Feedback generation of quantum entangled states

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Presentation outline

1. Feedback stabilization of single spin systems
2. Entanglement
3. Symmetric and anti-symmetric entangled states
4. General entangled states

Experience scheme

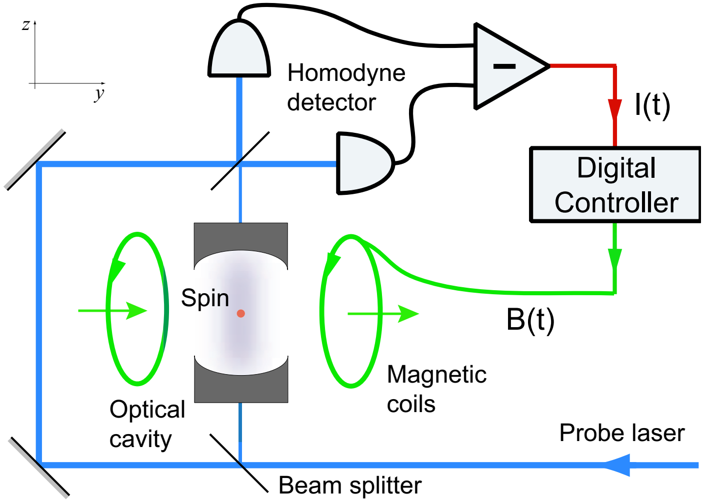


Figure: R. van Handel, J.K. Stockton, H. Mabuchi, IEEE Trans-AC, 2005

Quantum filter equation

State space: $\{\rho \in \mathbb{C}^{2\times 2} \mid \rho \geq 0,\ \text{Hermitian}\ , \mathrm{Tr}\,(\rho) = 1\}$.

$$
\begin{aligned}
d\rho_t = -iu_t[\sigma_y, \rho_t]dt + \frac{1}{2}(2\sigma_z\rho_t\sigma_z - \sigma_z^2\rho_t - \rho_t\sigma_z^2)dt \\
+(\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}\,(\sigma_z\rho_t)\,\rho_t)dW_t, \\
dW_t = dy_t - 2\,\mathrm{Tr}\,(\sigma_z\rho_t)\,dt
\end{aligned}
$$

where

$$
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

Stabilization problem

$$d\rho_t = \frac{1}{2}(2\sigma_z\rho_t\sigma_z - \sigma_z^2\rho_t - \rho_t\sigma_z^2)dt + (\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}\left(\sigma_z\rho_t\right)\rho_t)dW_t,$$

Equilibrium states

$$\rho_\uparrow := \Pi_{|\uparrow\rangle} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_\downarrow := \Pi_{|\downarrow\rangle} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem

To stabilize deterministically one of these states.

Stochastic Lyapunov theory

Stochastic system:

$$dx_t = f(x_t)dt + \sigma(x_t)dw_t, \quad x(t=0) = x_0,$$

Lyapunov funciton:

$$\frac{d}{dt}\mathbb{E}V(x_t) = \mathcal{L}V(x_t) = \frac{\partial V}{\partial x}(x_t)f(x_t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(x_t)\sigma(x_t)^2 \leq 0.$$

1. Doob's inequality: $\mathbb{P}\left(\sup_{0\leq t<\infty} V(x_t) \geq \alpha\right) \leq \frac{V(x_0)}{\alpha}$.
2. Convergence in probability towards the invariant set included in $\mathcal{L}V = 0$ (Kushner's theorem).

Spin-1/2 case

$$d\rho_t = -iu_t[\sigma_y, \rho_t]dt + \frac{1}{2}(2\sigma_z\rho_t\sigma_z - \sigma_z^2\rho_t - \rho_t\sigma_z^2)dt$$
$$+ (\sigma_z\rho_t + \rho_t\sigma_z - 2\mathrm{Tr}\left(\sigma_z\rho_t\right)\rho_t)dW_t.$$

Lyapunov function:

$$V(\rho_t) = 1 - \mathrm{Tr}\left(\rho_t\rho_\uparrow\right)^2$$

$$\mathcal{L}V(\rho_t) = 2u_t\mathrm{Tr}\left(i[\sigma_y, \rho_t]\rho_\uparrow\right)\} - 4\mathrm{Tr}\left(\rho_t\rho_\uparrow\right)^2\left(1 - \mathrm{Tr}\left(\sigma_z\rho_t\right)\right)^2.$$

A possible approach:

$$u_t = -\mathrm{Tr}\left(i[\sigma_y, \rho_t]\rho_\uparrow\right)$$

Spin-1/2 case (continued)

Invariant set

ρ_t converges in probability towards the set $\{\rho_\uparrow, \rho_\downarrow\}$. The Lyapunov function V takes it maximal value on $\rho_\downarrow$: $V(\rho_\downarrow) = 1 - \text{Tr}\,(\rho_\uparrow \rho_\downarrow)^2 = 1$.

Contrarily to the deterministic case, we do not have semi-global stabilization.

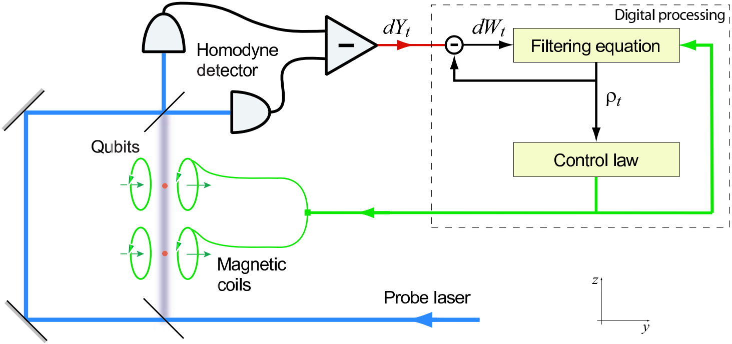
Change of strategy

1. $u_t = -\text{Tr}\left(i[\sigma_y, \rho_t]\rho_\uparrow\right)$ if $\text{Tr}\,(\rho_t \rho_\uparrow) \geq \gamma$;
2. $u_t = 1$ if $\text{Tr}\,(\rho_t \rho_\uparrow) \leq \gamma/2$;
3. if $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \text{Tr}\,(\rho\rho_\uparrow) < \gamma\}$, then $u_t = -\text{Tr}\left(i[\sigma_y, \rho_t]\rho_\uparrow\right)$ if the last entry of ρ_t into $\mathcal{B}$ has been via the boundary $\text{Tr}\,(\rho\rho_\uparrow) = \gamma$, and $u_t = 1$ if not.

M. Mirrahimi and R. van Handel, SIAM J. Cont. Optimization, 2007.

Presentation outline

Entangled states: experience scheme



Example: two qubits

The Hilbert state space for two qubits is given by $\mathbb{C}^2 \otimes \mathbb{C}^2$. One can think of an entangled state of the form

$$\Pi_{\frac{1}{\sqrt{2}}(|\uparrow\rangle\otimes|\downarrow\rangle+|\downarrow\rangle\otimes|\uparrow\rangle)}.$$

Quantum filter equation

$$d\rho_t = -iu_t^1[F_y^{(1)}, \rho_t]dt - iu_t^2[F_y^{(2)}, \rho_t]dt + \frac{1}{2}(2F_z\rho_t F_z - F_z^2\rho_t - \rho_t F_z^2)dt$$
$$+(F_z\rho_t + \rho_t F_z - 2\mathrm{Tr}\,(F_z\rho_t)\,\rho_t)dW_t,$$
$$dW_t = dy_t - 2\mathrm{Tr}\,(F_z\rho_t)\,dt$$

2**-qubit system:**

- The Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ of dimension 4,
- Angular momentum operators

$$F_y^{(1)} = \sigma_y \otimes \mathrm{Id},$$
$$F_y^{(2)} = \mathrm{Id} \otimes \sigma_y,$$
$$F_z = \sigma_z \otimes \mathrm{Id} + \mathrm{Id} \otimes \sigma_z.$$

Stabilization problem

$$d\rho_t = \frac{1}{2}(2F_z\rho_t F_z - F_z^2\rho_t - \rho_t F_z^2)dt + (F_z\rho_t + \rho_t F_z - 2\mathrm{Tr}\left(F_z\rho_t\right)\rho_t)dW_t,$$

Equilibrium states

$$\Pi_{|\uparrow\uparrow\rangle}, \qquad \Pi_{|\downarrow\downarrow\rangle}, \qquad \Pi_{\alpha|\uparrow\downarrow\rangle+\beta|\downarrow\uparrow\rangle} \quad (|\alpha|^2 + |\beta^2| = 1).$$

Main obstacle

Degeneracy of the measurement operator at the target state: taking the Lyapunov strategy as in the previous case, ρ_t can converge towards a state in the set $\{\Pi_{|\uparrow\uparrow\rangle}, \Pi_{|\downarrow\downarrow\rangle}, \Pi_{\alpha|\uparrow\downarrow\rangle+\beta|\downarrow\uparrow\rangle}\}$.

Presentation outline

Symmetric and anti-symmetric entangled states

Symmetric entangled state:

$$\rho_S = \Pi_{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}$$

Idea

We consider a dynamic stabilization by considering: $u_t^1 = 1 + v_t^1$ and $u_t^2 = -1 + v_t^2$. Therefore, the uncontrolled system writes:

$$d\rho_t = -i[F_y^{(1)} - F_y^{(2)}, \rho_t]dt$$
$$+ \frac{1}{2}(2F_z\rho_t F_z - F_z^2\rho_t - \rho_t F_z^2)dt + (F_z\rho_t + \rho_t F_z - 2\mathrm{Tr}\left(F_z\rho_t\right)\rho_t)dW_t,$$

The only equilibrium state: $\Pi_{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}$.

Symmetric and anti-symmetric entangled states

Symmetric case

The feedback law

1. $u_t^1 = 1 - \mathrm{Tr}\left(i[F_y^{(1)}, \rho_t]\rho_S\right), \; u_t^2 = -1 - \mathrm{Tr}\left(i[F_y^{(2)}, \rho_t]\rho_S\right)$ if $\mathrm{Tr}\left(\rho\rho_S\right) \geq \gamma$;
2. $u_t^1 = 1, \; u_t^2 = 0$ if $\mathrm{Tr}\left(\rho\rho_S\right) \leq \gamma$;

stabilize the symmetric entangled state $\rho_S = \Pi_{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}$.

Anti-symmetric case

The feedback law

1. $u_t^1 = 1 - \mathrm{Tr}\left(i[F_y^{(1)}, \rho_t]\rho_{AS}\right), \; u_t^2 = 1 - \mathrm{Tr}\left(i[F_y^{(2)}, \rho_t]\rho_{AS}\right)$ if $\mathrm{Tr}\left(\rho\rho_{AS}\right) \geq \gamma$;
2. $u_t^1 = 1, \; u_t^2 = 0$ if $\mathrm{Tr}\left(\rho\rho_{AS}\right) \leq \gamma$;

stabilize the anti-symmetric entangled state $\rho_{AS} = \Pi_{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)}$.

Presentation outline

General entangled states

$$d\rho_t = -iu_t^1[F_y^{(1)}, \rho_t]dt - iu_t^2[F_y^{(2)}, \rho_t]dt + \frac{1}{2}(2F_z\rho_t F_z - F_z^2\rho_t - \rho_t F_z^2)dt$$
$$+(F_z\rho_t + \rho_t F_z - 2\mathrm{Tr}\,(F_z\rho_t)\,\rho_t)dW_t,$$
$$dW_t = dy_t - 2\mathrm{Tr}\,(F_z\rho_t)\,dt$$

Target state: $\Pi_{\alpha|\uparrow\downarrow\rangle+\beta|\downarrow\uparrow\rangle}$.

Obstacle

We can not remove the degeneracy exactly at our target.

Here we propose a controllability-type result.

General entangled states

A simplification

If the system starts at ρ_0 of the form $\Pi_{\psi_0} = \psi_0\psi_0^*$ with $\psi_0 \in \mathbb{C}^2$, the above dynamics become equivalent to:

$$d\psi_t = -i(u_t^1 F_y^{(1)} + u_t^2 F_y^{(2)})\psi_t dt - \frac{1}{2}F_z^2\psi_t dt + F_z\psi_t dW_t,$$

$$\rho_t = \frac{\psi_t\psi_t^*}{\|\psi_t\|^2}.$$

Idea

By considering exciting control fields, we remove all equilibriums except a small neighborhood of the target state and we prove that, the trajectories, almost surely, hit this small neighborhood in finite time.

Support theorem

Strook-Varadhan support theorem

Consider the Ito SDE,

$$dx_t = f(x_t)dt + \sigma(x_t)dW_t,$$

and the associated deterministic controlled equation

$$\frac{d}{dt}x_t^u = f(x_t^u) - \frac{1}{2}\nabla\sigma(x_t^u)x_t^u + u(t)\sigma(x_t^u).$$

Consider $\mathcal{U}$ the set of all piecewise constant functions from $\mathbb{R}_+$ to $\mathbb{R}$, and define

$$\mathcal{S}_x = \overline{\{x_.^u \ : \ u \in \mathcal{U}\}}$$

the set of all controlled trajectories starting at x. The set $\mathcal{S}_x$ is the smallest set of the continuous trajectories starting at x such that

$$\mathbb{P}(\{\omega \in \Omega \mid x_.(\omega) \in \mathcal{S}_x\}) = 1.$$

Support theorem: application

$$d\psi_t = -i(u_t^1 F_y^{(1)} + u_t^2 F_y^{(2)})\psi_t dt - \frac{1}{2} F_z^2 \psi_t dt + F_z \psi_t dW_t,$$
$$\rho_t = \frac{\psi_t \psi_t^*}{\|\psi_t\|^2}.$$

We consider the controlled deterministic equation:

$$d\psi_t = -i(u_t^1 F_y^{(1)} + u_t^2 F_y^{(2)})\psi_t dt - F_z^2 \psi_t dt + U_t F_z \psi_t,$$
$$\rho_t = \frac{\psi_t \psi_t^*}{\|\psi_t\|^2}.$$

We show that by taking ($|\beta_1| \neq |\beta_2|$, $\left|\frac{\beta_1\beta_2(\alpha_1^2-\alpha_2^2)}{\alpha_1\alpha_2(\beta_1^2-\beta_2^2)}\right| \leq \frac{1}{2}$, $\alpha_1\alpha_2\beta_1\beta_2 < 0$)

$$u_t^1 = \epsilon\beta_1, \qquad u_t^2 = \epsilon\beta_2, \qquad U_t = 2\frac{\beta_1\beta_2(\alpha_1^2 - \alpha_2^2)}{\alpha_1\alpha_2(\beta_1^2 - \beta_2^2)},$$

the controlled system converges towards a state in an ϵ-neighborhood of the target state $\Pi_{\alpha_1|\uparrow\downarrow\rangle + \alpha_2|\downarrow\uparrow\rangle}$.

Stabilization

Theorem

The following feedback laws ensure the stabilization in an ϵ-neighborhood of the target state $\Pi_{\alpha|\uparrow\downarrow\rangle+\beta|\downarrow\uparrow\rangle}$:

$$u_t^{(k)} = \begin{cases} -c_k \mathrm{Tr}[i[F_y^{(k)}, \rho_t]\rho_f] & \text{for } V(\rho_t) \le \varepsilon \\ \epsilon\beta_k & \text{for } \varepsilon < V(\rho_t) \le 1-\delta \\ \gamma_k & \text{for } V(\rho_t) > 1-\delta \end{cases} \qquad k = 1, 2.$$

Where $\epsilon, c_1, c_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$, $c_k > 0$, $\epsilon > 0$ sufficiently small and

$$|\beta_1| \neq |\beta_2|, \quad \left|\frac{\beta_1\beta_2(\alpha_1^2 - \alpha_2^2)}{\alpha_1\alpha_2(\beta_1^2 - \beta_2^2)}\right| \le 1/2, \quad \alpha_1\alpha_2\beta_1\beta_2 < 0, \quad (\gamma_1, \gamma_2) \neq (\epsilon\beta_1, \epsilon\beta_2).$$

10 Random simulations of the 2-qubit system

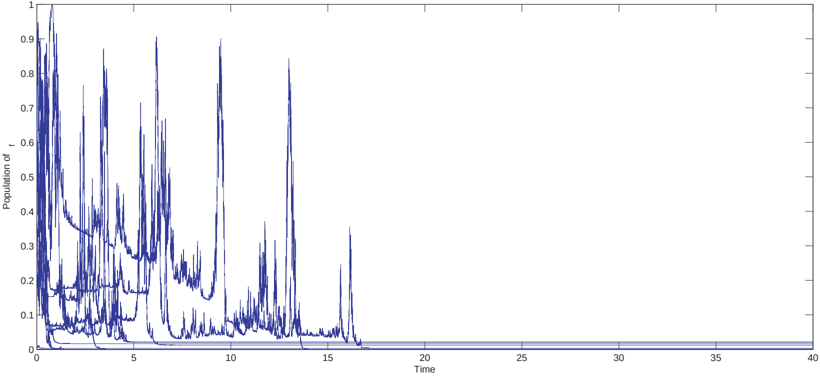


Figure: Approximate stabilization of the .03-neighborhood of $\frac{1}{\sqrt{5}}\left|\uparrow\downarrow\right\rangle + \frac{2}{\sqrt{5}}\left|\downarrow\uparrow\right\rangle$

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