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LNG portfolio optimization
approach by stochastic programming techniques

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Abstract

The work presented in this Ph.D dissertation is motivated by the problem of management of a fleet of cargos transporting liquefied natural gas (LNG) initially proposed by Total. The holder of the portfolio has to meet its commitments towards its counterparts while trying to generate profits through arbitrating different commodities market. Thus, the management of portfolio can be modeled as a stochastic, dynamic and integer optimization problem.

This Ph.D dissertation is organised as follows:

Chapter I  We first present the LNG portfolio management problem and give the mathematical model. Then we summarize the main results of this work.

Chapter II  We introduce a numerical method for solving continuous relaxation problem. We propose an algorithm based on the combination of the vectorial quantization method as discretization method and the dual dynamic programming approach. We show the convergence of numerical schema and give the error analysis on the discretization by quantization. Some numerical tests on real energy market problem are performed.

Chapter III  We also study the risk averse optimization by using conditional value at risk (CVaR) as criterion. We show that the algorithm proposed in chapter II is also adapted to such formulation. Furthermore, we propose the technique of changes in probability measure in stochastic programming in order to improve rare scenario simulation. Same numerical test as in Chapter II is performed in order to make comparison.

Chapter IV  We study the sensitivity of the portfolio with respect to several parameters in the market price model. We proposed a numerical method to compute sensitivity value based on Danskin’s theorem. The convergence of sensitivity value of discretized problem to the one of original problem is proved. Comparison between result obtained by algorithm in chapter II and other classical methods are provided.

Chapter V  We study the stochastic integer programming problem. The integrality cutting plane method is applied to approximate the integer problem. We show that it is impossible to converge to the integer solution because of the non convexity and discontinuity of the Bellman value function. We apply a heuristic method and propose a small improvement. Some numerical tests are also provided.
Keywords: stochastic programming, vectorial quantization method, dual dynamic programming, sensitivity analysis, integer programming, Fenchel cut, portfolio management.
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Chapter 1

Introduction

The work presented in the Ph.D thesis dissertation deals with numerical methods for portfolio optimization. This study is motivated by practical challenges encountered by Total, in liquefied natural gas (LNG) trading activity.

From a mathematical point of view, an LNG portfolio optimization is a multistage stochastic integer program. It is among the most challenging problems in mathematical programming because it combines two generally difficult classes of problems: stochastic programming and discrete optimization. Despite the fact that the mathematical formulation of LNG portfolio optimization has some particular structure, it is still far from being totally resolved. Moreover, approximation and heuristic method generally do not provide good solutions and they are not efficient in terms of algorithm complexity.

In the thesis, we focus on the stochastic programming aspect. In other words, we study the mathematical properties and the numerical methods associated to the continuous relaxation of the stochastic integer problem. It separates into three parts: computing the optimal value and the optimal strategy in risk neutral framework; computing the optimal value and the optimal strategy of risk averse optimization; sensitivity analysis of optimal value with respect to some parameters of the random process model. At the end, difficult points in integer programming have been discussed and some numerical methods have been applied in our problem when combining of stochastic programming techniques.

This chapter is a general introduction of the thesis dissertation. It is organized as follows: in section 1.1 we present in detail the LNG portfolio optimization problem and introduce the associated mathematical formulation. In section 1.2 we give a brief summary of the results of the thesis. Finally, in section 1.3 we present the structure of the thesis dissertation.

1.1 Context and motivation

Let us first introduce the characterises of a LNG trading portfolio. It is a shipping portfolio which is composed of supply and delivery contracts, and Total has to meet its commitments toward its counterparts while trying to generate profits through arbitrating
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different commodity markets. The figure 1.1 gives an illustration of a simplified LNG portfolio. This example is the main numerical example used throughout this thesis dissertation.

At each time stage before expiration of the contract, the portfolio holder observes the commodity spot price and then determines the optimal cargo number to allocate on each route. For economical reasons, cargos are always supposed to be fully charged. Then, the portfolio holder receives the difference between the price between supplying market and destination market times LNG quantity transported on route minus the shipping costs. The decision on the quantity to be transported is subject to volumetric constraints such as local constraints and global constraints at each port, as well as limitation of some routes.

1.1.1 Contract example

We give a brief example of supply purchase agreement (SPA). Contract is similar on the destination side.

Notation Before describing one contract, we introduce some notations.

- \( t \in \{0, 1, \ldots, T\} \) is the index of time step;
- \( s \in \mathcal{S} \) is the index of cargo size;
- \( cp_s \) is the capacity of cargo of type \( s \);
- \( lp \in \mathcal{P} \) is the index of supplying port;
- \( ap \in \mathcal{P} \) is the index of destination port;
- \( u_{t,s}^{lp-ap} \) is the number of cargo of type \( s \) send from \( lp \) to \( ap \) at time \( t \);
1.1. Context and motivation

- $\xi_t$ is the commodity index prices on different market at time $t$;
- $c^lp_t(\xi_t)$ is the running cost unit at time $t$ of supplying port $lp$, which is a function of commodity index prices $\xi_t$. Similarly we have $c^{ap}_t(\xi_t)$ at destination side.

**Monthly constraint**  One monthly constraint (or generally called local constraint) means the constraints on total LNG quantity leaving the port $lp$ at time step $t$:

$$\begin{align*}
U^lp_t &\leq \sum_s \sum_{ap} c_{ps} \cdot u^lp_{t,s} \leq U^lp_t, \tag{1.1}
\end{align*}$$

where $U^lp_t$ (resp. $U^lp_t$) is the lower bound (resp. the upper bound) of quantity leaving the port $lp$ at time step $t$.

Some contracts could also stipulate on the cargo number to be delivered:

$$\begin{align*}
U^lp_t &\leq \sum_s \sum_{ap} u^lp_{t,s} \leq U^lp_t, \tag{1.2}
\end{align*}$$

then $U^lp_t$ (resp. $U^lp_t$) is the lower bound (resp. the upper bound) of the number of cargos leaving the port $lp$ at time step $t$.

Mathematically, we can write the constraint in following form:

$$u_t \in \mathcal{U}_t \cap \mathbb{Z}^n \tag{1.3}$$

where $\mathcal{U}_t$ is a polyhedron set. We say that $u_t$ is a control variable.

**Annual constraint**  One annual constraint (or generally called global constraint) means the constraints related to the total LNG quantity leaving the port $lp$ in all horizon $[0, T - 1]$.

$$\begin{align*}
U^lp &\leq \sum_{t=0}^{T-1} \sum_s \sum_{ap} c_{ps} \cdot u^lp_{t,s} \leq U^lp, \tag{1.4}
\end{align*}$$

where $U^lp$ (resp. $U^lp$) is the lower bound (resp. the upper bound) of quantity leaving the port $lp$ in horizon $[0, T - 1]$.

Mathematically, we can write the constraint in following form:

$$\begin{align*}
\sum_{t=0}^{T-1} A_t u_t &\in \mathcal{X}_T \tag{1.5}
\end{align*}$$

where $\mathcal{X}_T$ is a polyhedron set. We can then naturally introduce the state variable $x_t = \sum_{s=0}^{t-1} A_s u_s$. The annual constraints (1.5) are equivalent to

$$\begin{align*}
x_{t+1} &= x_t + A_t u_t; \\
x_0 &= 0, \quad x_T \in \mathcal{X}_T. \tag{1.6}
\end{align*}$$
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Except the annual constraints, some other constraints such as seasonal constraint, can also be written in state constraint (1.6). Let us consider a seasonal constraint:

\[ U_{lp}^T_{0,T_1} \leq \sum_{t=T_0}^{T_1} \sum_s \sum_{ap} c_{p,s} \cdot u_{t,s}^{lp-ap} \leq U_{lp}^T_{0,T_1} \]  

We can still write the constraint as a state constraint (1.6) by just setting \( A_t = 0, t \notin [T_0, T_1] \).

Price formula The final element of a contract is the price formula. It is a function which takes commodity index at different market \( \xi_t \) as variable and defines the unit cost price \( c_{t}^{lp} \) at supplying port \( lp \) at time step \( t \). Then the total cost at port \( lp \) is

\[ \sum_{t=0}^{T-1} \sum_s \sum_{ap} c_{t}^{lp} \cdot c_{p,s} \cdot u_{t,s}^{lp-ap} \]  

Mathematically, we can write

\[ \sum_{t=0}^{T-1} c_t(\xi_t)u_t \]  

the total cost in period \([0, T-1]\).

1.1.2 LNG portfolio optimization problem

Thus, the LNG portfolio optimization problem maximizes the revenue subject to all local and global constraints. Mathematically, we can formulate the problem as follows. First of all, we have a discrete time Markov process \((\xi_t), 0 \leq t \leq T \) in the probability space \( L^2(\Omega, (F_t), P; R^d) \), where \( F_t \) denotes the canoncial filtration associated with \((\xi_t) : F_t := \sigma (\xi_s, 0 \leq s \leq t) \). The problem under consideration has the following expression:

\[ \inf \mathbb{E} \sum_{t=0}^{T-1} c_t(\xi_t) \cdot u_t + g(\xi_T, x_T) \]  

subject to \( u_t \in U_t \cap Z^n, F_t - \text{measurable}, \)

\[ x_{t+1} = x_t + A_tu_t, \]

\[ x_0 = 0, \ x_T \in X_T \ \text{almost surely}; \]

where \( u_t \in Z^n \) is the integer control variable, \( x_t \in R^n \) is the state variable, \( A_t \in R^{m \times n} \) is the technique matrix, \( c_t(\cdot) : R^d \rightarrow R^n \) is running cost unit, and finally \( g(\xi, x) : R^d \times R^m \rightarrow R \) is a penalty function for final state \( x_T \).

Remark 1.1.1. In fact, problem (1.10) has a particular structure. The most important point in the formulation (1.10) is that random variable is only involved in criteria, instead of right hand side of constraints in classical stochastic programming formulation. This consists an important advantage making the feasible set of control variable independent of the random variable. In other words, the feasible set is deterministic.
Remark 1.1.2. This mathematical formulation (1.10) is not limited to LNG portfolio optimization. Many energy/commodity market problems, or some problems in general financial market can be stated in this formulation.

- Gas supply contracts pricing (swing option). A swing option provides its holder the right to exercise one and only one call or put on any one of a number of specified exercise dates (this latter aspect is Bermudan). Penalties are imposed on the buyer if the net volume purchased exceeds or falls below specified upper and lower limits. It allows the purchaser to "swing" the price of the underlying asset. This option is primarily used in energy trading.  

- Gas storage contract problem. A gas storage contract gives its purchaser the right to use the gas storage. The contract holder could inject and withdraw gas at each time step subject to some volumetric constraints and physical operational constraints. We can view the contract as a option with three actions: inject, withdraw or do nothing.

- General multi exercise option.

Most examples are one dimensional (control variable and state variable), see Bernhart [22] for recent development. The LNG portfolio optimization can be view as a high dimensional multi exercise option pricing problem.

1.2 Summary of results

We summarize in this section the main mathematical properties and numerical methods studied in the thesis. The thesis focus on three parts around the LNG portfolio optimization. In the first part, we are interested in computing the optimal value and the optimal strategy by some particular numerical method for both risk neutral and risk averse perspective. In the second part, for the reason of trading activity, we are interested in the sensitivity information with respect to underlying commodity price model parameters. In the third part, we return to the stochastic integer problem and discuss the difficult points, and we finally give a heuristic method for the stochastic integer problem.

\[1\]This explication on swing option is taken from Wikipedia \url{http://en.wikipedia.org/wiki/Option_style}.
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1.2.1 Resolution of stochastic programming

In this part of work, we focus on finding a numerical algorithm adapted to the continuous relaxation problem of (1.10)

\[
\inf \mathbb{E} \left[ \sum_{t=0}^{T-1} c_t(\xi_t) \cdot u_t + g(\xi_T, x_T) \right]
\]

subject to \( u_t \in \mathcal{U}_t, \mathcal{F}_t \)-measurable,
\[
x_{t+1} = x_t + A_t u_t,
\]
\[
x_0 = 0, x_T \in \mathcal{X}_T \quad \text{almost surely;}
\]

(1.11)

Because high dimension of the control variable and state variable, classical methods in financial engineering such as PDE method (see for example Achdou and Pironneau [1]) and regression method (see for example Bertsekas [9] and Tsiatsikli [32, 33]) are not well adapted here. Thus, we turn to stochastic programming methods.

For surveys on stochastic programming, we refer to [10, 19, 30]. Unless the underlying random space has finite support, our model is an optimization problem over infinite-dimensional function spaces, which is difficult to solve. Analytical solutions are not available except for extremely simple and unrealistic cases. Numerical approximation methods have been largely studied in the literature.

Discretization

Most approximation schemes are based on the discretization of the underlying random space by a scenario tree. A survey and evaluation of popular scenario generation techniques is provided in Kaut and Wallace [20]. We choose the vectorial quantization method introduced by Pages [5, 4] to discretize the random process. They establish in [4] an error estimation of discretization by quantization on obstacle problem. The main idea of vectorial quantization is to approximate a random variable \( \xi \) taking value in an infinite set (or uncountable set) by a random variable \( \hat{\xi} \) taking value in a finite set \( \Gamma = \{\xi_1, \xi_2, \ldots, \xi_N\} \).

We say that the quantization is optimal if it minimizes the discretization error:

\[
\min \left\{ \|\xi - \hat{\xi}\| : \hat{\xi} \in \Gamma, \#(\Gamma) \leq N \right\}
\]

(1.12)

The vectorial quantization tree is an approximation of discrete time random process by a finite state Markov chain such that at each time step, the finite random variable \( \hat{\xi}_t \) is an optimal quantization of the original random variable \( \xi_t \). Pages provides in [4] a heuristic method to build an optimal quantization tree, called the competitive learning vector quantization (CLVQ) algorithm.

In the thesis, we briefly develop the approximation method by quantization. Let \( f \) be a function. The original idea is to approximate the function value at point \( \xi \) by the value on its nearest quantized point \( \hat{\xi} \):

\[
f(\xi) \leftarrow f(\hat{\xi}), \hat{\xi} = \text{proj}_\Gamma(\xi).
\]

(1.13)
In other words, \( f \) is approximated by a piecewise constant function. We give another idea of approximation by using Delaunay triangulation build by the quantized points \( \Gamma \). Then, the approximation now is:

\[
f(\xi) \leftarrow \bar{f}(\xi) := \sum_i \lambda_i f(\hat{\xi}^i)
\]  

(1.14)

where \( \hat{\xi}^i \) is the extreme point of the Delaunay triangle where find \( \text{proj}_{\text{conv}(\Gamma)}(\xi) \) and \( \lambda^i \) is the coefficient associated to \( \hat{\xi}^i \) of Delaunay triangle. The right hand side of approximation (1.14) is a continuous, piecewise linear function.

**Numerical method**

The problem under study (1.11) can be naturally decomposed stage-wisely and respect the dynamic programming principle. Let \( x_t = (x_0, x_1, \ldots, x_t) \) (resp \( \xi_t = (\xi_0, \xi_1, \ldots, \xi_t) \)) be the whole history of state variable (resp. of random variable) until time step \( t \). We denote \( Q(t, x_t, \xi_t) \) the Bellman value function (or cost-to-go function)

\[
Q(t, x_t, \xi_t) := \text{essinf} \left[ E \left[ \sum_{s=t}^{T-1} c_s(\xi_s) u_s + g(\xi_T, x_T) \right| F_t \right]
\]

subject to

\[
  u_s \in U_s, \quad F_s \text{ - measurable},
\]

\[
x_{s+1} = x_s + A_s u_s,
\]

\[
x_T \in X_T \text{ almost surely}.
\]

(1.15)

By Markov property of \((x_t, \xi_t)\), we first prove that \( Q \) is a function of \((x_t, \xi_t)\). Applying the dynamic programming principle, we can write

\[
Q(t, x_t, \xi_t) = \text{essinf} \left[ c_t(\xi_t) u_t + Q(t+1, x_{t+1}, \xi_t) \right| F_t
\]

subject to

\[
u_t \in U_t, \quad F_t \text{ - measurable},
\]

\[
x_{t+1} = x_t + A_t u_t,
\]

(1.16)

where \( Q(t+1, x_{t+1}, \xi_t) \) is the conditional expectation of \( Q(t+1, x_{t+1}, \xi_{t+1}) \)

\[
Q(t+1, x_{t+1}, \xi_t) := E \left[ Q(t+1, x_{t+1}, \xi_{t+1}) \right| F_t
\]

(1.17)

After discretizing by vectorial quantization method, we build a Markov chain \((\tilde{\xi}_t)\) based on the tree and apply stochastic programming techniques to solve the continuous relaxation problem (1.11). The method proposed is a combination of L-shape method and stochastic dual dynamic programming (SDDP) algorithm.

- Stochastic programming dates back to the pioneering study by Dantzig [14]. For the continuous stochastic programming problem with recourse, well-known L-shape method is first introduced by Van Slyke and Wets [31]. This approach is based upon Benders’ decomposition [8]. Now, L-shape method can be view as a classical...
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method for some cases of stochastic programming problem. We can refer to Birge and Louveaux [10] for more detail of the algorithm. The algorithm principle is based on feasibility cut to build local feasible set of control variable \( u_t \) and on optimality cut to approximate the Bellman value \( Q(t, x_t, \xi_t) \). In our problem, feasibility cuts are independent of random process as observed in remark 1.1.1 it can be read as

\[
\lambda_t^x x_t + \lambda_t^u u_t \leq \lambda_t^0, \quad (\lambda_t^x, \lambda_t^u, \lambda_t^0) \in \mathcal{I}_t. \tag{1.18}
\]

Optimality cuts can be generally written as

\[
\hat{Q}(t, x_t, \xi_t) \geq \lambda_t^x x_t + \lambda_t^0, \quad (\lambda_t^x, \lambda_t^0) \in \mathcal{O}(\xi_t). \tag{1.19}
\]

Furthermore, we can also approximate conditional expectation \( \hat{Q}(t, x_t, \xi_{t-1}) \) by optimality cuts

\[
\hat{Q}(t, x_t, \xi_{t-1}) \geq \vartheta(t, x_t, \xi_{t-1}, \mathcal{O}_t) := \sup\{\overline{\lambda}_t^x x_t + \overline{\lambda}_t^0 : (\overline{\lambda}_t^x, \overline{\lambda}_t^0) \in \mathcal{O}(\xi_{t-1})\}. \tag{1.20}
\]

Thus, the dynamic programming formulation (1.16) can be read as

\[
\hat{Q}(t, x_t, \xi_t) = \text{essinf} \quad c_t(\xi_t) \cdot u_t + \vartheta(t+1, x_{t+1}, \xi_t, \mathcal{O}_t)
\]

subject to \( x_{t+1} = x_t + A_t u_t \), \( u_t \in \mathcal{U}_t, \mathcal{F}_t \)-measurable,

(optimality cut) \( \lambda_t^x x_t + \lambda_t^u u_t \leq \lambda_t^0, \quad (\lambda_t^x, \lambda_t^u, \lambda_t^0) \in \mathcal{I}_t, \)

(1.21)

- SDDP algorithm has been first introduced by Pereira and Pinto [23] to solve the problem where randomness only appears on right-hand-side of constraint and is independent from one stage to another. Philpott and Guan [17] have studied the convergence of the method. Shapiro [29] has recently analysed the statistical properties and rate of convergence of the method.

In our context, the random variable is in the criteria instead of the right hand side of the state dynamic constraint. This difference makes the Bellman function \( Q(t, x_t, \xi_t) \) only partial convex, which means that it is only convex with respect to \( x_t \) and is non-convex with respect to \( \xi_t \). Therefore, it is impossible to approximate globally \( Q(t, x_t, \xi_t) \) by lower bound linear functions, and it is the key argument to discretize the random process. We have to build the optimality cuts with respect to \( x_t \) to approximate \( Q(t, \cdot, \xi_t) \) from below. After discretization, we are able to store the optimality cuts at the quantized points and approximate the optimality cuts on non-quantized points by technique presented in previous section.

The algorithm then follows from SDDP procedure. In the forward pass, we simulate numerically \( M_f \) scenarios of random process \( (\xi_t^m), m = 1, \ldots, M_f \), and compute
the optimal control \((u^m_t)\) associated to each random process. Then, the forward value \(\overline{v}\) is the statistical estimate of the value of these Monte Carlo samples, which is statistically an upper bound of the optimal value. In backward pass, we update the optimality cuts at each quantized point. The backward value \(\underline{v}\) is the optimal value associate to first stage, which is a lower bound of optimal value of the discretized problem. The algorithm stops when the upper bound value \(\overline{v}\) and the lower bound \(\underline{v}\) are close enough. Some discussion on stopping condition of SDDP algorithm is also provided in Shapiro [29].

- We prove the convergence result of the algorithm. The principle proof follows the finiteness of the optimality cuts and feasibility cuts on discretized problem, which is the main argument of finite convergence of L-Shape method. Then, using statistical argument, we obtain the convergence when the forward simulation follows discrete distribution. The convergence on the case where the forward sampling uses continuous distribution follows the theorem in Römisch [18] on the stability of optimal value with respect to small perturbation on the process distribution.

- We provide two numerical tests: one is the normalized swing option, another is the LNG portfolio optimization. Both tests shows good performance of the algorithm, even for large size problem.

### 1.2.2 Apply SDDP in risk aversion optimization

The multi stage stochastic programs (1.10) we have considered until now is under risk neutral approach. We also deal with the risk averse approach by using semi-continuous inferior convex risk measure as criterion. We focus on conditional value at risk CVaR in our work. Such risk measure is widely used in the practice of risk management.

\[
\inf (1 - \beta)E [v] + \beta \text{CVaR}_\alpha (v)
\]

subject to \(v = \sum_{t=0}^{T-1} c_t (\xi_t) u_t + g (\xi_T, x_T), \)

\(u_t \in U_t, \mathcal{F}_t - \text{measurable},\)

\(x_{t+1} = x_t + A_t u_t,\)

\(x_0 = 0, x_T \in \mathcal{X}_T,\)

where \(0 \leq \beta \leq 1\) and \(0 < \alpha \ll 1\) are some coefficient.

Such risk averse optimization problem has been largely studied in literature, in theory: see Ruszczyński and Shapiro [26], Dentecheva and Ruszczyński [15, 16], and in numerical method: see Ruszczyński [25], Ruszczyński and Shapiro [27], Philpott and de Matos [24].

The techniques of changes in probability measure (importance sampling) is very popular in variance reduction. Recently, Bardou et al. [7, 6] use Robbins-Monro algorithm to compute the best drift in importance sampling and apply it in CVaR hedging problem.
1. Introduction

- Following Rockafellar and Uryasev [33], CVaR can be reformulated as the optimal value of a stochastic optimization problem. Combined with (1.22), we obtain a similar problem as (1.11) but with two additional one dimensional state variables.

\[
\inf \beta z + \mathbb{E} \left[ (1 - \beta)v + \frac{\beta}{\alpha} (v - z)_+ \right]
\]

subject to \( v = y_T + g(\xi_T, x_T), \)
\( u_t \in U_t, \mathcal{F}_t \)-measurable,
\( x_{t+1} = x_t + A_t u_t, \)
\( x_0 = 0, x_T \in X_T, \)
\( y_{t+1} = y_t + c_t(\xi_t)u_t, y_0 = 0. \)

Therefore, our SDDP algorithm presented in previous section is still adapted to this problem.

- In order to increase the extreme scenarios in Monte Carlo simulation, we apply the changes in probability measure techniques into stochastic programming algorithm, relying on Girsanov theorem and theorem of changes in probability measure.

\[
\inf \beta z + \mathbb{E}^{Q^\phi} \left[ (1 - \beta)v + \frac{1}{Z_T^\phi} \frac{\beta}{\alpha} (v - z)_+ \right]
\]

subject to \( v = y_T + \frac{1}{Z_T^\phi} g(\xi_T, x_T), \)
\( u_t \in U_t, \mathcal{F}_t \)-measurable,
\( x_{t+1} = x_t + A_t u_t, \)
\( x_0 = 0, x_T \in X_T, \)
\( y_{t+1} = y_t + \frac{1}{Z_t^\phi} c_t(\xi_t)u_t, y_0 = 0, \)

where \( dQ^\phi = Z_T^\phi dP \) and

\[
Z_t^\phi = \exp \left( \sum_{s=1}^{t} \phi_s (W_s - W_{s-1}) - \frac{1}{2} \sum_{s=0}^{t} \phi_s^2 \right)
\]

The objective of the change is to reduce the variance of the Monte Carlo simulation. Because of some feature of our algorithm, modification of drift \((\phi_t)\) during SDDP procedure increases considerably the complexity of the algorithm. Therefore, we have to choose \((\phi_t)\) a priori, which is difficult. We have discuss several methods to compute \((\phi_t)\).

- Numerical tests have been done on swing option pricing problems, and it shows good performance of this algorithm without using technique of changes in probability measure.
1.2. Summary of results

Compared with risk neutral results, the results of risk averse problem give exactly what we expect: reduction of the extreme loss scenarios such that the option value of most scenarios do not exceed the VaR $\alpha$.

However numerical test using technique of changes in probability measure shows that they fail to reduce the variance of Monte Carlo simulation. We only arrive to increase the scenario number in the tail of the loss distribution.

1.2.3 Sensitivity analysis

The study on sensitivity analysis responds to the need of trading perspective. The values of sensitivities (called the greeks) allow traders to replicate the price variations originating from the contracts they have to manage on a regular basis.

We first assume that the random process, taking part in the modeling of future commodity spot prices, follows the celebrated Black model. The price model takes as parameters the market volatilities $\sigma$, correlation $\rho$ as well as the forward curve $F_0$.

$$\xi^i_t = (F_0^i)^{\frac{1}{2}} \exp \left( \sum_{s=0}^{t-1} \sigma^i W^i_s - \frac{1}{2} (\sigma^i)^2 T \right) \quad i = 1, \ldots, d$$

where $W^i_t$ is a standard normal distribution $\mathcal{N}(0, 1)$, with correlation $\text{corr}(W^i_s, W^j_s) = \rho_{ij}$.

We consider the continuous relaxation problem (1.11) where there is no nonlinear term $g(x_T, \xi_T)$ in criteria:

$$\inf \mathbb{E} \left[ \sum_{t=0}^{T-1} c_t(\xi_t) \cdot u_t \right].$$

Let us denote by $v(F_0, \sigma)$ the optimal value and by $U(F_0, \sigma)$ the set of optimal solutions. Then, the sensitivity analysis we are interested in is the derivative of the optimal value function with respect to $F_0$ and $\sigma$

$$D_{F_0,\sigma}v(F_0, \sigma).$$

The sensitivity analysis under stochastic optimization (or stochastic control) has not been much studied. Most of the articles in the literature focus on the procedure on how to solve the problem and hence how to get the contract’s price. In our study, we propose a method to estimate sensitivities based on Danskin’s theorem, which is well known among optimization practitioners.

- We prove that $v(F_0, \sigma)$ is Fréchet differentiable at almost every point $(F_0, \sigma)$. And on the point where $v(F_0, \sigma)$ is Fréchet differentiable, we have

$$v'(F_0, \sigma; dF_0, 0) = \mathbb{E} \left[ \sum_{t=0}^{T-1} D_{F_0} c_t dF_0 \cdot u_t^* \right]$$

$$v'(F_0, \sigma; 0, d\sigma) = \mathbb{E} \left[ \sum_{t=0}^{T-1} D_{\sigma} c_t d\sigma \cdot u_t^* \right]$$

(1.27)
where $u^* \in U(F_0, \sigma)$ is one optimal solution.

- Since $U(F_0, \sigma)$ is not available and we can only obtain an approximated optimal solution by our algorithm, we study the sensitivity values using optimal solution on the discretized problem. We provide two versions of discretized problem to model both forward and backward pass in SDDP algorithm. We prove the convergence result of sensitivity values of both discretized problem to the sensitivity values of the continuous problem.

- During numerical tests, we study the same two examples as in the first part: swing option and LNG portfolio optimization. Since swing option is a low dimension problem, comparisons between results obtained using SDDP method and other classical methods are provided and give evidence of good accuracy of the estimate of marginal prices.

### 1.2.4 Heuristic method for stochastic integer program

Finally, we study the multi stage stochastic integer program. First, we introduce a heuristic method to obtain a sub-optimal integer solution with the help of optimality cuts computed by continuous relaxation resolution.

\[
Q(t, x_t, \xi_t) = \text{essinf } c_t(\xi_t) \cdot u_t + \vartheta(t + 1, x_{t+1}, \xi_t)
\]

subject to  
\[
x_{s+1} = x_s + A_s u_s,
\]

\[
u_s \in U_s \cap \mathbb{Z}^n, \ F_s - \text{measurable}, \ s \geq t,
\]

\[
x_T \in X_T,
\]

(optimality cut)  
\[
\vartheta(t + 1, x_{t+1}, \xi_t) \geq \lambda^2_l x_t + \lambda^0_l, \ (\lambda^2_l, \lambda^0_l) \in O(\xi_t).
\]

This sub optimal strategy gives an upper bound estimate of optimal value of integer problem. The lower bound is still given by the lower bound of the continuous relaxation problem. Then, the objective is to reduce the gap between the upper bound and the lower bound.

The main tool we study is the integrality cutting plane technique. In literature, many various integrality cuts techniques have been proposed to solve the integer problem, see Cornuejols [13] for a survey. However, even in the deterministic case, it is very hard to obtain an optimal solution only by using the integrality cuts technique. It has been shown that combination of integrality cuts technique with branch and bound technique (called branch and cut method or cut and branch method) will greatly improve the convergence speed. However, applying branch and bound technique in the stochastic case will increase considerably the complexity of the algorithm and make problem numerically intractable.

Some studies using branch and bound style algorithm in stochastic programming are limited to two stages problem, see [28] [2] for survey. As to multi stage case, there are just a few publications. Even heuristic methods are rarely proposed and they are limited to some particular problem [21].
1.3 Organization of thesis dissertation

In the thesis, we discuss the difficulties of multi stage integer problem by a very simple example, and show that dual programming type algorithm is not well adapted to integer problem because of non-convexity and discontinuity of the Bellman value function $Q(t, x_t, \xi_t)$ with respect to $x_t$. Thus, our SDDP algorithm is not adapted to multistage integer programming. Finally, by analysing in detail where the continuous optimal solution $u_t$ is located, we propose a slight improvement, which will reduce the gap in some cases.

1.3 Organization of thesis dissertation

This manuscript is composed of three parts.

• Part I (chapter 2, 3) focuses on numerical method to solve the continuous relaxation of problem (1.10). Chapter 2 is related to a paper entitled *Energy contracts management by stochastic programming techniques*, written in collaboration with F. Bonnans and T. Christel, published as INRIA research report RR-7289 [11] and will appear in special issue of "Annal of operation research" on “stochastic programming”. We add a chapter 3 devoted to the analysis of the same type algorithm to risk aversion optimization problem.

• Part II (chapter 4) deals with the sensitivity analysis with respect to parameters in spot price model, based on Danskin’s theorem. It is based on paper entitled *Sensitivity analysis of energy contracts by stochastic programming techniques* written in collaboration with F.Bonnans and T.Christel, published as INRIA research report RR-7574 [12] and will appear in the book *Numerical Methods in Finance*, edited by R. Carmona, P. Del Moral, P. Hu, N. Oudjane, in series of Springer Proceeding in Mathematics.

• In part III (chapter 5), we study the multi-stage stochastic integer program.

• The appendix consists of two parts. First, we present some advancement of approximation method by quantization. Then, another chapter concentrates on various integrality cutting plane methods for integer solutions.

Bibliography


1. Introduction


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