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Controllability of Cubic Schroedinger Equation via Low-Dimensional Source Term

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Introduction

Main theme: Lie algebraic approach to *controllability of distributed parameter systems*.

Example of such approach - approximate controllability and controllability in finite-dimensional projections criteria* for 2D and 3D Navier-Stokes/Euler equation of fluid motion controlled by *low-dimensional forcing*.

Goal: develop similar technique for *cubic defocusing Schroedinger equation*

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x) \quad \text{(NLS)}$$

*A.Agrachev, A.Sarychev, S.Rodrigues,A.Shirikyan, H.Nersisyan controlled via low-dimensional source term F. We consider dimension 2 and periodic boundary conditions; $x \in \mathbb{T}^2$.

Problem setting is distinguished by the *type of control*; it is applied via source term, which is 'linear combination of few functions':

$$F(t,x) = \sum_{k \in \mathcal{K}^1} v_k(t) F^k(x), \ \mathcal{K}^1 \text{ is finite.}$$

In the periodic case we take $F^k(x) = e^{ik \cdot x}, \ k \in \mathbb{Z}^2, \ F$ being *trigonometric polynomial in x*.

The *control functions* $v_k(t), \ t \in [0,T], \ k \in \mathcal{K}^1$ are chosen freely from $L_\infty[0,T]$.

Preliminaries on existence and uniqueness of trajectories

For our goals it suffices to deal with NLS equation evolving in Sobolev space $H = H^2(\mathbb{T}^2)$. Our source term F is trigonometric polynomial in x and $t \mapsto F(t, x) = \sum_k v_k(t) e^{ik\cdot x}$ is measurable essentially bounded map in t.

Local existence of solutions in this setting is standard and is proved by fixed point argument for contracting map in $C([0, T]; H^2(\mathbb{T}^2))$. The same argument remains valid for equation with more general nonlinearity.

Preliminaries ctd.

Proposition 1. Given equation

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + P_2(u,\bar{u},t,x),$$

where $P_2(u,\bar{u},t)$ is second degree polynomial in $u,\bar{u}$ with coefficients $f(t,x)$ from $L_\infty([0,T],H^2(\mathbb{T}^2))$.

Then for each $B>0$ and each $\tilde{u}$ with $\|u\|_{H^2} \le B$ there exists $T_B > 0$ such that there $\exists$ **unique strong solution** $u(\cdot) \in C([0,T_B],H^2(\mathbb{T}^2))$ **of the Cauchy problem for (NLS)** with the initial condition $u(0)=\tilde{u}$. $\square$

Preliminaries-3

Global existence/uniqueness result for cubic NLS with source term:

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x),$$

Proposition 2. For the source term $F(t,x)$ from $L_\infty([0,T],H^2(\mathbb{T}^2))$. for each $\tilde{u} \in H^2$ the Cauchy problem with the initial condition $u(0) = \tilde{u}$ possesses **unique strong solution** $u(\cdot) \in C([0,T],H^2(\mathbb{T}^2))$. $\square$

(A stronger version of) results on continuous dependence of trajectories on the r.-h. side will appear later.

Controlled NLS equation: controllability problem settings

We will study controllability in finite-dimensional projections meaning that proper control $v_k(t),\ k \in \mathcal{K}^1$ may steer

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{k \in \mathcal{K}^1} v_k(t) e^{ik\cdot x}$$

in time $T > 0$ from $u_0 \in H^2(\mathbb{T}^2)$ to a point with preassigned orthogonal projection on a given finite-dimensional subspace $\mathcal{L} \subset H^2$;

and

approximate controllability, meaning that set of 'points' attainable from each $u_0 \in H^2(\mathbb{T}^2)$ is dense in L_2.

Controllability of NLS equation: main result

Theorem. **There exists set** $\mathcal{K} = \{m^1, m^2, m^3, m^4\}$**, consisting of** 4 **modes such that cubic defocusing Schroedinger equation**

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \sum_{\alpha=1}^{4} v_\alpha(t) e^{im^\alpha \cdot x}$$

is controllable in each finite-dimensional projection and approximately controllable. $\square$

Outline of approach from geometric control viewpoint

Our study of controllability of NLS equation is based (as well as previous work on Navier-Stokes/Euler equation) on method of *iterated Lie extensions*.

Lie extension of a control system $\dot{x} = f(x, u), \ u \in U$ allows us to join *('almost maintaining' controllability properties)* to the r.-h. side additional vector fields, which are expressed via Lie brackets of $f(\cdot, u)$ for various $u \in U$. If after a series of extensions one arrives to a system, which is then the original system also would be.

Controlled NLS equation is a particular type of infinite-dimensional control-affine system

$$\dot{u} = f^0(u) + \sum_{k \in \mathcal{K}} f^k(u) v_k(t).$$

We proceed with Lie extensions, at each step of which following Lie brackets appear:

$$[f^m,[f^m,f^0]],\ \ [f^n,[f^m,[f^m,f^0]]],\ \ m,n \in \mathcal{K}.$$

The 3rd-order Lie brackets $[f^m,[f^m,f^0]]$ are obstructions to controllability; they have to be 'compensated'. The 4th-order Lie bracket $[f^n,[f^m,[f^m,f^0]]]$ are directions along which the extended control acts.

Geometric control in infinite-dimension

Obstacles:

- r.-h. sides of equations ('vector fields') include unbounded operators

- instead of flows one often has to deal with semigroups of nonlinear operators;

- lack of adequate infinite-dimensional differential geometry: manifolds, distributions, integrability etc.

'In practice'

we use fast-oscillating controls, which underly Lie extensions method. Specially designed resonances between such controls result in a motion which provides (*approximates*) motion in *extending* direction, along a Lie bracket.

Choosing special coordinates (Fourier Ansatz) on torus we will feed fast-oscillating controls into the r.-h. sides of equations for the components $q_m, q_n, \ m, n \in \mathcal{K}^1 \subset \mathbb{Z}^2$ in such a way that it will produce effect of control for the dynamics of certain component q_ℓ with $\ell \notin \mathcal{K}^1$ and (asymptotically) will not affect the dynamics of other components.

This is called extension of control. Final result is obtained by (finite) iteration of such steps.

Cubic Schroedinger equation on $\mathbb{T}^2$ as infinite-dimensional system of ODE

Invoking Fourier Ansatz we seek for solution of the NLS equation

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x) \quad \text{(NLS)}$$

in the form of a series expansion

$$u(t,x) = \sum_{k \in \mathbb{Z}^2} q_k(t) e^{i(kx+|k|^2 t)}.$$

with respect to **modes** $\mathbf{e}_k = e^{i(kx+|k|^2 t)}$.

The source term can be represented as

$$F(t,x) = \sum_{k \in \mathcal{K}^1 \subset \mathbb{Z}^2} e^{i(kx+|k|^2 t)} v_k(t),$$

notation $v_k(t)$ is kept for controls. The set of *controlled modes* $\mathcal{K}^1$ is finite.

Substituting the expansions of u and F into NLS equation we get infinite system of ODE's for the coefficients $q(t)$:

$$i\partial_t q_k(t) = S_k(q,t) = -q_k|q_k|^2 + 2q_k \sum_{j\in\mathbb{Z}^2} |q_j|^2 +$$

$$+ \sum_{k_1-k_2+k_3=k;k\neq k_1,k_3} q_{k_1}\bar{q}_{k_2}q_{k_3}e^{i\omega(K)t}, \quad k \in \mathbb{Z}^2. \quad \textbf{(NLSODE)}$$

$$\omega(K) = |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2.$$

If we add controlling source term $\sum_{k\in\mathcal{K}^1} v_k(t)e^{ik^2t}e^{ikx}$, then controls $v_k(t)$ appear in the equations, indexed by $k \in \mathcal{K}^1$.

We proceed with **extension**.

Sketch of the extension step

Assume that the set of controlled modes is $\{m, n\} \subset \mathbb{Z}^2$. We will show how choosing in clever way controls in these modes, one gets an extended control for the mode $(2m - n) \in \mathbb{Z}^2$.

Feed into the r.-h. side of the ODEs for q_m, q_n control functions $\dot{v}_m(t) + \tilde{v}_n, \dot{v}_m(t) + \tilde{v}_n$ respectively, where $v_m(t), v_n(t)$ are Lipschitzian functions. We get

$$i\partial_t q_m(t) = S_m(q, t) + \dot{v}_m(t) + \tilde{v}_m,$$
$$i\partial_t q_n(t) = S_n(q, t) + \dot{v}_n(t) + \tilde{v}_n.$$

Introduce new variables q^*_ℓ by relations

$$q_m = q^*_m - iv_r(t), q_n = q^*_n - iv_n(t),\ q^*_k = q_k,\ \text{for } k \neq m, n,$$

or

$$q = q^* + V(t) = q^* + v_m(t)e_m + v_n(t)e_n. \quad \textbf{(SUB1)}$$

The equations for components of q^* are:

$$i\partial_t q_j^*(t) = \begin{cases} S_j(q + V(t), t) + \tilde{v}_j, & j \in \{m, n\}; \\ S_j(q + V(t), t), & j \notin \{m, n\}. \end{cases}$$

Impose isoperimetric constraints

$$v_m(0) = v_n(0) = 0, \; v_m(T) = v_n(T) = 0,$$

in order to preserve the end-points of the trajectory:

$$q(0) = q^*(0), \; q(T) = q^*(T)$$

.

Controllability of equations for q^* $\Rightarrow$ controllability of the original system.

Calculating $S_j(q+V(t),t)$ at the r-h. side we get

$$i\partial_t q_k^*(t) = -(q_k^* + \delta_{k,mn} v_k)|q_k^* + \delta_{k,mn} v_k|^2 +$$

$$+2(q_k^\star + \delta_{k,mn} v_k)\left(\|V\|^2 + \sum_{s\in\mathbb{Z}^2} |q_s^*|^2\right) +$$

$$+\sum (q_{k_1}^* + \delta_{k_1,mn} v_{k_1})(\bar{q}_{k_2}^* + \delta_{k_2,mn} \bar{v}_{k_2})(q_{k_3}^* + \delta_{k_3,mn} v_{k_3}) e^{i\omega(K)t},$$

$\delta_{k,mn} = 1$, whenever $k \in \{m,n\}$, otherwise $\delta_{k,mn} = 0$.

The result is cubic polynomial with respect to $v_m, v_n, \bar{v}_m, \bar{v}_n$.

Fast oscillations

Now we introduce fast-oscillations, choosing the controls $v_m(t), v_n(t)$ of the form

$$v_m(t) = e^{i(1+\varepsilon\rho)t/\varepsilon}\widehat{v}_m(t), v_n(t) = e^{i(2+\varepsilon\sigma)t/\varepsilon}\widehat{v}_n(t), \quad \textbf{(SUB2)}$$

where $\widehat{v}_m(t), \widehat{v}_n(t)$ are functions of bounded variation, ρ, σ will be specified later and $\varepsilon > 0$ is small parameter.

Taking all the monomials of degree ≤ 3 in $v_m, v_n, \bar{v}_m, \bar{v}_n$ we classify them into *resonant* and *non-resonant*. We call a monomial *non-resonant* if, after substitution of **(SUB2)** into it, we get a fast-oscillating factor $e^{i\beta t/\varepsilon}$, $\beta > 0$. All other monomials are *resonant*; they are classified as *bad resonances (obstructions)* and *good resonances - extending controls.*

Fast oscillations - ctd.

We get equation

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \tilde{v}_m(t)\mathbf{e_m} + \tilde{v}_n(t)\mathbf{e_n}$$
$$+ obstructions + extending\ control + \text{non-resonant terms}$$

We have to show how *obstructions can be compensated*; then we demonstrate how *extending control can be designed* and finally we prove that contribution of *non-resonant terms can be neglected*, whenever $\varepsilon > 0$ is small.

Obstructions Since the nonlinearity is cubic we study only monomials of degree ≤ 3. Direct computation shows that among these monomials the resonant ones are:

$$v_m\bar{v}_m = |v_m|^2,\ v_n\bar{v}_n = |v_n|^2,\ v_m^2\bar{v}_n.$$

The first two quadratic terms correspond to the quadruples

$$(m,m,k,k),(k,m,m,k),(n,n,k,k),(k,n,n,k)$$

in the set of indices for the sum representing cubic term. They are examples of *obstructions to controllability* in terminology of geometric control and appear at the r.-h. side of ODE for *each* q_k as:

$$2q_k^*\|V\|^2 = q_k^*\left(2|v_m|^2 + 2|v_n|^2\right),\ \text{for } k \neq m,n,$$
$$q_m^*\left(|v_m|^2 + 2|v_n|^2\right),\ q_n^*\left(2|v_m|^2 + |v_n|^2\right),\ \text{for } k \in \{m,n\}.$$

appear .

We can 'compensate' the obstructing quadratic terms by time-variant substitution for the variables q_k^*:

$$q_m^\star = q_m^* e^{-i\int_0^t (|v_m|^2+2|v_n|^2)(\tau)d\tau}, \ q_n^\star = q_n^* e^{-i\int_0^t (2|v_m|^2+|v_n|^2)(\tau)d\tau},$$
$$q_k^\star = q_k^* e^{-i\int_0^t (2|v_m|^2+2|v_n|^2)(\tau)d\tau}, \ k \neq r, s.$$

In order to guarantee $q^\star(T) = q^*(T) = q(T)$ we have to impose additional *(isoperimetric)* conditions on v_m, v_n:

$$\int_0^T |v_m(t)|^2 dt = \int_0^T |\hat{v}_m(t)|^2 dt = 2\pi N_m,$$
$$\int_0^T |v_n(t)|^2 dt = \int_0^T |\hat{v}_n(t)|^2 dt = 2\pi N_n, \ N_m, N_n \in \mathbb{Z}.$$

Extending control via resonance

Now we study **resonance cubic term** $v_m^2 v_n$ which corresponds to the quadruple

$$k_1 = k_3 = m, \ k_2 = n, \ k = k_1 + k_3 - k_2 = 2m - n.$$

with $\omega(K) = 2m^2 - n^2 - (2m - n)^2 = -(m - n)^2$ in the equation for $q^\star_{2m-n}$ of the system **(NLSODE)**: Then we get

$$e^{i(2\rho - \sigma + \omega(K))t} \widehat{v}_m^2(t) \bar{\widehat{v}}_n(t)$$

at the right-hand side of this equation.

We choose ρ, σ such that

$$2\rho - \sigma = -\omega(K) = (m - n)^2.$$

Then the resonant term $\widehat{v}_m^2(t)\bar{\widehat{v}}_n(t)$, appears as an *extending control* in the ODE for $q^\star_{2m-n}$.

Effect of non-resonant terms

Non resonant terms $\phi(t,x,u)$ at the r.-h. side of the modified NLS

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + \\ + \tilde{v}_m(t)\mathbf{e_m} + \tilde{v}_n(t)\mathbf{e_n} + v_{2m-n}(t)\mathbf{e_{2m-n}} + \underline{\phi(t,x,u)}$$

can be represented as

$$\phi(t,x,u) = W^0(t,x) + uW^{11}(t,x) + \bar{u}W^{12}(t,x) + \\ + u^2 W^{21}(t,x) + |u|^2 W^{22}(t,x).$$

For our choice of controls each $W^{ij}(t,x)$ is trigonometric polynomial in t:

$$W^{ij}(t,x) = \sum_r e^{i\beta_r t/\varepsilon} W_r^{ij}(x).$$

Global existence of solution of the equation

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x) + \phi(t,x,u,\varepsilon).(PERTURB$$

This can be done though if ϕ is fast oscillating ($\varepsilon > 0$ is small). Moreover solutions of this equation converge to the respective solutions of the 'limit equation'

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x),(LIMIT)$$

as $\varepsilon \to 0$.

This fact is part of relaxation result for NLS equation*

*adaptation of results by H.Frankowska (1990), H.Fattorini (1994), N.Ahmed (1987), on relaxation of evolution equations.

Relaxation

Relaxation seminorm $\|\cdot\|_b^{\mathrm{rx}}$ is defined by formula:

$$\|\phi\|_b^{\mathrm{rx}} = \max_{t,t' \in [0,T], x, \|u\| \le b} \left\| \int_t^{t'} \phi(\tau, x, u) d\tau \right\|. \tag{1}$$

The following theorem affirms continuous dependence of trajectories with respect to the r.-h. side in relaxation seminorm.

Continuous dependence of trajectories

Theorem. Let solution $\tilde{u}(t)$ of the (LIMIT) equation exist on $[0, T]$, belongs to $C([0, T], H)$ and $\sup_{t \in [0,T]} \|u(t)\| < b$. Then $\forall \varepsilon > 0 \exists \delta > 0$ such that whenever $\|\phi\|_b^{\text{rx}} < \delta$, then the solution $u(t)$ of the perturbed equation exists on the interval $[0, T]$, is unique and satisfies the bound

$$\sup_{t \in [0,T]} \|u(t) - \tilde{u}(t)\| < \varepsilon. \square$$

As a corollary we conclude that time-T attainable set of NLS equation, controlled by 2 controls, approximates similar attainable set for NLS equation controlled by 3 controls.

Solid controllability in projections

The latter equation is globally controllable in projection onto 3-dimensional linear subspace $\mathcal{L} = \text{Span}\{\mathbf{e_m}, \mathbf{e_n}, \mathbf{e_{2m-n}}\}$ and this property is stable **(solid controllability)**, then NLS equation, controlled by 2 controls is also controllable in projection onto $\mathcal{L}$.

Iterating the control extension procedure we are able to extend $\mathcal{L}$.

Extensions modeled in the space of modes $\mathbb{Z}^2$

We have established that having controls applied to the modes $\mathbf{e_m}, \mathbf{e_n}$ we can 'get' an extending control applied to the mode $\mathbf{e_{2m-n}}$, $m, n \in \mathbb{Z}^2$.

A nonvoid set $\mathcal{K}^1 \subset \mathbb{Z}^2$ is called saturating, if the only set $\mathcal{K} \supset \mathcal{K}^1$, invariant with respect to the operation $(m, n) \mapsto 2m - n$, is $\mathbb{Z}^2$ itself.

Then NLS equation with controls, applied to the modes from a saturating set $\mathcal{K}^1$, provide global controllability in each finite dimensional projection and approximate controllability.

Controllability via controls applied to 4 modes

The following result provides an example of saturating set $\mathcal{K}^1$.

Proposition. *Let* $m, n \in \mathbb{Z}^2$ *be such that* $m \wedge n = \pm 1$*. Then the set* $\{0, m, n, m+n\}$ *is saturating. NLS equation with controls applied to these modes is controllable in each finite-dimensional projection and approximately controllable.* $\square$