Approximate controllability of the bilinear Schrödinger equation

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With a piecewise constant $u : [0, T] \to U$ we associate the flow

$$t \mapsto \Upsilon_t^u = e^{(t-t_k)(A+u_k B)} \circ e^{(t_k-t_{k-1})(A+u_{k-1}B)} \circ \cdots \circ e^{t_1(A+u_1 B)}$$

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which is a unitary transformation of H.
We are interested to approximate controllability in the unit sphere of H.

Controllability result: reminder

$(\lambda_n)_{n\in\mathbf{N}}$ eigenvalues of A corresponding to $(\phi_n)_{n\in\mathbf{N}}$.

Theorem (Boscain, Caponigro, Chambrion, S.)

Assume that there exists $S \subset \mathbf{N}^2$ such that

- *S connects $\mathbf{N}$*
- *$\langle \phi_j, B\phi_k \rangle \neq 0$ for every $(j,k) \in S$*
- *each λ_j is simple*
- *$(j,k) \in S$ and $(j,k) \neq (m,l) \in \mathbf{N}^2 \Longrightarrow \lambda_j - \lambda_k \neq \lambda_m - \lambda_l$ or $\langle \phi_m, B\phi_l \rangle = 0$.*

Then

$$\frac{d\psi}{dt} = A(\psi) + uB(\psi), \qquad u \in [0,\delta],$$

is approximately controllable for every $\delta > 0$.

Remarks on the approximate controllability result

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- upper and lower bound on the L^1 norm of the control
- lower bound on the controllability time
- controllability extends to density matrices and simultaneous control (or tracking: approximating any prescribed unfeasible trajectory arbitrarily well up to the phases)

Rotating bipolar molecule

$$i\frac{\partial\psi(\theta,t)}{\partial t} = \left(-\frac{\partial^2}{\partial\theta^2} + u_1(t)\cos(\theta) + u_2(t)\sin(\theta)\right)\psi(\theta,t), \quad \theta \in S^1$$

- θ rotational degree of freedom of a bipolar rigid molecule confined to a plane
- controlled fields pointing in the directions $(0,1)$ and $(1,0)$
- $u_1(t), u_2(t) \in [0,\delta]$
- if $u_2 \equiv 0$ controllability between even wavefunctions and no transfer of probability between odd and even ones
- in [Boscain, Chambrion, Mason, Sigalotti, Sugny, 48th IEEE CDC, 2009] we studied the problem of controlling simultaneously the even and the odd part, with $u_2 \equiv 0$

Rotating bipolar molecule: controllability with 1D controls

For every $\alpha \in S^1$, consider the splitting

$$H = H_e^\alpha \oplus H_o^\alpha$$

with H_e^α and H_o^α the Hilbert spaces of, respectively, even and odd functions with respect to α.
Our first result is the generalization of the partial controllability with $u_2 \equiv 0$.

Lemma

Let $\alpha \in [0, \pi/2]$. Then the system restricted to H_e^α (or H_o^α) with controls in

$$\mathcal{U}_\alpha = \{u : \mathbf{R} \to [0, \delta]^2 \cap \mathbf{R}(\cos\alpha, \sin\alpha) \mid u \textit{ piecewise constant}\}$$

is approximately controllable.

Rotating bipolar molecule: controllability with 1D controls

The control system with 1D controls can be rewritten as

$$i\frac{\partial\psi(\theta,t)}{\partial t} = \left(-\frac{\partial^2}{\partial\theta^2} + v(t)\cos(\theta-\alpha)\right)\psi(\theta,t)$$

$$v \in \left(0, \delta\sqrt{1+\min\{\tan\alpha, \operatorname{cotan}\alpha\}^2}\right).$$

Complete orthonormal systems for H_e^α and H_o^α of eigenfuntions of A are given by $\{\cos(k(\cdot-\alpha))/\sqrt{\pi}\}_{k=0}^{\infty}$ and $\{\sin(k(\cdot-\alpha))/\sqrt{\pi}\}_{k=1}^{\infty}$, respectively.

The sufficient conditions for controllability are easily tested:

$A = \mathrm{diag}(k^2/\pi)$

$$B = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \cdots & & & \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & \cdots & & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}$$

$S = \{(j,k) \mid |j-k| = 1\}$

Rotating bipolar molecule: controllability between eigenfunctions

Remark: Any eigenfunction is even with respect to some $\alpha \in [0, \pi/2]$

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Controllability with 1D controls

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Any eigenfunction can be approximately driven to the ground state $\phi(x) \equiv 1/\sqrt{2\pi}$

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Any eigenfunction can be approximately driven to the ground state $\phi(x) \equiv 1/\sqrt{2\pi}$

For the same reason, the ground state can be approximately driven to any other eigenfunction.

Concatenating the two controls we get approximate controllability between eigenfunctions.

Rotating bipolar molecule: approximate controllability

Let us prove approximate controllability with 2D controls.
It is enough to prove that every wavefunction $\psi \in H$ of norm one can be steered ε-close to the constant $1/\sqrt{2\pi}$, for $\varepsilon > 0$ arbitrary (time-reversibility).

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It is enough to prove that every wavefunction $\psi \in H$ of norm one can be steered ε-close to the constant $1/\sqrt{2\pi}$, for $\varepsilon > 0$ arbitrary (time-reversibility).
Take $\alpha \in (0, \pi/2)$.
Using controls in $\mathcal{U}_\alpha$, we can steer the α-even part of ψ in a ε-neighborhood of $\|\psi_e^\alpha\|/\sqrt{2\pi}$.
Then ψ goes to

$$\tilde{\psi} \simeq \frac{\|\psi_e^\alpha\|}{\sqrt{2\pi}} + \phi_1, \quad \text{with } \phi_1 \in H_o^\alpha.$$

If $\|\phi_1\|$ is smaller than ε then we are done.

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If $\|\phi_1\|$ is smaller than ε then we are done.
Assume then that $\|\phi_1\| \geq \varepsilon$ and consider, for every $\beta \in S^1$, $\tau_\beta = \|(\phi_1)_e^\beta\|^2$.

Rotating bipolar molecule: approximate controllability

$$\tilde{\psi} \simeq \frac{\|\psi_e^\alpha\|}{\sqrt{2\pi}} + \phi_1, \quad \phi_1 \in H_o^\alpha,\ \|\phi_1\| \geq \varepsilon,\ \tau_\beta = \|(\phi_1)_e^\beta\|^2.$$

A computation shows that there exists $c > 0$ independent on k and α and there exists $\beta \in (0, \pi/2)$ such that

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We can repeat the step of controlling the even part towards a constant using controls in $\mathcal{U}_\beta$.
At every step we end up with a wavefunction whose even part with respect to some angle is approximately constant, and the constant grows by some uniform amount.
Iterating the procedure finitely many times, the final wavefunction is ε-close to the constant $1/\sqrt{2\pi}$.

Example: 1D potential well

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} + u(t)x\right)\psi(x,t), \quad x \in (0,1),$$

with $\psi(0,t) = \psi(1,t) = 0$.
Exact controllability between regular enough wavefunctions:
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The spectrum of $-\Delta + V$ is

$$\left\{\lambda_k = ik^2\pi^2 \mid k \geq 1\right\}$$

with $\phi_k(x) = \sqrt{2}\sin(n\pi x)$.
$\int_0^1 x\phi_j(x)\phi_k(x)dx \neq 0$ if and only if $j-k$ is odd.
There is no non-resonant connectedness chain, since
$\lambda_j - \lambda_k \neq \lambda_m - \lambda_l$ for all $(m,l) \neq (j,k),(k,j)$ only if
$(j,k) = \left(\frac{p\pm1}{2}, \frac{p\mp1}{2}\right)$ with p prime.

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$(j,k) = \left(\frac{p\pm1}{2}, \frac{p\mp1}{2}\right)$ with p prime.
SOLUTION: perturbation.

Perturbation theory for the 1D potential well

IDEA: Let $A_\eta = A + \eta B$, $\eta \in [0,\delta]$. System

$$\dot{\psi} = A + uB, \;\; u \in [0,\delta]$$

can be rewritten as

$$\dot{\psi} = A_\eta \psi + vB\psi, \;\; v \in [-\eta, \delta - \eta].$$

Since $\eta \mapsto A_\eta$ is analytic, there exist $\phi_k(\cdot)$ and $\lambda_k(\cdot)$ analytic such that $(\lambda_k(\eta), \phi_k(\eta))_{k\in\mathbf{N}}$ is a complete system of eigenpairs for A_η (Rellich-Kato theorem).

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Following the computations in [Beauchard & Mirrahimi,2009],

$$\lambda_k(\eta) = k^2\pi^2 + \left(\frac{1}{12\pi^2 k^2} - \frac{5}{4\pi^4 k^4} \right) \eta^2 + \mathrm{o}(\eta^2).$$

We easily get that

$$\lambda_j''(0) - \lambda_k''(0) = \lambda_m''(0) - \lambda_l''(0) \Longrightarrow (j, k) = (m, l).$$

For almost all $\eta \in (0, \delta)$, $\{(j, k) \in \mathbf{N}^2 \mid j - k \text{ odd}\}$ is a non-resonant connectedness chain $\Longrightarrow$ approximate controllability

Perturbation theory for the 1D potential well: more general control potentials

$$i\frac{\partial\psi(x,t)}{\partial t} = \left(-\frac{\partial^2}{\partial x^2} + u(t)W(x)\right)\psi(x,t), \qquad \psi(0,t) = \psi(1,t) = 0$$

The derivative of $\lambda_k(u)$ with respect to u at $u = 0$ is

$$\lambda_k'(0) = \int_{\mathbf{R}} W(x)\phi_k(x)^2 dx = 2\int_{\mathbf{R}} W(x)\sin(k\pi x)^2 dx.$$

We look for W such that the non-resonance properties are satisfied by $(\lambda_k'(0))_{k\in\mathbf{N}}$.

For instance, one easily check by direct computation that, for almost every $\alpha \in \mathbf{R}$, $W(x) = e^{\alpha x}$ allows to control.

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Since $\phi_k(x)^2 = 2\sin(k\pi x)^2$ are linearly independent functions, for most $W \in L^\infty(0,1)$ the system is controllable.

Generic: frequent and robust

AIM: prove genericity of the sufficient conditions for controllability (related results in [Nersesyan, 2010])

Recall that a property is generic with respect to some parameter belonging to a metric space, if it is true for a dense set of parameters which is intersection of countably many open sets.

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$$i\dot{\psi} = -\Delta\psi + V\psi + uW\psi$$

Ω bounded domain of $\mathbf{R}^d$ or $\Omega = \mathbf{R}^d$; $H = L^2(\Omega)$

PARAMETERS: $V, W : \Omega \to \mathbf{R}$ and also Ω in the bounded case

$-\Delta + V$ has discrete spectrum if $\lim_{x\to\infty} V(x) = +\infty$

Baire spaces and topologies

$$\Omega \;\rightarrow\; \Sigma_m = \{\Omega \mid \Omega \text{ bounded domain with } \mathcal{C}^m \text{ boundary}\}, m \in \mathbf{N}$$

$$V \;\rightarrow\; \mathcal{V}(\Omega) = \begin{cases} L^\infty(\Omega) & \Omega \text{ bdd} \\ \{V \in L^\infty_{loc} \mid \lim_{x\to\infty} V(x) = +\infty\} & \Omega = \mathbf{R}^d \end{cases}$$

$$W \;\rightarrow\; \mathcal{W}(\Omega) = \begin{cases} L^\infty(\Omega) & \Omega \text{ bdd} \\ \{W \in L^\infty_{loc} \mid \limsup_{x\to\infty} \frac{\log(|W(x)|+1)}{\|x\|} < \infty\} & \Omega = \mathbf{R}^d \end{cases}$$

$$(V, W) \;\rightarrow\; \mathcal{Z}(\Omega) = \{(V, W) \in \mathcal{V}(\Omega)\times\mathcal{W}(\Omega) \mid V+uW \in \mathcal{V}(\Omega) \quad \forall u \in [0, \delta]\}$$

We endow these spaces with the $\mathcal{C}^m$, L^∞ and $L^\infty \times L^\infty$ topology

Analytic dependence

Theorem (Rellich, Kato)

Let I be an interval of $\mathbf{R}$ and Ω be a bounded domain or $\mathbf{R}^d$. Let $V \in \mathcal{V}(\Omega)$ and $\mu \mapsto W_\mu$ an analytic function from I into $L^\infty(\Omega, \mathbf{R})$. Then, there exist

$$(\Lambda_k : I \to \mathbf{R})_{k \in \mathbf{N}}$$

$$(\Phi_k : I \to L^2(\Omega, \mathbf{R}))_{k \in \mathbf{N}}$$

families of analytic functions such that for any $\mu \in I$ the sequence $(\Lambda_k(\mu))_{k \in \mathbf{N}}$ is the family of eigenvalues of $-\Delta + V + W_\mu$ counted according to their multiplicities and $(\Phi_k(\mu))_{k \in \mathbf{N}}$ is an orthonormal basis of corresponding eigenfunctions.

Analytic propagation of non-vanishing conditions and the role of the Laplace–Dirichelet operator when Ω is bounded

Let Ω be bounded.

Fix Ω and V satisfying:

$$(\lambda_k(V,\Omega))_{k\in\mathbf{N}} \text{ non-resonant (all gaps are different)}$$

Then generically w.r.t. W the system is approximately controllable, since every condition

$$\int_\Omega W\phi_k(V,\Omega)\phi_{k+1}(V,\Omega)dx \neq 0$$

defines an open dense subset of $\mathcal{W}$.

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Now fix Ω such that the non-resonance condition is true for $(\lambda_k(0,\Omega))_k$. Then, by analytic perturbation, $(\lambda_k(\mu V,\Omega))_k$ is non-resonant for a generic $\mu\in\mathbf{R}$. In particular, generically w.r.t. (V,W) the system is approximately controllable.

Analytic propagation of non-vanishing conditions and the role of the Laplace–Dirichelet operator when Ω is bounded

Similarly, fix Ω such that each $\lambda_k(0,\Omega)$ is simple and $(\phi_k(0,\Omega)^2)_k$ are linearly independent. Then, thanks to

$$\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V,\Omega)=\int_\Omega V\phi_k(0,\Omega)^2$$

generically with respect to V the sequence $\frac{d}{d\mu}|_{\mu=0}\lambda_k(\mu V,\Omega)$ is non-resonant. This would imply that generically w.r.t. μ the same is true for $\lambda_k(\mu V,\Omega)$. Again, generically w.r.t. (V,W) the system is approximately controllable.

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Resuming: if Ω is such that either $(\lambda_k(0,\Omega))_k$ is non-resonant or $(\phi_k(0,\Omega)^2)_k$ is a free family, then generically w.r.t. (V,W) the system is approximately controllable.

Generic approximate controllability

Theorem (Y. Privat, M. S.)

Generically with respect to $\Omega \in \Sigma_m$, $(\phi_k(0,\Omega)^2)_k$ is free and (for $d > 1$) $(\lambda_k(0,\Omega))_k$ is non-resonant.

Corollary

Generically with respect to $\{(\Omega, V, W) \mid \Omega \in \Sigma_m,\ (V, W) \in \mathcal{Z}(\Omega)\}$ the Schrödinger equation

$$i\dot{\psi} = -\Delta\psi + V\psi + uW\psi, \quad \psi|_{\partial\Omega} = 0, \quad u \in [0, \delta]$$

is approximately controllable for every $\delta > 0$.

Techniques

The openness of the sets of parameters (here, domains Ω) corresponding to each non-resonance condition follows from standard continuity results. The hard point is their density.

Global perturbation

If one Ω satisfying the non-resonance can be found, consider any analytic path starting from Ω in order to *propagate* the good property. The property will be true for all but countably many points of the path, hence, for almost every domain with the same topology as Ω.

Local step

Use local perturbations to get a domain Ω with a prescribed topology satisfying the desired non-resonance property

Tricky point of the global perturbation analysis: intersection of eigenvalues

If λ_2 crosses λ_3 along the analytic perturbation, then the condition $\lambda_4 - \lambda_2 \neq \lambda_5 - \lambda_4$ becomes $\lambda_4 - \lambda_3 \neq \lambda_5 - \lambda_4$.

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This is possible because eigenvalue intersections is a somehow rare phenomenon: the eigevalues of

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

are double if $a = c$ and $b = 0$, two conditions on three parameters! (Von Neumann-Wigner [1929], Lupo-Micheletti [1995], Lamberti-Lanza de Cristoforis [2006]).

Back to the Schrödinger equation

It is possible to obtain stronger genericity results for the Schrödinger equation for any fixed Ω bounded domain or $\Omega = \mathbf{R}^d$.

Proposition (P. Mason, M. S.)

Fix Ω. Then, generically with respect to V, $(\lambda_k(V,\Omega))_k$ is non-resonant.

Corollary

Fix Ω. Generically with respect to $(V,W) \in \mathcal{Z}(\Omega)$ the Schrödinger equation is approximately controllable.

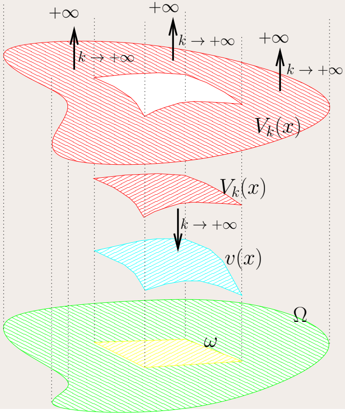
The potential well lemma

Lemma

Fix Ω (bdd or $\mathbf{R}^d$). Let ω be a compactly contained subdomain of Ω with Lipschitz boundary, $v \in L^\infty(\omega)$ and $(V_k)_{k\in\mathbf{N}} \subset \mathcal{V}(\Omega)$ such that

$$V_k|_\omega \to v \quad \text{in } L^\infty(\omega)$$

$$\lim_{k\to\infty} \inf_{\Omega\setminus\omega} V_k = +\infty$$



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$V_k|_\omega \to v \quad$ *in* $L^\infty(\omega)$

$\lim_{k\to\infty} \inf_{\Omega\setminus\omega} V_k = +\infty$

Then, for every $j \in \mathbf{N}$,

$$\lambda_j(\Omega, V_k) \to \lambda_j(\omega, v) \quad \textit{in } \mathbf{R}.$$

Moreover, if $\lambda_j(\omega, v)$ is simple then (up to the sign)

$$\phi_j(\Omega, V_k) \to \phi_j(\omega, v), \sqrt{V_k}\phi_j(\Omega, V_k) \to \sqrt{v}\phi_j(\omega, v) \quad \textit{in } L^2(\Omega, \mathbf{C}).$$

Genericity with respect to one single argument

Theorem (P. Mason, M. S.)

Fix Ω bdd or $\mathbf{R}^d$ and $W \in \mathcal{W}(\Omega)$ absolutely continuous and non-constant. Generically with respect to V in $\{Z \in \mathcal{V}(\Omega) \mid (Z, W) \in \mathcal{Z}(\Omega)\}$ the Schrödinger equation is approximately controllable.

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