} i\langle \mu\varphi_1, \varphi_k\rangle \left(\int_0^T v(t) e^{i(\lambda_k - \lambda_1)t} dt\right) e^{-i\lambda_k T}\varphi_k.$$

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$\Psi(T) = \Psi_f$ is equivalent to the trigonometric moment problem

$$\int_0^T v(t) e^{i(\lambda_k-\lambda_1)t} dt = d_{k-1}(\Psi_f) := \frac{\langle \Psi_f, \varphi_k\rangle e^{i\lambda_k T}}{i\langle \mu\varphi_1, \varphi_k\rangle}, \forall k \in \mathbb{N}^*. \qquad (3)$$

By Ingham theorem : $\forall T > 0; \Psi_f \in H^3_{(0)}(]0,1[$ there exists one $v \in L^2(]0,T[)$ solution. (if $T = 2/\pi$, it is only Fourier series in time)

Ingham Theorem

Theorem (Ingham, Haraux)

Let $N \in \mathbb{N}$, $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence of real numbers such that

$$\omega_{k+1} - \omega_k \geqslant \gamma > 0, \forall k \in \mathbb{Z}, |k| \geqslant N,$$

$$\omega_{k+1} - \omega_k \geqslant \rho > 0, \forall k \in \mathbb{Z},$$

and $T > 2\pi/\gamma$. The map

$$\begin{array}{llcc} J: & F := Clos_{L^2(]0,T[)}(Span\{e^{i\omega_k t}; k \in \mathbb{Z}\}) & \to & l^2(\mathbb{Z},\mathbb{C}) \\ & v & \mapsto & \left(\int_0^T v(t)e^{i\omega_k t}dt\right)_{k\in\mathbb{Z}} \end{array}$$

is an isomorphism.

This is a kind of Fourier decomposition for "not exactly orthogonal basis" (Riesz basis).

Proof of the "regularizing" effect

$$\int_0^t e^{-i\partial_x^2 s} f(s)ds = \sum_{k=1}^{\infty} \left(\int_0^t \langle f(s), \varphi_k \rangle_{L^2_x} e^{i\lambda_k s} ds \right) \varphi_k = \sum_{k=1}^{\infty} x_k(t)\varphi_k.$$

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$$\begin{aligned}
\langle f(s), \varphi_k \rangle_{L_x^2} &= \int_0^1 f(s,x)\sin(k\pi x)dx \\
&= -\frac{1}{(k\pi)^2} \int_0^1 f''(s,x)\sin(k\pi x)dx \\
&= \frac{1}{(k\pi)^3} \left((-1)^k f''(s,1) - f''(s,0) \right) \\
&\quad - \frac{1}{(k\pi)^3} \int_0^1 f'''(s,x)\cos(k\pi x)dx.
\end{aligned}$$

Proof of the "regularizing" effect

$$\begin{aligned}
\|x_k(t)\|_{h^3}^2 &\lesssim C\sum_{k=1}^{\infty}\left|\int_0^t f''(s,1)e^{i\lambda_k s}ds\right|^2 + idem \\
&\quad + \sum_{k=1}^{\infty}\left|\int_0^t\int_0^1 f'''(s,x)\cos(k\pi x)e^{i\lambda_k s}dxds\right|^2 \\
&\lesssim C\left\|f''(.,1)\right\|_{L^2(]0,2/\pi[)} + idem + t\left\|f'''\right\|_{L^2([0,T],L^2)}
\end{aligned}$$

from Plancherel (in time) formula on $]0,2/\pi[$ (first estimate) and Cauchy Schwartz (second estimate).

Other results

The method is quite robust and can be applied to other problems :

- Nonlinear Schödinger equation near constant in space solution
- Linear and nonlinear wave equation near constant solution

Control smoother data with smoother control

Theorem (with K. Beauchard)

Let $T > 0$ *and* $\mu \in H^5(]0,1[,\mathbb{R})$ *satisfying (2) There exists* $\delta > 0$ *such that for any* $\psi_f \in \mathcal{S} \cap H^5_{(0)}(]0,1[,\mathbb{C})$ *with* $\|\psi_f - \psi_1(T)\|_{H^5} < \delta$ *there exists a control* $u \in H^1_0(]0,T[,\mathbb{R})$ *s.t. the solution of (1) with initial condition*

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Rq : Actually, we prove that the solution fulfills $\partial_x^2\psi + u(t)\mu\psi \in C^0([0,T], H^3_{(0)})$. Therefore, $\psi(t)$ does not, in general, belong to $H^5_{(0)}(]0,1[)$ for $t \in (0,T)$ (OK if $u(t) = 0$).

3D ball with radial data

We prove similar results for the linear Schrödinger equation on the 3D ball with radial data : same eigenvalues and behavior is "one dimensional".

Control of nonlinear Schrödinger equation

Nonlinear Schrödinger equation on $]0,1[$ with Neumann boundary conditions

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) + |\psi|^2\psi(t,x) - u(t)\mu(x)\psi(t,x) \\ \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0. \end{cases} \tag{4}$$

We control around the trajectory $\psi(t) = e^{-it}$

Theorem (with K. Beauchard)

Let $T > 0$ and $\mu \in H^2(0,1)$ be such that

$$\exists c > 0 \text{ such that } \left| \int_0^1 \mu(x)\cos(k\pi x)dx \right| \geqslant \frac{c}{\max\{1,k\}^2}, \forall k \in \mathbb{N}. \tag{5}$$

There exists $\delta > 0$ such that for any $\psi_f \in \mathcal{S} \cap H^2_{(0,N)}(]0,1[,\mathbb{C})$ with $\|\psi_f - e^{-iT}\|_{H^2} < \delta$ there exists a control $u \in L^2(]0,T[,\mathbb{R})$ s.t. the solution of (4) with initial condition $\psi(0) = \varphi_1$ and control u satisfies $\psi(T) = \psi_f$.

Nonlinear wave equations

Nonlinear wave equation on $]0,1[$ with Neumann boundary conditions

$$\begin{cases} w_{tt} = w_{xx} + f(w, w_t) + u(t)\mu(x)(w + w_t) \\ w_x(t,0) = w_x(t,1) = 0, \end{cases} \tag{6}$$

We assume $f \in C^3(\mathbb{R}^2, \mathbb{R})$ such that $f(1,0) = 0$ (the constant $w \equiv 1$ is solution) and $\nabla f(1,0) = 0$ (the linearized around 1 is the linear wave equation).

Theorem

Let $T > 2$, $\mu \in H^2((0,1),\mathbb{R})$ be such that (5) holds There exists $\delta > 0$ such that for any $(w_f, \dot{w}_f) \in H^3_{(0,N)} \times H^2_{(0,N)}(]0,1[,\mathbb{R})$ with $\|w_f - 1\|_{H^3} + \|\dot{w}_f\|_{H^2} < \eta$ there exists a control $u \in L^2(]0,T[,\mathbb{R})$ s.t. the solution of (6) with initial data $(w, w_t)(0,x) = (1,0)$ and control u satisfies $(w, w_t)(T) = (w_f, \dot{w}_f)$.

Further problems

- Higher dimensions : but the spectral gap used to apply Ingham theorem is no more guarranted.
- May be some negative results more precise than Ball-Marsden-Slemrod using microlocal analysis

THANK YOU FOR YOUR ATTENTION ! ! ! ! !