AIP conference

Construction of indistinguishable conductivity perturbations for the point electrode model in EIT

Lucas Chesnel[1]

Coll. with N. Hyvönen[2] and S. Staboulis[3].

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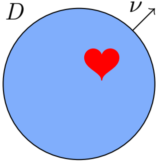
Electrical Impedance Tomography (EIT)

Goal of the EIT: to reconstruct the conductivity inside a body from boundary measurements of current and potential.



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$\sigma : D \to \mathbb{R}$ a uniformly positive conductivity.

- Define the current-to-voltage (Neumann-to-Dirichlet) map

$$\begin{array}{rccc} \Lambda^\sigma : & \mathrm{H}_\diamond^{-1/2}(\partial D) & \to & \mathrm{H}^{1/2}(\partial D)/\mathbb{R} \\ & f & \mapsto & u \end{array}$$

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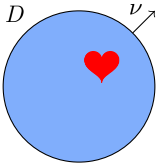

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Here, $\mathrm{H}_\diamond^{-1/2}(\partial D) := \{ f \in \mathrm{H}^{-1/2}(\partial D) \,|\, \langle f, 1 \rangle_{\partial D} = 0 \}$.

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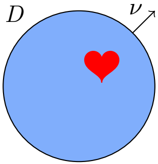

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→ The knowledge of Λ^σ uniquely determines $\sigma \in \mathrm{L}_+^\infty(D)$ (d=2, Astala, Päivärinta 06) or $\sigma \in \mathrm{W}_+^{1,\infty}(D)$ ($d \geq 3$, Haberman, Tataru 13).

→ Uniqueness results when the Cauchy data are known on a continuous subset of $\partial D \times \partial D$ also exist (Imanuvilov, Uhlmann, Yamamoto 10).

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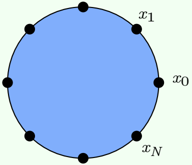

Reference conductivity $\sigma \equiv 1$

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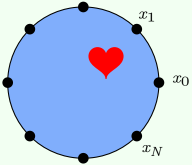

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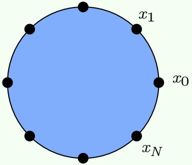

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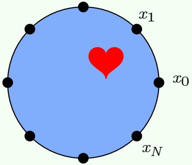

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$$(u_n - u_n^0)(x_m), \qquad \forall m, n = 0, \ldots, N.$$

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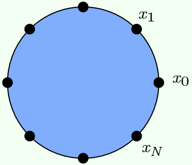

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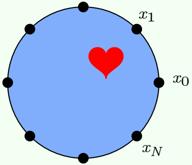

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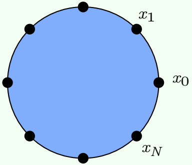

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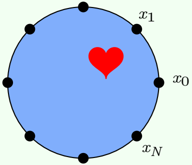

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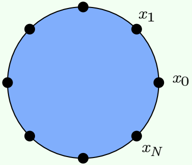

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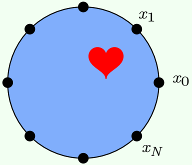

PERTURBED CONDUCTIVITY

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- Note that $\Lambda^\sigma - \Lambda^1 : \mathscr{D}'_\diamond(\partial D) \to \mathscr{D}(\partial D)/\mathbb{R}$ when $\mathrm{supp}(\sigma - 1) \Subset D$ so that the latter quantities are well-defined.

Matrix of relative measurements

- Define the matrix of relative measurements $\mathscr{M}(\sigma) \in \mathbb{R}^{N \times N}$ such that

$$\mathscr{M}(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D}.$$

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In this talk, we build some $\sigma \not\equiv 1$, with $\mathrm{supp}(\sigma - 1) \Subset D$, s. t. $\boldsymbol{\mathscr{M}(\sigma) = 0}$.

These perturbations of the reference conductivity cannot be detected with our measurements.

Outline of the talk

1. General scheme
2. Application to our problem
3. Numerical experiments

Origin of the method

- We will work as in the proof of the implicit functions theorem.

• This idea was used in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

• It has been adapted in Bonnet-Ben Dhia & Nazarov 13 to build invisible perturbations of waveguides (see also Bonnet-Ben Dhia, Nazarov & Taskinen 14 for an application to a water-wave problem).

• In Bonnet-Ben Dhia, Chesnel & Nazarov 15 it has been used to construct invisible inclusions for an observer sending plane waves and measuring the resulting scattered field at infinity in a finite number of directions.

Sketch of the method

- Define $\rho = \sigma - 1$ and gather the measurements in the vector
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- Let $\Omega \neq \emptyset$ be some Lipschitz domain such that $\Omega \Subset D$ ($\overline{\Omega}$ will correspond to the support of the perturbation which can be chosen arbitrarily).

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- We look for small perturbations of the reference medium: $\rho = \varepsilon\kappa$ where $\varepsilon > 0$ is a small parameter and where κ has be to determined.

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where $\vec{\tau} = (\tau_1, \ldots, \tau_K)^\top$

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- Take $\kappa = \kappa_0 + \sum_{k=1}^{K} \tau_k \kappa_k$ where the τ_k are real parameters to set:

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Sketch of the method

- Define $\rho = \sigma - 1$ and gather the measurements in the vector
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If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$. Set $\rho^{\mathrm{sol}} := \varepsilon\kappa^{\mathrm{sol}}$. We have $F(\rho^{\mathrm{sol}}) = 0$ (invisible perturbation).

Calculus of $dF(0)$

- For our problem, we have ($\rho = \sigma - 1$)

$$F(\rho) = (\mathscr{M}(\sigma)_{mn})_{1 \le m \le n \le N}.$$

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Is $dF(0) : \mathrm{L}^\infty(\Omega) \to \mathbb{R}^K$ onto ?

Construction of the shape functions

$$dF(0)(\kappa) = \Big(-\int_\Omega \kappa \nabla u_m^0 \cdot \nabla u_n^0 \, d\boldsymbol{x} \Big)_{1 \le m \le n \le N}$$

1. Using classical results concerning Gram matrices, we can prove that

$$\mathscr{S} := \{\nabla u_m^0 \cdot \nabla u_n^0\}_{1 \le m \le n \le N} \in \mathscr{C}^\infty(\overline{\Omega})^K$$

is a family of linearly independent functions

$\Leftrightarrow$ there are $\kappa_{mn} \in \operatorname{span}(\mathscr{S})$ s.t. $-\int_\Omega \kappa_{mn} \nabla u_{m'}^0 \cdot \nabla u_{n'}^0 \, d\boldsymbol{x} = \left| \begin{array}{l} 1 \text{ if (m,n)=(m',n')} \\ 0 \text{ else} \end{array} \right.$

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2. We need to construct some $\kappa_0 \in \ker dF(0)$, *i.e.* some κ_0 satisfying

$$\int_{\Omega} \kappa_0 \nabla u_m^0 \cdot \nabla u_n^0 \, d\boldsymbol{x} = 0, \qquad \forall m, n = 1, \ldots, N.$$

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2. We take

$$\kappa_0 = \kappa_0^\# - \sum_{1\le m\le n\le N} \left(\int_\Omega \kappa_{mn} \kappa_0^\# \, d\boldsymbol{x} \right) \kappa_{mn}$$

where $\kappa_0^\# \notin \mathrm{span}\{\kappa_{mn}\}_{1\le m\le n\le N}$.

Prop. Assume that $\{\nabla u_m^0 \cdot \nabla u_n^0\}_{1\le m\le n\le N} \in \mathscr{C}^\infty(\overline{\Omega})^K$ is a family of linearly independent functions. For $\boldsymbol{\varepsilon}$ **small enough**, define $\sigma^{\,\mathrm{sol}} = 1 + \varepsilon\kappa^{\,\mathrm{sol}}\mathbb{1}_\Omega$ with

$$\kappa^{\,\mathrm{sol}} = \kappa_0 + \sum_{1\le m\le n\le N} \tau_{mn}^{\,\mathrm{sol}}\,\kappa_{mn}.$$

Then, we have

$$\mathscr{M}(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda^{\sigma^{\,\mathrm{sol}}} - \Lambda^1)(\delta_n - \delta_0)\rangle_{\partial D} = 0, \qquad \forall m, n = 1, \dots, N,$$

so that the conductivity perturbation is invisible.

Comments:

→ We need $\boldsymbol{\varepsilon}$ to be **small enough** to prove that G^ε is a contraction.

→ We have $\kappa^{\,\mathrm{sol}} \not\equiv 0$ (non trivial perturbation). To see it, compute $dF(0)(\kappa^{\,\mathrm{sol}})$.

Main result 2/2

It remains to prove that $\{\nabla u_m^0 \cdot \nabla u_n^0\}_{1\leq m\leq n\leq N} \in \mathscr{C}^\infty(\overline{\Omega})^K$ is a family of linearly independent functions.

Main result

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By definition, we have

$$\left|\begin{array}{l} \Delta u_n^0 = 0 \\ \nu \cdot \nabla u_n^0 = \delta_n - \delta_0. \end{array}\right.$$

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1. When D is a 2D disk, there holds

$$u_n^0(x) = \frac{1}{\pi} \ln|x - x_0| - \frac{1}{\pi} \ln|x - x_n|$$

and the result can be proved doing explicit computations.

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2. Then, we deduce that the result is also true for general 2D smooth domains using conformal mapping techniques.

Main result 2/2

THM. Let $D \subset \mathbb{R}^2$ be a smooth domain and Ω a nonempty Lipschitz domain such that $\Omega \Subset D$. For $\boldsymbol{\varepsilon}$ **small enough**, define $\sigma^{\,\mathrm{sol}} = 1 + \varepsilon\kappa^{\,\mathrm{sol}}\mathbb{1}_{\Omega}$ with

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Comments:

→ The 3D case is open.

→ The existence of invisible inclusions may appear not so surprising since $\mathscr{M}(\sigma) \in \mathbb{R}^{N\times N}$, $\sigma \in \mathrm{L}^\infty(\Omega)$.

Main result 2/2

THM. Let $D \subset \mathbb{R}^2$ be a smooth domain and Ω a nonempty Lipschitz domain such that $\Omega \Subset D$. For ε **small enough**, define $\sigma^{\,\mathrm{sol}} = 1 + \varepsilon\kappa^{\,\mathrm{sol}}\mathbb{1}_\Omega$ with

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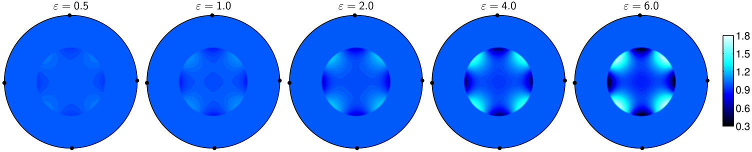
COMMENTS:

→ The 3D case is open.

→ The existence of invisible inclusions may appear not so surprising since $\mathscr{M}(\sigma) \in \mathbb{R}^{N\times N}$, $\sigma \in \mathrm{L}^\infty(\Omega)$. However, for an analogous problem in scattering theory, this result does not hold ...

Influence of the choice of ε

- Examples of conductivities (at the end of the fixed point iteration) which provide the same measurements as the reference conductivity $\sigma \equiv 1$.



(The dots correspond to the positions of the electrodes.)

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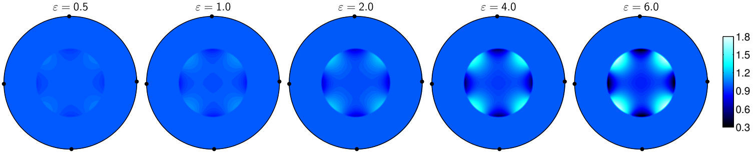

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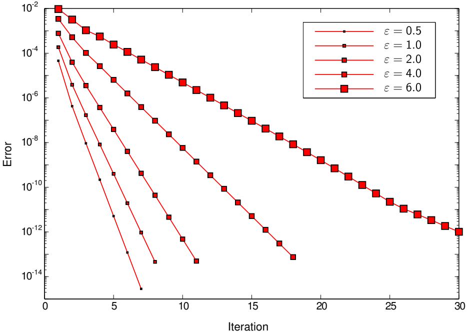
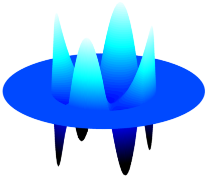
- Convergence of the fixed point iteration with respect to the choice of ε.

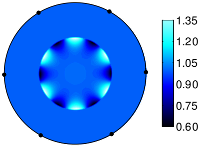




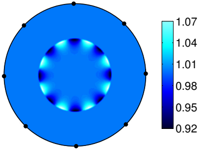
- 3D view of σ for $\varepsilon = 0.6$

Influence of the number of electrodes

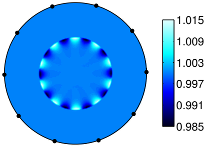
- The dots correspond to the position of the $N+1$ electrodes.



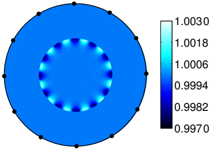
$N=5$



$N=7$



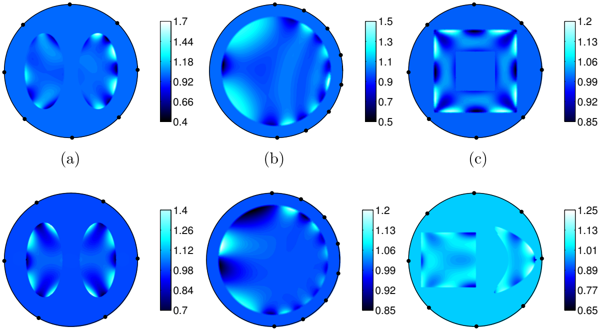
$N=9$



$N=11$

When the number of electrodes increases, the obtained perturbation of the reference conductivity $\sigma \equiv 1$ becomes smaller and smaller.

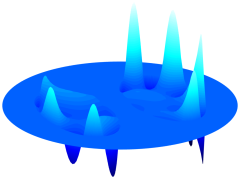
Influence of the choice of $\kappa_0^\#$ and of the shape



(a) (b) (c)

(d) (e) (f)

	ε	$\kappa_0^\#(x,y)$
(a)	4.0	$x+y+1$
(b)	2.0	$\exp(-(x+0.5)^2-y^2)$
(c)	0.25	1
(d)	6.0	1
(e)	0.5	$-y$
(f)	2.0	x



- 3D view of σ for case (a)

Conclusion

What we did

- We explained how to construct invisible conductivity perturbations for the Point Electrode Model.
- The proof is rigorous for the 2D setting with $\sigma^0 \equiv 1$.

Open questions

1) Can we prove that $\{\nabla u_m^0 \cdot \nabla u_n^0\}_{1 \le m \le n \le N}$ is a family of linearly independent functions in 3D?
2) Can we justify the construction of invisible conductivity perturbations when $\sigma^0 \not\equiv 1$?
3) Can we reiterate the process to construct larger invisible perturbations of the reference conductivity?
4) Can we construct invisible conductivity perturbations for other models (Complete Electrode Model)?

Kiitos!