Global exact controllability in infinite time of Schrödinger equation

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IHP, December 9, 2010
Introduction

Controlled Schrödinger equation:

\[ i \dot{z} = -\Delta z + V(x)z + u(t)Q(x)z, \quad x \in D, \]
\[ z|_{\partial D} = 0, \]
\[ z(0, x) = z_0(x), \]

where \( D \subseteq \mathbb{R}^d, \partial D \in C^\infty, \ d \geq 1, \ V, \ Q \in C^\infty(\overline{D}, \mathbb{R}) \) are given functions, \( u \) is the control, \( z \) is the state.
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Let \( \mathcal{U}_t(\cdot, u) : L^2 \to L^2 \), \( u \in L^1_{loc}([0, \infty), \mathbb{R}) \) be the resolving operator, i.e. \( \mathcal{U}_t(z_0, u) = z(t) \).
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\[ \|\mathcal{U}_t(z_0, u)\|_{L^2} = \|z_0\|_{L^2}, \quad t \geq 0. \]
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\[ \|\mathcal{U}_t(z_0, u)\|_{L^2} = \|z_0\|_{L^2}, \quad t \geq 0. \]

Let \( S := \{ z \in L^2 : \|z\|_{L^2} = 1 \} \).
Main result

The system is globally exactly controllable in infinite time generically in $V$ and $Q$. 
Introduction

Non-controllability result
Controllability of linearized system
Controllability of nonlinear system

Main result

The system is globally exactly controllable in infinite time generically in $V$ and $Q$.

For any $z_0, z_1 \in S \cap H^k$ there is a control $u \in H^s(\mathbb{R}_+, \mathbb{R})$ and a sequence $T_n \rightarrow +\infty$ such $\mathcal{U}_{T_n}(z_0, u) \rightarrow z_1$ in $H^k$. 
Plan of the talk

1. Non-controllability results
2. Controllability of linearized system
3. Controllability of nonlinear system
Previous results

Ramakrishna, Salapaka, Dahleh, Rabitz, Pierce, Turinici, Altafini, Albertini, D’Alessandro, …

Beauchard, Coron, Laurent

Chambrion, Mason, Sigalotti, Boscain

Mirrahimi, Beauchard, V.N.

V.N.
Non-controllability result
The Schrödinger equation is not exactly controllable in finite time in Sobolev space $H^2$ with controls $L_{loc}^p([0, +\infty), \mathbb{R})$, i.e., for any $z_0 \in S$ the set

$$\{ \mathcal{U}_t(z_0, u) : t \in [0, +\infty), u \in L_{loc}^p([0, +\infty), \mathbb{R}) \text{ for some } p > 1 \}$$

does not contain a ball of the space $H^2$. 
The Schrödinger equation is not exactly controllable in finite time in Sobolev spaces $H^k$, $k < d$ with controls $H^1_{loc}([0, +\infty), \mathbb{R})$, i.e., for any $z_0 \in S$ the set
\[
\{U_t(z_0, u) : t \in [0, +\infty), u \in H^1_{loc}([0, +\infty), \mathbb{R})\}
\]
does not contain a ball of the space $H^k$.

Proof is based on the ideas of Shirikyan introduced to prove non-controllability of Euler equation.
Controllability of linearized system
Let us linearize the system around trajectory $U_t(\tilde{z}_0, 0)$:

$$i\dot{z} = -\Delta z + V(x)z + u(t)Q(x)U_t(\tilde{z}_0, 0),$$

$$z|_{\partial D} = 0,$$

$$z(0) = z_0.$$
Let us linearize the system around trajectory $\mathcal{U}_t(\tilde{z}_0, 0)$:

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Preliminaries

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z|_{\partial D} = 0,
\]

\[
z(0) = 0.
\]

Let us rewrite this problem in the Duhamel form

\[
z(t) = -i \int_0^t e^{i(t-s)(\Delta-V)} u(s)Q(x)U_s(\tilde{z}_0, 0)ds.
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$$z(t) = -i \int_0^t e^{i(t-s)(\Delta-V)}u(s)Q(x)\mathcal{U}_s(\tilde{z}_0, 0)ds.$$

Let $\mathcal{R}_t$ be the resolving operator.
Preliminaries

\[ \langle \mathcal{R}_t(0, u), e_m \rangle = -i \sum_{k=1}^{+\infty} e^{-i\lambda_m t} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^t e^{i\omega_{mk}s} u(s) ds, \quad m \geq 1, \]

where \( \omega_{mk} = \lambda_m - \lambda_k \) and \( Q_{mk} := \langle Q e_m, e_k \rangle \).
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where \( \omega_{mk} = \lambda_m - \lambda_k \) and \( Q_{mk} := \langle Qe_m, e_k \rangle \). For any \( u \in L^1(\mathbb{R}_+, \mathbb{R}) \) the following limit exists

\[
\mathcal{R}_\infty(0, u) := \lim_{n \to +\infty} \mathcal{R}_{T_n}(0, u).
\]

Vahagn Nersesyan

Exact controllability in infinite time of Schrödinger equation
\[ \langle R_t(0, u), e_m \rangle = -i \sum_{k=1}^{+\infty} e^{-i\lambda_m t} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^t e^{i\omega_{mk} s} u(s) ds, \ m \geq 1, \]

where \( \omega_{mk} = \lambda_m - \lambda_k \) and \( Q_{mk} := \langle Q e_m, e_k \rangle \). For any \( u \in L^1(\mathbb{R}_+, \mathbb{R}) \) the following limit exists

\[ R_\infty(0, u) := \lim_{n \to +\infty} R_{T_n}(0, u). \]

The choice of the sequence \( T_n \) implies that

\[ \langle R_\infty(0, u), e_m \rangle = -i \sum_{k=1}^{+\infty} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^{+\infty} e^{i\omega_{mk} s} u(s) ds. \]
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$$\tilde{u}(\omega) := \int_{0}^{+\infty} e^{i\omega s} u(s) ds.$$
Preliminaries

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\[ \tilde{u}(\omega) := \int_{0}^{+\infty} e^{i\omega s} u(s) ds. \]

The set of admissible controls is the Banach space

\[ \Theta := u \in L^1(\mathbb{R}_+, \mathbb{R}) \cap H^s(\mathbb{R}_+, \mathbb{R}) \cap C \]

where \( s \geq 1 \) is any fixed constant and

\[ C := \{ u \in L^1(\mathbb{R}_+, \mathbb{R}) : \{ \tilde{u}(\omega_{mk}) \} \in \ell^2 \}. \]
Condition 1

Let $V(x_1, \ldots, x_d) = V_1(x_1) + \ldots + V_d(x_d)$ and $D \subset \mathbb{R}^d$ is a rectangle. The functions $V, Q \in C^\infty(D, \mathbb{R})$ are such that

(i) $\inf_{p_1, j_1, \ldots, p_d, j_d \geq 1} |(p_1 j_1 \cdot \ldots \cdot p_d j_d)^3 Q_{pj}| > 0$, $Q_{pj} := \langle Q e_{p_1}, \ldots, p_d, e_{j_1}, \ldots, j_d \rangle$,

(ii) $\lambda_i - \lambda_j \neq \lambda_p - \lambda_q$ for all $i, j, p, q \geq 1$ such that $\{i, j\} \neq \{p, q\}$ and $i \neq j$. 

Vahagn Nersesyan

Exact controllability in infinite time of Schrödinger equation
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Privat and Sigalotti; Mason and Sigalotti; V.N.
Let us introduce the set

\[ \mathcal{E} := \{ z \in S : \exists p, q \geq 1, p \neq q, z = c_p e_p + c_q e_q, \]
\[ |c_p|^2 \langle Q e_p, e_p \rangle - |c_q|^2 \langle Q e_q, e_q \rangle = 0 \}. \]

**Theorem**

*Under Condition 1, for any $\tilde{z}_0 \in S \cap H^3 \setminus \mathcal{E}$, the mapping $\mathcal{R}_\infty(0, \cdot) : \Theta \to H^3$ admits a continuous right inverse. If $\tilde{z}_0 \in S \cap H^3 \cap \mathcal{E}$, then $\mathcal{R}_\infty(0, \cdot)$ is not invertible.*
Case 1. Let suppose that $\tilde{z}_0 \in \mathcal{E}$, i.e., $\tilde{z}_0 = c_p e_p + c_q e_q$ with $|c_p|^2 \langle Q e_p, e_p \rangle - |c_q|^2 \langle Q e_q, e_q \rangle = 0$. 
Case 1. Let suppose that $\tilde{z}_0 \in \mathcal{E}$, i.e., $\tilde{z}_0 = c_p e_p + c_q e_q$ with $|c_p|^2 \langle Q e_p, e_p \rangle - |c_q|^2 \langle Q e_q, e_q \rangle = 0$. By Beauchard and Coron

$$\text{Im} \langle \mathcal{R}_t(0, u), c_p e^{-i\lambda_p t} e_p - c_q e^{-i\lambda_q t} e_q \rangle = \text{const}.$$
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$$\text{Im} \langle \mathcal{R}_t(0, u), c_p e^{-i\lambda_p t} e_p - c_q e^{-i\lambda_q t} e_q \rangle = \text{const}.$$ 

Thus the system is not controllable.
Case 2. Let $\tilde{z}_0 \in S \cap H^3 \setminus \mathcal{E}$. 

$$\langle \mathcal{R}_\infty(0, u), e_m \rangle = -i \sum_{k=1}^{+\infty} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^{+\infty} e^{i\omega_{mk}s} u(s) ds.$$
Proof

Case 2. Let $\tilde{z}_0 \in S \cap H^3 \setminus \mathcal{E}$.

$$\langle R_\infty(0, u), e_m \rangle = -i \sum_{k=1}^{+\infty} \langle \tilde{z}_0, e_k \rangle Q_{mk} \int_0^{+\infty} e^{i\omega_{mk}s} u(s)ds.$$ 

This system is equivalent to the following moment problem

$$\int_0^{+\infty} e^{i\omega_{mk}s} u(s)ds = d_{mk}, \quad d_{mk} \in \ell^2. \quad (1)$$
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**Proposition**

For any $d_{mk} \in \ell^2$ Problem (1) admits a solution $u \in \Theta$. 

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Vahagn Nersesyan

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Controllability of nonlinear system
Main result

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\[ \mathcal{U}_t(\cdot, u) : L^2 \to L^2, \quad u \in L^1_{loc}([0, \infty), \mathbb{R}) \] is the resolving operator.
Under Condition 1, the system is globally exactly controllable in infinite time in $S \cap H^3$ with controls $u \in \Theta$. 
Proof

Let $\mathcal{U}_\infty(z_0, u)$ be the $H^3$-weak $\omega$-limit set of the trajectory corresponding to $u \in \Theta$ and $z_0 \in H^3$:

$$\mathcal{U}_\infty(z_0, u) := \{ z \in H^3 : \mathcal{U}_{T_{n_k}}(z_0, u) \rightharpoonup z \text{ in } H^3 \text{ for some } n_k \to +\infty \}.$$
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Let $U_\infty(z_0, u)$ be the $H^3$-weak $\omega$-limit set of the trajectory corresponding to $u \in \Theta$ and $z_0 \in H^3$:

$$U_\infty(z_0, u) := \{ z \in H^3 : U_{T_{n_k}}(z_0, u) \rightarrow z \text{ in } H^3 \text{ for some } n_k \rightarrow +\infty \}.$$

**Lemma**

*For any $u \in \Theta$ and $z_0 \in H^3$, the trajectory $U_{T_n}(z_0, u)$ is bounded in $H^3$.***
Proof

Let $\mathcal{U}_\infty(z_0, u)$ be the $H^3$-weak $\omega$-limit set of the trajectory corresponding to $u \in \Theta$ and $z_0 \in H^3$:

$$\mathcal{U}_\infty(z_0, u) := \{ z \in H^3 : \mathcal{U}_{T_{n_k}}(z_0, u) \rightharpoonup z \text{ in } H^3 \text{ for some } n_k \to +\infty \}.$$

Lemma

For any $u \in \Theta$ and $z_0 \in H^3$, the trajectory $\mathcal{U}_{T_n}(z_0, u)$ is bounded in $H^3$.

Thus $\mathcal{U}_\infty(z_0, u)$ is non-empty subset of $H^3$. 
Consider the multivalued function

$$U_\infty(\cdot, \cdot) : S \cap H^3 \times \Theta \to 2^{S \cap H^3},$$

$$(z_0, u) \mapsto U_\infty(z_0, u).$$
Consider the multivalued function

\[ \mathcal{U}_\infty(\cdot, \cdot) : S \cap H^3 \times \Theta \to 2^{S \cap H^3}, \]

\[ (z_0, u) \mapsto \mathcal{U}_\infty(z_0, u). \]

We apply the inverse function theorem for this mapping.
Inverse function theorem for multivalued functions

Let $X$ and $Y$ be Banach spaces. Define the Hausdorff distance

$$d(x, D) = \inf_{y \in D} \|x - y\|_X,$$

$$e(C, D) = \max\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(y, C)\}.$$
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**Definition**

A multifunction $F : X \to 2^Y$ is said to be strictly differentiable at $(x_0, y_0)$ if there exists some continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there exist $\delta > 0$ for which

\[ e(F(x) - A(x), F(x') - A(x')) \leq \varepsilon \|x - x'|_X, \]

whenever $x, x' \in B(x_0, \delta)$. $A$ is called a derivative of $F$ at $(x_0, y_0)$. 
Inverse function theorem for multivalued functions

**Theorem (Nachi and Penot)**

Let $F$ be a multifunction from an open set $X_0 \subset X$ to $Y$ with non-empty closed non-empty values. Suppose $F$ is strictly differentiable at $(x_0, y_0) \in \text{Gr}(F)$, and some derivative $A$ of $F$ at $(x_0, y_0)$ has a right inverse. Then for any neighborhood $U$ of $x_0$ there exists a neighborhood $V$ of $y_0$ such that $V \subset F(U)$. 
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$U_\infty(z_0, u)$ is a non-empty and closed.
Inverse function theorem for multivalued functions

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$U_\infty(z_0, u)$ is a non-empty and closed. The construction of the sequence $T_n$ implies that $U_\infty(z_0, 0) = z_0$. 

Vahagn Nersesyan

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$U_\infty(z_0, u)$ is a non-empty and closed. The construction of the sequence $T_n$ implies that $U_\infty(z_0, 0) = z_0$. $U_\infty(z_0, u)$ is strictly differentiable at $(z_0, 0)$ with derivative $R_\infty$. 
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$U_\infty(z_0, u)$ is a non-empty and closed. The construction of the sequence $T_n$ implies that $U_\infty(z_0, 0) = z_0$. $U_\infty(z_0, u)$ is strictly differentiable at $(z_0, 0)$ with derivative $R_\infty$. Since the linearized system is controllable for $z_0 \notin E$, we get the controllability near $z_0$. 

□
Theorem (Nachi and Penot)

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$U_\infty(z_0, u)$ is a non-empty and closed. The construction of the sequence $T_n$ implies that $U_\infty(z_0, 0) = z_0$. $U_\infty(z_0, u)$ is strictly differentiable at $(z_0, 0)$ with derivative $R_\infty$. Since the linearized system is controllable for $z_0 \notin \mathcal{E}$, we get the controllability near $z_0$. If $z_0 \in \mathcal{E}$, controllability is proved by the arguments of Beauchard and Coron. □
Generalization

The proof works also for the defocusing nonlinear Schrödinger equation:

$$i \dot{z} = -\Delta z + V(x)z + |z|^{2p}z + u(t)Q(x)z, \quad x \in \mathbb{T}^d,$$

where $p \in \mathbb{N}^*$ and $d \geq 1$ are such that the equation is globally well posed in $H^1$. 
The proof works also for the defocusing nonlinear Schrödinger equation:

\[ i\dot{z} = -\Delta z + V(x)z + |z|^{2p}z + u(t)Q(x)z, \quad x \in \mathbb{T}^d, \]

where \( p \in \mathbb{N}^* \) and \( d \geq 1 \) are such that the equation is globally well posed in \( H^1 \).

**Theorem**

The nonlinear Schrödinger equation is exactly controllable in infinite time near the stationary solutions.