Approximate controllability of the bilinear Schrödinger equation

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Workshop on Quantum Control
Institut Henri Poincaré, Paris
10/12/2010
Abstract controllability problem: reminder

\[ \frac{d\psi}{dt} = A(\psi) + uB(\psi), \quad u \in U, \]

with

- \( H \) complex Hilbert space;
- \( U \subset \mathbb{R} \).
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With a piecewise constant \( u : [0, T] \rightarrow U \) we associate the flow

\[ t \mapsto \Upsilon_t^u = e^{(t-t_k)(A+u_kB)} \circ e^{(t_k-t_{k-1})(A+u_{k-1}B)} \circ \ldots \circ e^{t_1(A+u_1B)} \]

which is a unitary transformation of \( H \).
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\]

which is a unitary transformation of \( H \).

We are interested to approximate controllability in the unit sphere of \( H \).
Controllability result: reminder

$$(\lambda_n)_{n \in \mathbb{N}}$$ eigenvalues of $A$ corresponding to $$(\phi_n)_{n \in \mathbb{N}}$$.

Theorem (Boscain, Caponigro, Chambrion, S.)

Assume that there exists $S \subset \mathbb{N}^2$ such that

- $S$ connects $\mathbb{N}$
- $\langle \phi_j, B\phi_k \rangle \neq 0$ for every $(j, k) \in S$
- each $\lambda_j$ is simple
- $(j, k) \in S$ and $(j, k) \neq (m, l) \in \mathbb{N}^2$ $\implies$ $\lambda_j - \lambda_k \neq \lambda_m - \lambda_l$ or $\langle \phi_m, B\phi_l \rangle = 0$.

Then

$$\frac{d\psi}{dt} = A(\psi) + uB(\psi), \quad u \in [0, \delta],$$

is approximately controllable for every $\delta > 0$. 
Remarks on the approximate controllability result

- the proof is based on control analysis of Galerkin approximations with respect to the basis $(\phi_k)_{k \in \mathbb{N}}$
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- the proof is based on control analysis of Galerkin approximations with respect to the basis \( (\phi_k)_{k \in \mathbb{N}} \)
- upper and lower bound on the \( L^1 \) norm of the control
- lower bound on the controllability time
- controllability extends to density matrices and simultaneous control (or tracking: approximating any prescribed unfeasible trajectory arbitrarily well up to the phases)
Rotating bipolar molecule

\[ i \frac{\partial \psi(\theta, t)}{\partial t} = \left( - \frac{\partial^2}{\partial \theta^2} + u_1(t) \cos(\theta) + u_2(t) \sin(\theta) \right) \psi(\theta, t), \quad \theta \in S^1 \]

- θ rotational degree of freedom of a bipolar rigid molecule confined to a plane
- controlled fields pointing in the directions (0, 1) and (1, 0)
- \( u_1(t), u_2(t) \in [0, \delta] \)
- if \( u_2 \equiv 0 \) controllability between even wavefunctions and no transfer of probability between odd and even ones
- in [Boscain, Chambrian, Mason, Sigalotti, Sugny, 48th IEEE CDC, 2009] we studied the problem of controlling simultaneously the even and the odd part, with \( u_2 \equiv 0 \)
For every $\alpha \in S^1$, consider the splitting

$$H = H^\alpha_e \oplus H^\alpha_o$$

with $H^\alpha_e$ and $H^\alpha_o$ the Hilbert spaces of, respectively, even and odd functions with respect to $\alpha$.

Our first result is the generalization of the partial controllability with $u_2 \equiv 0$.

**Lemma**

*Let $\alpha \in [0, \pi/2]$. Then the system restricted to $H^\alpha_e$ (or $H^\alpha_o$) with controls in

$$U_\alpha = \{ u : \mathbb{R} \rightarrow [0, \delta]^2 \cap \mathbb{R} \langle \cos \alpha, \sin \alpha \rangle \mid u \text{ piecewise constant} \}$$

is approximately controllable.*
The control system with 1D controls can be rewritten as
\[
   i \frac{\partial \psi(\theta, t)}{\partial t} = \left( -\frac{\partial^2}{\partial \theta^2} + v(t) \cos(\theta - \alpha) \right) \psi(\theta, t)
\]
\[
   v \in \left( 0, \delta \sqrt{1 + \min\{\tan \alpha, \cotan \alpha\}^2} \right).
\]

Complete orthonormal systems for $H^\alpha_e$ and $H^\alpha_o$ of eigenfunctions of $A$ are given by
\[
   \left\{ \cos((k \cdot - \alpha))/\sqrt{\pi} \right\}_{k=0}^{\infty} \text{ and } \left\{ \sin((k \cdot - \alpha))/\sqrt{\pi} \right\}_{k=1}^{\infty},
\]
respectively.

The sufficient conditions for controllability are easily tested:
\[
   A = \text{diag}(k^2/\pi)
\]
\[
   B = \begin{pmatrix}
   0 & \frac{1}{\sqrt{2}} & 0 & \cdots \\
   \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & \cdots \\
   0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\
   \vdots & \ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}
\]
\[
   S = \{(j, k) \mid |j - k| = 1\}
Remark: Any eigenfunction is even with respect to some \( \alpha \in [0, \pi/2] \).
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Controllability with 1D controls

$\downarrow$

Any eigenfunction can be approximately driven to the ground state

$\phi(x) \equiv 1/\sqrt{2\pi}$
Rotating bipolar molecule: controllability between eigenfunctions

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Controllability with 1D controls

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\phi(x) \equiv 1/\sqrt{2\pi}
\]

For the same reason, the ground state can be approximately driven to any other eigenfunction.

Concatenating the two controls we get approximate controllability between eigenfunctions.
Let us prove approximate controllability with 2D controls. It is enough to prove that every wavefunction \( \psi \in H \) of norm one can be steered \( \varepsilon \)-close to the constant \( 1/\sqrt{2\pi} \), for \( \varepsilon > 0 \) arbitrary (time-reversibility).
Let us prove approximate controllability with 2D controls.
It is enough to prove that every wavefunction $\psi \in H$ of norm one can be steered $\varepsilon$-close to the constant $1/\sqrt{2\pi}$, for $\varepsilon > 0$ arbitrary (time-reversibility).
Take $\alpha \in (0, \pi/2)$.
Using controls in $U_\alpha$, we can steer the $\alpha$-even part of $\psi$ in a $\varepsilon$-neighborhood of $\|\psi_\alpha^T\|/\sqrt{2\pi}$.
Then $\psi$ goes to

$$\tilde{\psi} \simeq \frac{\|\psi_\alpha^T\|}{\sqrt{2\pi}} + \phi_1,$$
with $\phi_1 \in H_0^\alpha$.

If $\|\phi_1\|$ is smaller than $\varepsilon$ then we are done.
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with $\phi_1 \in H_0^\alpha$.

If $||\phi_1||$ is smaller than $\varepsilon$ then we are done.

Assume then that $||\phi_1|| \geq \varepsilon$ and consider, for every $\beta \in S^1$,

$$\tau_\beta = ||(\phi_1)_e^\beta||^2.$$
Rotating bipolar molecule: approximate controllability

\[ \tilde{\psi} \simeq \frac{\|\psi_\alpha^e\|}{\sqrt{2\pi}} + \phi_1, \quad \phi_1 \in H_0^\alpha, \quad \|\phi_1\| \geq \varepsilon, \quad \tau_\beta = \|(\phi_1)_e^\beta\|^2. \]

A computation shows that there exists \( c > 0 \) independent on \( k \) and \( \alpha \) and there exists \( \beta \in (0, \pi/2) \) such that

\[ \tau_\beta \geq c\varepsilon^2. \]
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Example: 1D potential well

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} + u(t)x \right) \psi(x, t), \quad x \in (0, 1), \]

with \( \psi(0, t) = \psi(1, t) = 0 \).

Exact controllability between regular enough wavefunctions: Beauchard, Beauchard-Coron, Beauchard-Laurent
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Exact controllability between regular enough wavefunctions: Beauchard, Beauchard-Coron, Beauchard-Laurent

The spectrum of \(-\Delta + V\) is

\[ \{ \lambda_k = ik^2\pi^2 | k \geq 1 \} \]

with \( \phi_k(x) = \sqrt{2} \sin(n\pi x). \)

\[ \int_0^1 x\phi_j(x)\phi_k(x)dx \neq 0 \text{ if and only if } j - k \text{ is odd.} \]

There is no non-resonant connectedness chain, since \( \lambda_j - \lambda_k \neq \lambda_m - \lambda_l \) for all \((m, l) \neq (j, k), (k, j)\) only if \((j, k) = \left( \frac{p\pm1}{2}, \frac{p\mp1}{2} \right) \text{ with } p \text{ prime.}\)
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**Solution:** perturbation.
**Idea:** Let $A_\eta = A + \eta B$, $\eta \in [0, \delta]$. System

\[
\dot{\psi} = A + uB, \quad u \in [0, \delta]
\]

can be rewritten as

\[
\dot{\psi} = A_\eta \psi + vB \psi, \quad v \in [-\eta, \delta - \eta].
\]

Since $\eta \mapsto A_\eta$ is analytic, there exist $\phi_k(\cdot)$ and $\lambda_k(\cdot)$ analytic such that $(\lambda_k(\eta), \phi_k(\eta))_{k \in \mathbb{N}}$ is a complete system of eigenpairs for $A_\eta$ (Rellich-Kato theorem).
**Perturbation theory for the 1D potential well**

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Following the computations in [Beauchard & Mirrahimi, 2009],

$$\lambda_k(\eta) = k^2 \pi^2 + \left( \frac{1}{12 \pi^2 k^2} - \frac{5}{4 \pi^4 k^4} \right) \eta^2 + o(\eta^2).$$

We easily get that

$$\lambda_j''(0) - \lambda_k''(0) = \lambda_m''(0) - \lambda_l''(0) \implies (j, k) = (m, l).$$

For almost all $\eta \in (0, \delta)$, \{\((j, k) \in \mathbb{N}^2 \mid j - k \text{ odd}\}\} is a non-resonant connectedness chain $\implies$ approximate controllability.
Perturbation theory for the 1D potential well: more general control potentials

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} + u(t)W(x) \right) \psi(x, t), \quad \psi(0, t) = \psi(1, t) = 0 \]

The derivative of \( \lambda_k(u) \) with respect to \( u \) at \( u = 0 \) is

\[ \lambda_k'(0) = \int_{\mathbb{R}} W(x)\phi_k(x)^2 dx = 2 \int_{\mathbb{R}} W(x)\sin(k\pi x)^2 dx. \]

We look for \( W \) such that the non-resonance properties are satisfied by \( (\lambda_k'(0))_{k \in \mathbb{N}} \).

For instance, one easily check by direct computation that, for almost every \( \alpha \in \mathbb{R} \), \( W(x) = e^{\alpha x} \) allows to control.
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Since \( \phi_k(x)^2 = 2 \sin(k\pi x)^2 \) are linearly independent functions, for most \( W \in L^\infty(0, 1) \) the system is controllable.
**Generic: frequent and robust**

**Aim:** prove genericity of the sufficient conditions for controllability (related results in [Nersesyan, 2010])

Recall that a property is **generic** with respect to some parameter belonging to a metric space, if it is true for a **dense** set of parameters which is intersection of countably many **open** sets.
**Aim:** prove genericity of the sufficient conditions for controllability (related results in [Nersesyan, 2010])

Recall that a property is **generic** with respect to some parameter belonging to a metric space, if it is true for a dense set of parameters which is intersection of countably many open sets.

\[ i\dot{\psi} = -\Delta \psi + V\psi + uW\psi \]

Ω bounded domain of \(\mathbb{R}^d\) or \(\Omega = \mathbb{R}^d\); \(H = L^2(\Omega)\)

**Parameters:** \(V, W : \Omega \rightarrow \mathbb{R}\) and also \(\Omega\) in the bounded case

\(-\Delta + V\) has discrete spectrum if \(\lim_{x \rightarrow \infty} V(x) = +\infty\)
Baire spaces and topologies

\[ \Omega \rightarrow \Sigma_m = \{ \Omega \mid \Omega \text{ bounded domain with } C^m \text{ boundary} \}, \ m \in \mathbb{N} \]

\[ V \rightarrow V(\Omega) = \left\{ \begin{array}{l}
L^\infty(\Omega) \\
\{ V \in L_{loc}^\infty \mid \lim_{x \to \infty} V(x) = +\infty \}\end{array} \right\} \quad \Omega \text{ bdd} \]

\[ W \rightarrow W(\Omega) = \left\{ \begin{array}{l}
L^\infty(\Omega) \\
\{ W \in L_{loc}^\infty \mid \limsup_{x \to \infty} \frac{\log(|W(x)|+1)}{\|x\|} < \infty \}\end{array} \right\} \quad \Omega \text{ bdd} \]

\[ (V, W) \rightarrow Z(\Omega) = \{(V, W) \in V(\Omega) \times W(\Omega) \mid V + uW \in V(\Omega) \quad \forall u \in [0, \delta]\} \]

We endow these spaces with the $C^m$, $L^\infty$ and $L^\infty \times L^\infty$ topology.
Theorem (Rellich, Kato)

Let $I$ be an interval of $\mathbb{R}$ and $\Omega$ be a bounded domain or $\mathbb{R}^d$. Let $V \in \mathcal{V}(\Omega)$ and $\mu \mapsto W_\mu$ an analytic function from $I$ into $L^\infty(\Omega, \mathbb{R})$. Then, there exist $(\Lambda_k : I \to \mathbb{R})_{k \in \mathbb{N}}$

$(\Phi_k : I \to L^2(\Omega, \mathbb{R}))_{k \in \mathbb{N}}$

families of analytic functions such that for any $\mu \in I$ the sequence $(\Lambda_k(\mu))_{k \in \mathbb{N}}$ is the family of eigenvalues of $-\Delta + V + W_\mu$ counted according to their multiplicities and $(\Phi_k(\mu))_{k \in \mathbb{N}}$ is an orthonormal basis of corresponding eigenfunctions.
Let $\Omega$ be bounded.

Fix $\Omega$ and $V$ satisfying:

$$(\lambda_k(V, \Omega))_{k \in \mathbb{N}} \text{ non-resonant (all gaps are different)}$$

Then generically w.r.t. $W$ the system is approximately controllable, since every condition

$$\int_{\Omega} W \phi_k(V, \Omega) \phi_{k+1}(V, \Omega) dx \neq 0$$

defines an open dense subset of $\mathcal{W}$. 
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Then generically w.r.t. $W$ the system is approximately controllable, since every condition

$$\int_{\Omega} W\phi_k(V,\Omega)\phi_{k+1}(V,\Omega)dx \neq 0$$

defines an open dense subset of $\mathcal{W}$.

Now fix $\Omega$ such that the non-resonance condition is true for $(\lambda_k(0,\Omega))_k$. Then, by analytic perturbation, $(\lambda_k(\mu V,\Omega))_k$ is non-resonant for a generic $\mu \in \mathbb{R}$. In particular, generically w.r.t. $(V, W)$ the system is approximately controllable.
Similarly, fix $\Omega$ such that each $\lambda_k(0, \Omega)$ is simple and $(\phi_k(0, \Omega)^2)_k$ are linearly independent. Then, thanks to

$$\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V, \Omega) = \int_{\Omega} V \phi_k(0, \Omega)^2$$

generically with respect to $V$ the sequence $\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V, \Omega)$ is non-resonant. This would imply that generically w.r.t. $\mu$ the same is true for $\lambda_k(\mu V, \Omega)$. Again, generically w.r.t. $(V, W)$ the system is approximately controllable.
Similarly, fix \( \Omega \) such that each \( \lambda_k(0, \Omega) \) is simple and \((\phi_k(0, \Omega)^2)_k\) are linearly independent. Then, thanks to

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generically with respect to \( V \) the sequence \( \frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V, \Omega) \) is non-resonant. This would imply that generically w.r.t. \( \mu \) the same is true for \( \lambda_k(\mu V, \Omega) \). Again, generically w.r.t. \((V, W)\) the system is approximately controllable.

Resuming: if \( \Omega \) is such that either \((\lambda_k(0, \Omega))_k\) is non-resonant or \((\phi_k(0, \Omega)^2)_k\) is a free family, then generically w.r.t. \((V, W)\) the system is approximately controllable.
Theorem (Y. Privat, M. S.)

Generically with respect to $\Omega \in \Sigma_m$, $(\phi_k(0, \Omega)^2)_k$ is free and (for $d > 1$) $(\lambda_k(0, \Omega))_k$ is non-resonant.

Corollary

Generically with respect to

\[ \{(\Omega, V, W) \mid \Omega \in \Sigma_m, (V, W) \in \mathcal{Z}(\Omega)\} \] the Schrödinger equation

\[ i\dot{\psi} = -\Delta \psi + V \psi + uW\psi, \quad \psi|_{\partial\Omega} = 0, \quad u \in [0, \delta] \]

is approximately controllable for every $\delta > 0$. 

The openness of the sets of parameters (here, domains $\Omega$) corresponding to each non-resonance condition follows from standard continuity results. The hard point is their density.

**Global perturbation**
If one $\Omega$ satisfying the non-resonance can be found, consider any analytic path starting from $\Omega$ in order to propagate the good property. The property will be true for all but countably many points of the path, hence, for almost every domain with the same topology as $\Omega$.

**Local step**
Use local perturbations to get a domain $\Omega$ with a prescribed topology satisfying the desired non-resonance property
If $\lambda_2$ crosses $\lambda_3$ along the analytic perturbation, then the condition $
abla 4 - \lambda_2 \neq \lambda_5 - \lambda_4$ becomes $\lambda_4 - \lambda_3 \neq \lambda_5 - \lambda_4$. 
If $\lambda_2$ crosses $\lambda_3$ along the analytic perturbation, then the condition $\lambda_4 - \lambda_2 \neq \lambda_5 - \lambda_4$ becomes $\lambda_4 - \lambda_3 \neq \lambda_5 - \lambda_4$.

Strategy to avoid the bad effect of eigenvalue rearrangement along the path: elude intersections by small modifications of the analytic path (Arnold, Colin de Verdière, Teytel [1999]).
If $\lambda_2$ crosses $\lambda_3$ along the analytic perturbation, then the condition $\lambda_4 - \lambda_2 \neq \lambda_5 - \lambda_4$ becomes $\lambda_4 - \lambda_3 \neq \lambda_5 - \lambda_4$.

Strategy to avoid the bad effect of eigenvalue rearrangement along the path: elude intersections by small modifications of the analytic path (Arnold, Colin de Verdière, Teytel [1999]).

This is possible because eigenvalue intersections is a somehow rare phenomenon: the eigenvalues of

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

are double if $a = c$ and $b = 0$, two conditions on three parameters! (Von Neumann-Wigner [1929], Lupo-Micheletti [1995], Lamberti-Lanza de Cristoforis [2006]).
It is possible to obtain stronger genericity results for the Schrödinger equation for any fixed $\Omega$ bounded domain or $\Omega = \mathbb{R}^d$.

**Proposition (P. Mason, M. S.)**

Fix $\Omega$. Then, generically with respect to $V$, $(\lambda_k(V, \Omega))_k$ is non-resonant.

**Corollary**

Fix $\Omega$. Generically with respect to $(V, W) \in \mathcal{Z}(\Omega)$ the Schrödinger equation is approximately controllable.
The potential well lemma

Lemma

Fix $\Omega$ (bdd or $\mathbb{R}^d$). Let $\omega$ be a compactly contained subdomain of $\Omega$ with Lipschitz boundary, $v \in L^\infty(\omega)$ and $(V_k)_{k \in \mathbb{N}} \subset \mathcal{V}(\Omega)$ such that

$$
V_k|_\omega \rightarrow v \quad \text{in } L^\infty(\omega)
$$

$$
\lim_{k \rightarrow \infty} \inf_{\Omega \setminus \omega} V_k = +\infty
$$
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Fix $\Omega$ (bdd or $\mathbb{R}^d$). Let $\omega$ be a compactly contained subdomain of $\Omega$ with Lipschitz boundary, $\nu \in L^\infty(\omega)$ and $(V_k)_{k \in \mathbb{N}} \subset \mathcal{V}(\Omega)$ such that

$$V_k|_{\omega} \to \nu \quad \text{in } L^\infty(\omega)$$

$$\lim_{k \to \infty} \inf_{\Omega \setminus \omega} V_k = +\infty$$

Then, for every $j \in \mathbb{N}$,

$$\lambda_j(\Omega, V_k) \to \lambda_j(\omega, \nu) \quad \text{in } \mathbb{R}.$$  

Moreover, if $\lambda_j(\omega, \nu)$ is simple then (up to the sign)

$$\phi_j(\Omega, V_k) \to \phi_j(\omega, \nu), \sqrt{V_k} \phi_j(\Omega, V_k) \to \sqrt{\nu} \phi_j(\omega, \nu) \quad \text{in } L^2(\Omega, \mathbb{C}).$$
Theorem (P. Mason, M. S.)

Fix $\Omega$ bdd or $\mathbb{R}^d$ and $W \in \mathcal{W}(\Omega)$ absolutely continuous and non-constant. Generically with respect to $V$ in

$\{Z \in \mathcal{V}(\Omega) \mid (V, Z) \in \mathcal{Z}(\Omega)\}$ the Schrödinger equation is approximately controllable.
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Theorem (P. Mason, M. S.)

Fix $\Omega$ bdd or $\mathbb{R}^d$ and $V \in \mathcal{V}(\Omega)$ absolutely continuous. Generically with respect to $W \in \{Z \in \mathcal{V}(\Omega) \mid (V, Z) \in \mathcal{Z}(\Omega)\}$ the Schrödinger equation is approximately controllable.