Geometric Models with Co-occurrence Groups

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Abstract. A geometric model of sparse signal representations is introduced for classes of signals. It is computed by optimizing co-occurrence groups with a maximum likelihood estimate calculated with a Bernoulli mixture model. Applications to face image compression and MNIST digit classification illustrate the applicability of this model.

1 Introduction

Finding image representations with a dimensionality reduction while maintaining relevant information for classification, remains a major issue. Effective approaches have recently been developed based on locally orderless representations as proposed by Koendering and Van Doom [1]. They observed that high frequency structures are important for recognition but do not need to be precisely located. This idea has inspired a family of descriptors such as SIFT [2] or HOG [3], which delocalize the image information over large neighborhoods, by only recording histogram information. These histograms are usually computed over wavelet like coefficients, providing a multiscale image representation with several wavelets having different orientation tunings.

This paper introduces a new geometric image representation obtained by grouping coefficients that have co-occurrence properties across an image class. It provides a locally orderless representation where sparse descriptors are delocalized over groups which optimize the coefficient co-occurrences, and can be interpreted as a form of parcellization [4]. Section 2 reviews wavelet image representations and the notion of sparse geometry through significant sets. Section 3 introduces our co-occurrence grouping model which is optimized with a maximum likelihood approach. Groups are computed from a training sequence in Section 4, using a Bernoulli mixture approximation. Applications to face image compression are shown in Section 5 and the application of this representation is illustrated for MNIST image classifications in Section 6.

2 Geometric Significance Set

Sparse signal representations are obtained by decomposing signals over bases or frames $\{\phi_p\}_{p \in \bar{y}}$ which take advantage of the signal regularity to produce many zero coefficients. A sparse representation is obtained by keeping the significant coefficients above a threshold T,

$$y = \{p \in \bar{y} \ \ : \ \ |\langle f, \phi_p \rangle| > T \ \}$$

The original signal can be reconstructed with a dual family $f = \sum_{p\in\bar{y}} \langle f, \phi_p \rangle \tilde{\phi}_p$, and the resulting sparse approximation is $f_y = \sum_{p\in y} \langle f, \phi_p \rangle \tilde{\phi}_p$.

Wavelet transforms compute signal inner products with several mother wavelets ψ^d having a specific direction tuning, and which are dilated by 2^j and translated by $2^j n$: $\phi_p = \psi^d_{j,n}$. Separable wavelet bases are obtained with 3 mother wavelets [5], in which case the total number $|\bar{y}|$ of wavelets is equal to the image size.

Let $|y|$ be the cardinal of the set y. In absence of prior information on y, the number of bits needed to code y in $\bar{y}$ is $R_0 = \log_2 \binom{|\bar{y}|}{|y|}$.

One can also verify [5] that the number of bits required to encode the values of coefficients in y is proportional to $|y|$ and is smaller than R_0 so that the coding budget is indeed dominated by R_0 which carries most of the image information.

3 Co-occurrence Groups

In a supervised classification problem, a geometric model defines a prior model of the probability distribution $q(y)$. There is a huge number $2^{|\bar{y}|}$ of subsets y in $\bar{y}$. Estimating the probability $q(y)$ from a limited training set thus requires using a simplified prior model.

A signal class is represented by a random vector whose realizations are within the class and whose significance sets y are included in $\bar{y}$. A mixture model is introduced with co-occurrence groups $\theta(k)$ of constant size s, which define a partition of the overall index set $\bar{y}$

$$\bar{y} = \cup_k \theta(k) \;\text{ with }\; |\theta(k)| = s \;\text{ and }\; \theta(k) \cap \theta(k') = \emptyset \;\text{ if }\; k \neq k' \;.$$

Co-occurrence groups $\theta(k)$ are optimized by enforcing that all coefficients have a similar behavior in a group and hence that $y \cap \theta(k)$ is either almost empty or almost equal to $\theta(k)$ with a high probability. The mixture model assumes that the distributions of the components $y \cap \theta(k)$ are independent. The distribution $q(y \cap \theta(k))$ is assumed to be uniform among all subsets of $\theta(k)$ of cardinal $z(k) = |y \cap \theta(k)|$. Let $q_k(z(k))$ be its distribution,

$$q(y|\theta) = \prod_k q(y \cap \theta(k)) = \prod_k q_k(z(k)) \binom{s}{z(k)}^{-1}$$

This co-occurrence model is identified with a maximum log-likelihood approach which computes

$$\arg\max_\theta \sum_k \left(-\log_2 \binom{s}{z(k)} + \log_2 q_k(z(k)) \right) \;.$$

4 Group Estimation with Bernoulli Mixtures

Given a training sequence of images $\{f_l\}_{l \le L}$ that belong to a class, we optimize the group co-occurrence by approximating the maximum likelihood with a Bernoulli mixture.

Let y_l be the significant set of f_l. The log likelihood is calculated with

$$\mathcal{L}(y,\theta) = \sum_l \sum_k \Big(-\log_2 \binom{s}{z_l(k)} + \log_2 q_k(z_l(k)) \Big) \text{ with } z_l(k) = |y_l \cap \theta(k)| \ . \quad (1)$$

The maximization of this expression is obtained using the Stirling formula which approximates the first term by the entropy of a Bernoulli distribution. Let us write $q_{k,l}(0) = z_l(k)/s$ and $q_{k,l}(1) = 1 - z_l(k)/s$, the Bernoulli probability distribution associated to $z_l(k)/s$. Let us specify the groups $\theta(k)$ by the inverse variables $k(p)$ such that $p \in \theta(k(p))$. It results that

$$\begin{aligned} \sum_k -\log_2 \binom{s}{z_l(k)} &\approx z_l(k)\ \log_2 \left(\frac{z_l(k)}{s}\right) + (s - z_l(k)) \log_2 \left(1 - \frac{z_l(k)}{s}\right) \\ &= \sum_{p \in \bar{y}} \log_2 q_{k(p),l}(1_{y_l}(p)) \ . \end{aligned}$$

The distribution q_k is generally unknown and must therefore be estimated. The estimation is regularized by approximating this distribution with a piecewise constant distribution $\hat{q}_k$ over a fixed number of quantization bins, that is small relatively to the number of realizations L. The likelihood (1) is thus approximated by a likelihood over the Bernoulli mixture, which is optimized over all parameters:

$$\arg \min_{\theta, z_l, \hat{q}_k} - \sum_l (\sum_{p \in \bar{y}} \log q_{k(p),l}(1_{y_l}(p)) + \sum_k \log_2 \hat{q}_k(z_l(k))) \ . \quad (2)$$

The following algorithm, minimizes (2) by updating separately the Bernoulli parameters $z_l(k)$, the distribution $\hat{q}_k$ and the grouping variables $k(p)$.

The minimization algorithm begins with a random initialization of groups $\theta(k)$ of same size s. The empirical histograms $\hat{q}_k$ are initialized to uniform distributions. The algorithm iterates the following steps:

- Step 1: Given $\{\theta(k)\}_k$ and $\{\hat{q}_k\}_k$ compute $\{z_l(k)\}_{l,k}$ which minimizes (2) by minimizing

$$-\log_2 \hat{q}_k(z_l(k)) - z_l(k)\ \log_2 \frac{z_l(k)}{s} - (s - z_l(k)) \log_2 (1 - \frac{z_l(k)}{s})) \ . \quad (3)$$

- Step 2: Update $\{\hat{q}_k\}_k$ to minimize (2) as the normalized histogram of the updated parameters $\{z_l(k)\}_l$ over a predefined number of bins.
- Step 3: Update the group indexes $\{k(p)\}_p$ to minimize (2) by minimizing

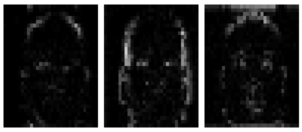
$$-\sum_{p \in \bar{y}} \log q_{k(p),l}(1_{y_l}(p)) \ , \quad (4)$$

  for groups of constant size $|\theta(k)| = s$.

This algorithm is guaranteed to converge to a local maxima because each step further increases the log-likelihood. In fact, it is the equivalent of the K-means algorithm adapted to the mixture model considered here.

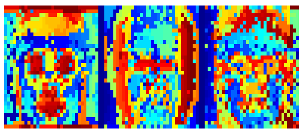
5 Face Compression

To illustrate the efficiency of this grouping strategy, it is first applied to the compression of face images that have been approximately registered. A database of 170 face images were used for training and a different set of 30 face images were used for testing. Figure 1 shows the optimal co-occurrence groups obtained over wavelet coefficients by applying the maximum log-likelihood algorithm on the training set. The encoding cost of the significance map using the optimized model is equal to minus the log-likelihood of this model. Figure 2 shows the evolution of the average bit budget needed to encode the significance maps with the Bernoulli mixture over optimized co-occurrence groups, depending upon the groups size s. The optimal group size which maximizes the log-likelihood and hence minimizes the encoding cost over all group sizes is $s = 16$.



(a)

(b)



(c)

Fig. 1: (a): Images of wavelet coefficients $|\langle f, \psi^d_{j,n}\rangle|$ for three directions $d = 1, 2, 3$ at a scale $2^j = 2^2$ (b): thresholded coefficients, defining the significance maps y_l. (c): grouping obtained with optimal group size $s = 16$. The stable geometric features are clearly visible.

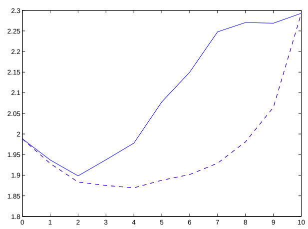


Fig. 2: *Solid*: bit rate using fixed square groups of size s as a function of $\log_2 s$. *Dashed*: bit rate (equal to minus the log likelihood, in bits per pixel) using the optimal groups of size s as a function of $\log_2 s$.

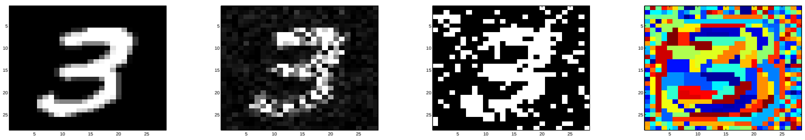


Fig. 3: Digit example. *From left to right*: original digit taken from MNIST database, random digit, significance maps, and grouping obtained by using the described algorithm.

When s is equal to the image size, there is a single group and the encoding is thus equivalent to a standard image coding using no prior information on the class. The bit rate is also compared with a Bernoulli mixture computed with a partition into square groups $\theta(k)$, as a function of s. Figure 2 shows that the optimized co-occurrence grouping improves the bit rate by 20 % relatively to the case where there is a single group, and also with respect to the fixed square groups, which means that the optimal grouping provides a geometric information which is stable across the image class. The optimal group size $s = 16$ also gives an estimation of the image deformations that are due to variations of scaling and eye positions and to intrinsic variations of faces in the database.

6 Random MNIST Digit Classification

This section shows the classification ability of our geometric representation despite the presence of strong variability in the images. The test is performed using the standard MNIST database of digits. This database is relatively simple and without any modification of the image representation an SVM classifier can reach 1.4% of error with a training set of 60,000 images. This section shows that our geometric co-occurence model can learn with much less training elements and for more complex images.

To take into account texture variation phenomena, which are a central difficulty for geometric models, a white noise texture is introduced. A digit image $f[n]$ is transformed into a random digit $\tilde{f}[n] = (f[n] + C)W[n]$ where $W[n]$ is a normalized Gaussian white noise. The significance maps of these digits are simply obtained with a thresholding as shown in Figure 3. It yields a binary image with a low density binary texture on the digit background and high density texture on the digit support. Visually, the digit is still perfectly recognizable despite the texture variability. With 4000 training images an SVM with a polynomial kernel yields a very low recognition rate of **21%** on a different set of 10000 test images.

Figure 3 shows the optimal co-occurrence groups of size 14 computed with the minimization algorithm of Section 4. Despite the geometric variability, the algorithm is able to extract co-occurrence groups that do correspond to the digit structures and their deformations. To each digit $0 \leq d \leq 9$, corresponds an optimized co-occurrence grouping θ_d. Let $\mathcal{L}(y, \theta_d)$ be the likelihood of the significance map y of f with the grouping model θ_d. An SVM classifier is trained on the feature vector $\{\mathcal{L}(y, \theta_d(k))\}_{0 \le d \le 9, 0 \le k \le K}$, of dimension $10 \cdot 56$ with groups of size 14. With 4000 training images this classifier yields a recognition rate of **9%** on a different set of 10000 test images. A simple maximum likelihood classifier (MAP) associates to each test image f the digit class

$$\tilde{d} = \arg \max_{0 \le d \le 9} \mathcal{L}(f, \theta_d) \ .$$

With 4000 training examples, this simple classifier yields a recognition rates of 18% for random digits, which is already better than the SVM applied on the original pixels.

7 Conclusion

This paper introduces a new approach to define the geometry of a class of images computed over a sparse representation, using co-occurrence groups. These co-occurrence groups are computed with a maximum log likelihood estimation calculated over optimized Bernoulli mixture model. An algorithm is introduced to optimize the group computation. The application to face image compression shows the efficiency of this encoding approach, and the ability to compute co-occurrence groups that provide stable information across the class. A classification test is performed over textured digits, which shows that the algorithm can take into account texture geometry and provide much better classification rates than a standard pixel based image representation.

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