Global exact controllability in infnite time of Schrödinger equation

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IHP, December 9, 2010

Introduction

V. N., H. Nersisyan, Global exact controllability in infinite time of Schrödinger equation, arXiv:1006.2602, 2010.

Introduction

Controlled Schrödinger equation:

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\begin{aligned}
i\dot{z} &= -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D,\\
z|_{\partial D} &= 0,\\
z(0,x) &= z_0(x),
\end{aligned}
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where $D \Subset \mathbb{R}^d$, $\partial D \in C^\infty$, $d \geq 1$, $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ are given functions, u is the control, z is the state.

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Let $\mathcal{U}_t(\cdot, u) : L^2 \to L^2$, $u \in L^1_{loc}([0,\infty), \mathbb{R})$ be the resolving operator, i.e. $\mathcal{U}_t(z_0, u) = z(t)$.

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$$
\|\mathcal{U}_t(z_0, u)\|_{L^2} = \|z_0\|_{L^2}, \quad t \geq 0.
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Let $S := \{z \in L^2 : \|z\|_{L^2} = 1\}$.

Introduction

Main result

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For any $z_0, z_1 \in S \cap H^k$ there is a control $u \in H^s(\mathbb{R}_+, \mathbb{R})$ and a sequence $T_n \to +\infty$ such $\mathcal{U}_{T_n}(z_0, u) \rightharpoonup z_1$ in H^k.

Plan of the talk

1. Non-controllability results
2. Controllability of linearized system
3. Controllability of nonlinear system

Previous results

Ramakrishna, Salapaka, Dahleh, Rabitz, Pierce, Turinici, Altafini, Albertini, D'Alessandro, ...

Beauchard, Coron, Laurent

Chambrion, Mason, Sigalotti, Boscain

Mirrahimi, Beauchard, V.N.

V.N.

Non-controllability result

Non-controllability

Theorem (Ball, Marsden, Slemrod, 82)

The Schrödinger equation is not exactly controllable in finite time in Sobolev space H^2 with controls $L^p_{loc}([0,+\infty),\mathbb{R})$, i.e., for any $z_0 \in S$ the set

$$\{\mathcal{U}_t(z_0,u) : t \in [0,+\infty), u \in L^p_{loc}([0,+\infty),\mathbb{R}) \text{ for some } p > 1\}$$

does not contain a ball of the space H^2.

Non-controllability

Theorem

The Schrödinger equation is not exactly controllable in finite time in Sobolev spaces $H^k, k < d$ with controls $H^1_{loc}([0, +\infty), \mathbb{R})$, i.e., for any $z_0 \in S$ the set

$$\{\mathcal{U}_t(z_0, u) : t \in [0, +\infty), u \in H^1_{loc}([0, +\infty), \mathbb{R})\}$$

does not contain a ball of the space H^k.

Proof is based on the ideas of Shirikyan introduced to prove non-controllability of Euler equation.

Controllability of linearized system

Previous results

Let us linearize the system around trajectory $\mathcal{U}_t(\tilde{z}_0, 0)$:

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\begin{aligned}
i\dot{z} &= -\Delta z + V(x)z + u(t)Q(x)\mathcal{U}_t(\tilde{z}_0, 0),\\
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Beauchard, Chitour, Kateb, Long

Preliminaries

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$$

Let us rewrite this problem in the Duhamel form

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z(t) = -i\int_0^t e^{i(t-s)(\Delta - V)}u(s)Q(x)\mathcal{U}_s(\tilde{z}_0, 0)\mathrm{d}s.
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$$z(t) = -i \int_0^t e^{i(t-s)(\Delta - V)} u(s)Q(x)\mathcal{U}_s(\tilde{z}_0, 0)ds.$$

Let $\mathcal{R}_t$ be the resolving operator.

Preliminaries

$$\langle \mathcal{R}_t(0,u), e_m \rangle = -i \sum_{k=1}^{+\infty} e^{-i\lambda_m t} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^t e^{i\omega_{mk} s} u(s) \mathrm{d}s, \; m \geq 1,$$

where $\omega_{mk} = \lambda_m - \lambda_k$ and $Q_{mk} := \langle Q e_m, e_k \rangle$.

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where $\omega_{mk} = \lambda_m - \lambda_k$ and $Q_{mk} := \langle Qe_m, e_k\rangle$. For any $u \in L^1(\mathbb{R}_+, \mathbb{R})$ the following limit exists

$$\mathcal{R}_\infty(0,u) := \lim_{n\to+\infty} \mathcal{R}_{T_n}(0,u).$$

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where $\omega_{mk}=\lambda_m-\lambda_k$ and $Q_{mk}:=\langle Qe_m, e_k\rangle$. For any $u\in L^1(\mathbb{R}_+,\mathbb{R})$ the following limit exists

$$\mathcal{R}_\infty(0,u):=\lim_{n\to+\infty}\mathcal{R}_{T_n}(0,u).$$

The choice of the sequence T_n implies that

$$\langle \mathcal{R}_\infty(0,u), e_m\rangle = -i\sum_{k=1}^{+\infty}\langle \tilde{z}_0, e_k\rangle Q_{mk}\int_0^{+\infty} e^{i\omega_{mk}s}u(s)\mathrm{d}s.$$

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Preliminaries

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$$\check{u}(\omega) := \int_0^{+\infty} e^{i\omega s} u(s) ds.$$

The set of admissible controls is the Banach space

$$\Theta := u \in L^1(\mathbb{R}_+, \mathbb{R}) \cap H^s(\mathbb{R}_+, \mathbb{R}) \cap \mathcal{C}$$

where $s \geq 1$ is any fixed constant and

$$\mathcal{C} := \{u \in L^1(\mathbb{R}_+, \mathbb{R}) : \{\check{u}(\omega_{mk})\} \in \ell^2\}.$$

Preliminaries

Condition 1

Let $V(x_1, \ldots, x_d) = V_1(x_1) + \ldots + V_d(x_d)$ *and* $D \subset \mathbb{R}^d$ *is a rectangle. The functions* $V, Q \in C^\infty(\overline{D}, \mathbb{R})$ *are such that*

(i) $\inf_{p_1, j_1, \ldots, p_d, j_d \geq 1} |(p_1 j_1 \cdot \ldots \cdot p_d j_d)^3 Q_{pj}| > 0, Q_{pj} := \langle Q e_{p_1, \ldots, p_d}, e_{j_1, \ldots, j_d} \rangle$,

(ii) $\lambda_i - \lambda_j \neq \lambda_p - \lambda_q$ *for all* $i, j, p, q \geq 1$ *such that* $\{i, j\} \neq \{p, q\}$ *and* $i \neq j$.

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(i) $\inf_{p_1,j_1,\dots,p_d,j_d\geq 1}|(p_1j_1\cdot\ldots\cdot p_dj_d)^3Q_{pj}|>0, Q_{pj}:=\langle Qe_{p_1,\dots,p_d},e_{j_1,\dots,j_d}\rangle,$

(ii) $\lambda_i-\lambda_j\neq\lambda_p-\lambda_q$ *for all* $i,j,p,q\geq 1$ *such that* $\{i,j\}\neq\{p,q\}$ *and* $i\neq j$.

Privat and Sigalotti; Mason and Sigalotti; V.N.

Main result

Let us introduce the set

$$\mathcal{E} := \{z \in S : \exists p, q \geq 1, p \neq q, z = c_p e_p + c_q e_q,\\ |c_p|^2 \langle Q e_p, e_p \rangle - |c_q|^2 \langle Q e_q, e_q \rangle = 0\}.$$

Theorem

Under Condition 1, for any $\tilde{z}_0 \in S \cap H^3 \setminus \mathcal{E}$*, the mapping* $\mathcal{R}_\infty(0, \cdot) : \Theta \to H^3$ *admits a continuous right inverse. If* $\tilde{z}_0 \in S \cap H^3 \cap \mathcal{E}$*, then* $\mathcal{R}_\infty(0, \cdot)$ *is not invertible.*

Proof

Case 1. Let suppose that $\tilde{z}_0 \in \mathcal{E}$, i.e., $\tilde{z}_0 = c_p e_p + c_q e_q$ with $|c_p|^2 \langle Q e_p, e_p \rangle - |c_q|^2 \langle Q e_q, e_q \rangle = 0$.

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$$\mathrm{Im}\,\langle \mathcal{R}_t(0,u), c_p e^{-i\lambda_p t} e_p - c_q e^{-i\lambda_q t} e_q \rangle = const.$$

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$$\mathrm{Im}\, \langle \mathcal{R}_t(0,u), c_p e^{-i\lambda_p t} e_p - c_q e^{-i\lambda_q t} e_q \rangle = const.$$

Thus the system is not controllable.

Proof

Case 2. Let $\tilde{z}_0 \in S \cap H^3 \setminus \mathcal{E}$.

$$\langle \mathcal{R}_\infty(0,u), e_m \rangle = -i \sum_{k=1}^{+\infty} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^{+\infty} e^{i\omega_{mk}s} u(s) \mathrm{d}s.$$

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This system is equivalent to the following moment problem

$$\int_0^{+\infty} e^{i\omega_{mk}s} u(s) \mathrm{d}s = d_{mk}, \qquad d_{mk} \in \ell^2. \tag{1}$$

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Proposition

For any $d_{mk} \in \ell^2$ Problem (1) admits a solution $u \in \Theta$.

Controllability of nonlinear system

Main result

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Proof of main result

Main result

Under Condition 1, the system is globally exactly controllable in infinite time in $S \cap H^3$ with controls $u \in \Theta$.

Proof

Let $\mathcal{U}_\infty(z_0, u)$ be the H^3-weak ω-limit set of the trajectory corresponding to $u \in \Theta$ and $z_0 \in H^3$:

$$\mathcal{U}_\infty(z_0, u) := \{z \in H^3 : \mathcal{U}_{T_{n_k}}(z_0, u) \rightharpoonup z \text{ in } H^3 \text{ for some } n_k \to +\infty\}.$$

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Lemma

For any $u \in \Theta$ and $z_0 \in H^3$, the trajectory $\mathcal{U}_{T_n}(z_0, u)$ is bounded in H^3.

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Lemma

For any $u \in \Theta$ and $z_0 \in H^3$, the trajectory $\mathcal{U}_{T_n}(z_0, u)$ is bounded in H^3.

Thus $\mathcal{U}_\infty(z_0, u)$ is non-empty subset of H^3.

Proof of main result

Consider the multivalued function

$$\mathcal{U}_\infty(\cdot,\cdot) : S \cap H^3 \times \Theta \rightarrow 2^{S \cap H^3},$$
$$(z_0, u) \rightarrow \mathcal{U}_\infty(z_0, u).$$

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We apply the inverse function theorem for this mapping.

Inverse function theorem for multivalued functions

Let X and Y be Banach spaces. Define the Hausdorff distance

$$d(x, D) = \inf_{y \in D} \|x - y\|_X,$$

$$e(C, D) = \max\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(y, C)\}.$$

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Definition

A multifunction $F : X \to 2^Y$ is said to be strictly differentiable at (x_0, y_0) if there exists some continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there exist $\delta > 0$ for which

$$e(F(x) - A(x), F(x') - A(x')) \leq \varepsilon \|x - x'\|_X,$$

whenever $x, x' \in B(x_0, \delta)$. A is called a derivative of F at (x_0, y_0).

Inverse function theorem for multivalued functions

Theorem (Nachi and Penot)

Let F be a multifunction from an open set $X_0 \subset X$ to Y with non-empty closed non-empty values. Suppose F is strictly differentiable at $(x_0, y_0) \in Gr(F)$, and some derivative A of F at (x_0, y_0) has a right inverse. Then for any neighborhood U of x_0 there exists a neighborhood V of y_0 such that $V \subset F(U)$.

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$\mathcal{U}_\infty(z_0, u)$ is a non-empty and closed. The construction of the sequence T_n implies that $\mathcal{U}_\infty(z_0, 0) = z_0$.

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Generalization

The proof works also for the defocusing nonlinear Schrödinger equation:

$$i\dot{z} = -\Delta z + V(x)z + |z|^{2p}z + u(t)Q(x)z, \qquad x \in \mathbb{T}^d,$$

where $p \in \mathbb{N}^*$ and $d \geq 1$ are such that the equation is globally well posed in H^1.

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where $p \in \mathbb{N}^*$ and $d \geq 1$ are such that the equation is globally well posed in H^1.

Theorem

The nonlinear Schrödinger equation is exactly controllable in infinite time near the stationary solutions.