Construction of indistinguishable conductivity perturbations for the point electrode model in EIT

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Electrical Impedance Tomography (EIT)

Goal of the EIT: to reconstruct the conductivity inside a body from boundary measurements of current and potential.

\[ D \subset \mathbb{R}^d, \quad d \geq 2, \text{ is a bounded domain with smooth boundary.} \]
\[ \sigma : D \to \mathbb{R} \text{ a uniformly positive conductivity.} \]

- Define the current-to-voltage (Neumann-to-Dirichlet) map

\[ \Lambda^\sigma : H^{-1/2}_0(\partial D) \to H^{1/2}(\partial D)/\mathbb{R} \]
\[ f \mapsto u \]

where \( u \) is the solution to

\[ \text{div} (\sigma \nabla u) = 0 \quad \text{in} \ D; \quad \sigma \nabla u \cdot \nu = f \quad \text{on} \ \partial D. \]
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Here, \( H^{-1/2}(\partial D) := \{ f \in H^{-1/2}(\partial D) \mid \langle f, 1 \rangle_{\partial D} = 0 \} \).
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→ The knowledge of \( \Lambda^\sigma \) uniquely determines \( \sigma \in L^\infty_+(D) \) (d=2, Astala, Päivärinta 06) or \( \sigma \in W^{1,\infty}_+(D) \) (d ≥ 3, Haberman, Tataru 13).
→ Uniqueness results when the Cauchy data are known on a continuous subset of \( \partial D \times \partial D \) also exist (Imanuvilov, Uhlmann, Yamamoto 10).
Point Electrode Model

- This continuum model is mathematically favorable in its simplicity. In practice EIT measurements are performed with a finite number of electrodes.

- If the electrodes are small, the Point Electrode Model is a good model (Hanke, Harrach, Hyvönen 11).
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- Assume that the electrodes are located at \( x_0, \ldots, x_N \in \partial D \). Denote \( \delta_n \) the Dirac distribution at \( x_n \) and \( u^0_n, u_n \in H^{- (d - 4) / 2 - 1}(D) \) the solutions to

\[
\begin{align*}
\Delta u^0_n &= 0 \\
\nu \cdot \nabla u^0_n &= \delta_n - \delta_0
\end{align*}
\]

**Reference conductivity** \( \sigma \equiv 1 \)

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  - In the PEM, the observer measures the quantities

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(u_n - u^0_n)(x_m), \quad \forall m, n = 0, \ldots, N.
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  \[
  \begin{aligned}
  \Delta u_n &= 0 \\
  \nu \cdot \nabla u_n &= \frac{\delta_n}{\sigma} - \frac{\delta_0}{\sigma}
  \end{aligned}
  \]

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  (u_n - u_n^0)(x_m) - (u_n - u_n^0)(x_0), \quad \forall m, n = 1, \ldots, N.
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\langle \delta_m - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D}, \quad \forall m, n = 1, \ldots, N.
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    \langle \delta_m - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D}, \quad \forall m, n = 1, \ldots, N.
    \]

  - Note that \( \Lambda^\sigma - \Lambda^1 : D'_c(\partial D) \to \mathcal{D}(\partial D) / \mathbb{R} \) when \( \text{supp}(\sigma - 1) \subseteq D \) so that the latter quantities are well-defined.
Define the matrix of relative measurements $\mathcal{M}(\sigma) \in \mathbb{R}^{N \times N}$ such that

$$\mathcal{M}(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D}.$$
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- Note that \( \mathcal{M}(\sigma) = 0 \) when there is no perturbation \( (\sigma \equiv 1) \Rightarrow \text{“relative”} \).
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We have

$$\langle \delta_m - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D} = \langle \delta_n - \delta_0, (\Lambda^\sigma - \Lambda^1)(\delta_m - \delta_0) \rangle_{\partial D}$$

so $\mathcal{M}(\sigma)$ is symmetric.
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so \( \mathcal{M}(\sigma) \) is symmetric \( \Rightarrow \) \( K := \frac{N(N + 1)}{2} \) degrees of freedom.

In this talk, we build some \( \sigma \not\equiv 1 \), with \( \text{supp}(\sigma - 1) \subseteq D \), s. t. \( \mathcal{M}(\sigma) = 0 \).

These perturbations of the reference conductivity cannot be detected with our measurements.
Outline of the talk

1. General scheme
2. Application to our problem
3. Numerical experiments
1 General scheme

2 Application to our problem

3 Numerical experiments
Origin of the method

We will work as in the proof of the implicit functions theorem.

- This idea was used in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

- It has been adapted in Bonnet-Ben Dhia & Nazarov 13 to build invisible perturbations of waveguides (see also Bonnet-Ben Dhia, Nazarov & Taskinen 14 for an application to a water-wave problem).

- In Bonnet-Ben Dhia, Chesnel & Nazarov 15 it has been used to construct invisible inclusions for an observer sending plane waves and measuring the resulting scattered field at infinity in a finite number of directions.
Sketch of the method

▸ Define $\rho = \sigma - 1$ and gather the measurements in the vector

$$F(\rho) = (F_1(\rho), \ldots, F_K(\rho))^\top \in \mathbb{R}^K.$$
Sketch of the method

- Define $\rho = \sigma - 1$ and gather the measurements in the vector
  \[ F(\rho) = (F_1(\rho), \ldots, F_K(\rho))^\top \in \mathbb{R}^K. \]

- No perturbation leads to null measurements $\Rightarrow F(0) = 0$. 

- We look for small perturbations of the reference medium: $\rho = \varepsilon \kappa$ where $\varepsilon > 0$ is a small parameter and where $\kappa$ has to be determined.

- Assume that $dF(0): L^\infty(\Omega) \to \mathbb{R}^K$ is onto.

- There exist $\kappa_0, \kappa_1, \ldots, \kappa_K \in L^\infty(\Omega)$ such that $dF(0)(\kappa_0) = 0$ and $dF(0)(\kappa_1), \ldots, dF(0)(\kappa_K) = Id_K$.

- Take $\kappa = \kappa_0 + K \sum_{k=1}^K \tau_k \kappa_k$ where the $\tau_k$ are real parameters to set,

\[ 0 = F(\varepsilon \kappa) \iff 0 = \varepsilon K \sum_{k=1}^K \tau_k dF(0)(\kappa_k) + \varepsilon^2 \tilde{F}_\varepsilon(\kappa). \]

- If $G_\varepsilon$ is a contraction, the fixed-point equation has a unique solution $\vec{\tau}_{\text{sol}}$.

- Set $\rho_{\text{sol}} = \varepsilon \kappa_{\text{sol}}$. We have $F(\rho_{\text{sol}}) = 0$ (invisible perturbation).
Sketch of the method

- Define $\rho = \sigma - 1$ and gather the measurements in the vector
  \[ F(\rho) = (F_1(\rho), \ldots, F_K(\rho))\top \in \mathbb{R}^K. \]

- Let $\Omega \neq \emptyset$ be some Lipschitz domain such that $\Omega \subseteq D$ ($\overline{\Omega}$ will correspond to the support of the perturbation which can be chosen arbitrarily).
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Define $\rho = \sigma - 1$ and gather the measurements in the vector

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Our goal: to find $\rho \in L^\infty(\Omega)$ such that $F(\rho) = 0$ (with $\rho \neq 0$).
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- Taylor: $F(\varepsilon\kappa) = F(0) + \varepsilon dF(0)(\kappa) + \varepsilon^2 \tilde{F}^\varepsilon(\kappa)$. 

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Assume that $dF(0) : L^\infty(\Omega) \to \mathbb{R}^K$ is onto.
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- Take $\kappa = \kappa_0 + \sum_{k=1}^{K} \tau_k \kappa_k$ where the $\tau_k$ are real parameters to set:
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- Define $\rho = \sigma - 1$ and gather the measurements in the vector $F(\rho) = (F_1(\rho), \ldots, F_K(\rho))^\top \in \mathbb{R}^K$.

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$$0 = F(\varepsilon \kappa) \iff$$
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Assume that $dF(0) : L^\infty(\Omega) \to \mathbb{R}^K$ is onto.

Existence of $\kappa_0, \kappa_1, \ldots, \kappa_K \in L^\infty(\Omega)$ such that $dF(0)(\kappa_0) = 0$, $[dF(0)(\kappa_1), \ldots, dF(0)(\kappa_K)] = Id_K$.

Take $\kappa = \kappa_0 + \sum_{k=1}^{K} \tau_k \kappa_k$ where the $\tau_k$ are real parameters to set:

$$0 = F(\varepsilon \kappa) \Leftrightarrow 0 = \varepsilon \bar{\tau} + \varepsilon^2 \tilde{F}^\varepsilon(\kappa)$$
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Take $\kappa = \kappa_0 + \sum_{k=1}^K \tau_k \kappa_k$ where the $\tau_k$ are real parameters to set:

$$0 = F(\varepsilon \kappa) \iff \bar{\tau} = G^\varepsilon(\bar{\tau})$$

where $\bar{\tau} = (\tau_1, \ldots, \tau_K)^\top$.
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- Define $\rho = \sigma - 1$ and gather the measurements in the vector $F(\rho) = (F_1(\rho), \ldots, F_K(\rho))^{\top} \in \mathbb{R}^K$.

Our goal: to find $\rho \in L^\infty(\Omega)$ such that $F(\rho) = 0$ (with $\rho \neq 0$).

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Assume that $dF(0) : L^\infty(\Omega) \rightarrow \mathbb{R}^K$ is onto.

\[ \exists \kappa_0, \kappa_1, \ldots, \kappa_K \in L^\infty(\Omega) \text{ s.t. } \begin{cases} dF(0)(\kappa_0) = 0 \\ [dF(0)(\kappa_1), \ldots, dF(0)(\kappa_K)] = Id_K. \end{cases} \]

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where $\bar{\tau} = (\tau_1, \ldots, \tau_K)^{\top}$ and $G^\epsilon(\bar{\tau}) = -\epsilon \tilde{F}^\epsilon(\kappa)$. 

If $G^\epsilon$ is a contraction, the fixed-point equation has a unique solution $\bar{\tau}^{\text{sol}}$. Set $\rho^{\text{sol}} := \epsilon \kappa^{\text{sol}}$. We have $F(\rho^{\text{sol}}) = 0$ (invisible perturbation).
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  \[ 0 = F(\varepsilon \kappa) \quad \Leftrightarrow \quad \tilde{\tau} = G^\varepsilon(\tilde{\tau}) \]

  where $\tilde{\tau} = (\tau_1, \ldots, \tau_K)^\top$ and $G^\varepsilon(\tilde{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\kappa)$.

  If $G^\varepsilon$ is a contraction, the fixed-point equation has a unique solution $\tilde{\tau}^{\text{sol}}$. 
Sketch of the method

► Define $\rho = \sigma - 1$ and gather the measurements in the vector
\[ F(\rho) = (F_1(\rho), \ldots, F_K(\rho))^\top \in \mathbb{R}^K. \]

Our goal: to find $\rho \in L^\infty(\Omega)$ such that $F(\rho) = 0$ (with $\rho \neq 0$).

► Taylor: $F(\varepsilon \kappa) = \varepsilon dF(0)(\kappa) + \varepsilon^2 \tilde{F}^\varepsilon(\kappa)$.

Assume that $dF(0) : L^\infty(\Omega) \to \mathbb{R}^K$ is onto.

\[ \exists \kappa_0, \kappa_1, \ldots, \kappa_K \in L^\infty(\Omega) \text{ s.t.} \quad \begin{bmatrix} dF(0)(\kappa_0) = 0 \\ [dF(0)(\kappa_1), \ldots, dF(0)(\kappa_K)] = \text{Id}_K. \end{bmatrix} \]

► Take $\kappa = \kappa_0 + \sum_{k=1}^K \tau_k \kappa_k$ where the $\tau_k$ are real parameters to set:
\[ 0 = F(\varepsilon \kappa) \quad \Leftrightarrow \quad \vec{\tau} = G^\varepsilon(\vec{\tau}) \]

where $\vec{\tau} = (\tau_1, \ldots, \tau_K)^\top$ and $G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\kappa)$.

If $G^\varepsilon$ is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{sol}$.
Set $\rho^{sol} := \varepsilon \kappa^{sol}$. We have $F(\rho^{sol}) = 0$ (invisible perturbation).
1 General scheme

2 Application to our problem

3 Numerical experiments
Calculus of $dF(0)$

- For our problem, we have ($\rho = \sigma - 1$)

$$F(\rho) = (\mathcal{M}(\sigma)_{mn})_{1 \leq m \leq n \leq N}.$$

Is $dF(0) : L_\infty(\Omega) \rightarrow \mathbb{R}$ onto?
Calculus of $dF(0)$

- For our problem, we have $(\rho = \sigma - 1)$

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To compute $dF(0)(\kappa)$, we take $\sigma^\varepsilon = 1 + \varepsilon \kappa$ with $\kappa$ supported in $\Omega$. 

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To compute $dF(0)(\kappa)$, we take $\sigma^\varepsilon = 1 + \varepsilon \kappa$ with $\kappa$ supported in $\overline{\Omega}$.

- We denote $u_n^\varepsilon$ the functions satisfying

$$\begin{align*}
\text{div} (\sigma^\varepsilon \nabla u_n^\varepsilon) &= 0 \\
\nu \cdot \sigma^\varepsilon \nabla u_n^\varepsilon &= \delta_n - \delta_0
\end{align*}$$
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- $${\mathcal M}(\sigma)_{mn} = \int_{\Omega} (1 - \sigma^\varepsilon) \nabla u^\varepsilon_m \cdot \nabla u^0_n \, d\mathbf{x}.$$
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- We can prove that $u^\varepsilon_m = u^0_m + O(\varepsilon)$. 
Calculus of \( dF(0) \)

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- Thus, we find

$$dF(0)(\kappa) = \left( - \int_{\Omega} \kappa \nabla u^0_m \cdot \nabla u^0_n \, dx \right)_{1 \leq m \leq n \leq N}$$
Calculus of $dF(0)$

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Is $dF(0) : L^\infty(\Omega) \to \mathbb{R}^K$ onto?
Construction of the shape functions

\[ \begin{align*} dF(0)(\kappa) &= \left( - \int_{\Omega} \kappa \nabla u_0^m \cdot \nabla u_0^n \, dx \right)_{1 \leq m \leq n \leq N} \end{align*} \]

1. Using classical results concerning Gram matrices, we can prove that 

\[ \mathcal{J} := \{ \nabla u_0^m \cdot \nabla u_0^n \}_{1 \leq m \leq n \leq N} \in C^\infty(\overline{\Omega})^K \]

is a family of linearly independent functions

\[ \iff \text{there are } \kappa_{mn} \in \text{span}(\mathcal{J}) \text{ s.t. } - \int_{\Omega} \kappa_{mn} \nabla u_0^{m'} \cdot \nabla u_0^{n'} \, dx = \begin{cases} 1 & \text{if } (m,n) = (m',n') \\ 0 & \text{else} \end{cases} \]
Construction of the shape functions

\[ dF(0)(\kappa) = \left( - \int_{\Omega} \kappa \nabla u_0^m \cdot \nabla u_0^n \, dx \right)_{1 \leq m \leq n \leq N} \]

1. Using classical results concerning Gram matrices, we can prove that

\[ \mathcal{S} := \{ \nabla u_0^m \cdot \nabla u_0^n \}_{1 \leq m \leq n \leq N} \in C^\infty(\overline{\Omega})^K \]

is a family of linearly independent functions if

\[ \begin{align*}
\text{there are } \kappa_{mn} \in \text{span}(\mathcal{S}) \text{ s.t. } & \quad - \int_{\Omega} \kappa_{mn} \nabla u_0^{m'} \cdot \nabla u_0^{n'} \, dx = 1 \text{ if } (m,n) = (m',n') \\
& \quad 0 \text{ else}
\end{align*} \]

\[ \iff \ dF(0) : L^\infty(\Omega) \to \mathbb{R}^K \text{ is onto.} \]
Construction of the shape functions

\[ dF(0)(\kappa) = \left( -\int_{\Omega} \kappa \nabla u^0_m \cdot \nabla u^0_n \, dx \right)_{1 \leq m \leq n \leq N} \]

1. Using classical results concerning Gram matrices, we can prove that

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\[ \iff dF(0) : L^\infty(\Omega) \to \mathbb{R}^K \text{ is onto.} \]

2. We need to construct some \( \kappa_0 \in \ker dF(0) \), i.e. some \( \kappa_0 \) satisfying

\[ \int_{\Omega} \kappa_0 \nabla u^0_m \cdot \nabla u^0_n \, dx = 0, \quad \forall m, n = 1, \ldots, N. \]
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\[ \iff dF(0) : L^\infty(\Omega) \to \mathbb{R}^K \text{ is onto.} \]

2. We take

\[ \kappa_0 = \kappa_0^\# - \sum_{1 \leq m \leq n \leq N} \left( \int_{\Omega} \kappa_{mn} \kappa_0^\# \, dx \right) \kappa_{mn} \]

where \( \kappa_0^\# \notin \text{span}\{\kappa_{mn}\}_{1 \leq m \leq n \leq N} \).
PROP. Assume that \( \{ \nabla u_m^0 \cdot \nabla u_n^0 \}_{1 \leq m \leq n \leq N} \in \mathscr{C}^\infty(\overline{\Omega})^K \) is a family of linearly independent functions. For \( \varepsilon \) small enough, define \( \sigma^{\text{sol}} = 1 + \varepsilon \kappa^{\text{sol}} \mathbb{1}_\Omega \) with

\[
\kappa^{\text{sol}} = \kappa_0 + \sum_{1 \leq m \leq n \leq N} \tau_{mn}^{\text{sol}} \kappa_{mn}.
\]

Then, we have

\[
\mathcal{M}(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda \sigma^{\text{sol}} - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D} = 0, \quad \forall m, n = 1, \ldots, N,
\]

so that the conductivity perturbation is invisible.

**COMMENTS:**

→ We need \( \varepsilon \) to be small enough to prove that \( G^{\varepsilon} \) is a contraction.

→ We have \( \kappa^{\text{sol}} \neq 0 \) (non trivial perturbation). To see it, compute \( dF(0)(\kappa^{\text{sol}}) \).
Main result

It remains to prove that \( \{\nabla u_m^0 \cdot \nabla u_n^0\}_{1 \leq m \leq n \leq N} \in C^\infty(\overline{\Omega})^K \) is a family of linearly independent functions.
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By definition, we have
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\begin{align*}
\Delta u_n^0 &= 0 \\
\nu \cdot \nabla u_n^0 &= \delta_n - \delta_0.
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It remains to prove that \( \{ \nabla u^0_m \cdot \nabla u^0_n \}_{1 \leq m \leq n \leq N} \in C^\infty(\overline{\Omega})^K \) is a family of linearly independent functions.

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1. When \( D \) is a 2D disk, there holds

\[
u^n(x) = \frac{1}{\pi} \ln |x - x_0| - \frac{1}{\pi} \ln |x - x_n|
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and the result can be proved doing explicit computations.
It remains to prove that \( \{\nabla u_m \cdot \nabla u_n\}_{1 \leq m \leq n \leq N} \in \mathcal{C}^\infty(\overline{\Omega})^K \) is a family of linearly independent functions.

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and the result can be proved doing explicit computations.

2. Then, we deduce that the result is also true for general 2D smooth domains using conformal mapping techniques.
**Thm.** Let $D \subset \mathbb{R}^2$ be a smooth domain and $\Omega$ a nonempty Lipschitz domain such that $\Omega \subset D$. For $\varepsilon$ small enough, define $\sigma^\text{sol} = 1 + \varepsilon \kappa^\text{sol} 1_\Omega$ with

$$
\kappa^\text{sol} = \kappa_0 + \sum_{1 \leq m \leq n \leq N} \tau_{mn}^\text{sol} \kappa_{mn}.
$$

Then, we have

$$
\mathcal{M}(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda \sigma^\text{sol} - \Lambda^1)(\delta_n - \delta_0) \rangle_{\partial D} = 0, \quad \forall m, n = 1, \ldots, N,
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so that the conductivity perturbation is invisible.
Thm. Let $D \subset \mathbb{R}^2$ be a smooth domain and $\Omega$ a nonempty Lipschitz domain such that $\Omega \subseteq D$. For $\varepsilon$ small enough, define $\sigma^{\text{sol}} = 1 + \varepsilon \kappa^{\text{sol}} 1_\Omega$ with
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\]
so that the conductivity perturbation is invisible.

Comments:

→ The 3D case is open.

→ The existence of invisible inclusions may appear not so surprising since $\mathcal{M}(\sigma) \in \mathbb{R}^{N \times N}$, $\sigma \in \mathcal{L}^\infty(\Omega)$. 
**Main result**

**THM.** Let $D \subset \mathbb{R}^2$ be a smooth domain and $\Omega$ a nonempty Lipschitz domain such that $\Omega \Subset D$. For $\varepsilon$ small enough, define $\sigma^{\text{sol}} = 1 + \varepsilon \kappa^{\text{sol}} 1_\Omega$ with

$$\kappa^{\text{sol}} = \kappa_0 + \sum_{1 \leq m \leq n \leq N} \tau^{\text{sol}}_{mn} \kappa_{mn}.$$ 

Then, we have

$$M(\sigma)_{mn} = \langle \delta_m - \delta_0, (\Lambda^{\sigma^{\text{sol}}_m} - \Lambda^1) (\delta_n - \delta_0) \rangle_{\partial D} = 0, \quad \forall m, n = 1, \ldots, N,$$

so that the conductivity perturbation is invisible.

**Comments:**

→ The 3D case is open.

→ The existence of invisible inclusions may appear not so surprising since $M(\sigma) \in \mathbb{R}^{N \times N}, \sigma \in L^\infty(\Omega)$. However, for an analogous problem in scattering theory, this result does not hold ...
1 General scheme

2 Application to our problem

3 Numerical experiments
Influence of the choice of $\varepsilon$

- Examples of conductivities (at the end of the fixed point iteration) which provide the same measurements as the reference conductivity $\sigma \equiv 1$.

(The dots correspond to the positions of the electrodes.)
Influence of the choice of $\varepsilon$

- Examples of conductivities (at the end of the fixed point iteration) which provide the same measurements as the reference conductivity $\sigma \equiv 1$.

- Convergence of the fixed point iteration with respect to the choice of $\varepsilon$.

- 3D view of $\sigma$ for $\varepsilon = 0.6$
Influence of the number of electrodes

The dots correspond to the position of the $N + 1$ electrodes.

When the number of electrodes increases, the obtained perturbation of the reference conductivity $\sigma \equiv 1$ becomes smaller and smaller.
Influence of the choice of $\kappa_0^\#$ and of the shape

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon$</th>
<th>$\kappa_0^#(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>4.0</td>
<td>$x + y + 1$</td>
</tr>
<tr>
<td>(b)</td>
<td>2.0</td>
<td>$\exp(-(x + 0.5)^2 - y^2)$</td>
</tr>
<tr>
<td>(c)</td>
<td>0.25</td>
<td>1</td>
</tr>
<tr>
<td>(d)</td>
<td>6.0</td>
<td>1</td>
</tr>
<tr>
<td>(e)</td>
<td>0.5</td>
<td>$-y$</td>
</tr>
<tr>
<td>(f)</td>
<td>2.0</td>
<td>$x$</td>
</tr>
</tbody>
</table>

3D view of $\sigma$ for case (a)
1 General scheme

2 Application to our problem

3 Numerical experiments
Conclusion

What we did

♠ We explained how to construct invisible conductivity perturbations for the Point Electrode Model.

♠ The proof is rigorous for the 2D setting with $\sigma^0 \equiv 1$.

Open questions

1) Can we prove that $\{\nabla u_m^0 \cdot \nabla u_n^0\}_{1 \leq m \leq n \leq N}$ is a family of linearly independent functions in 3D?

2) Can we justify the construction of invisible conductivity perturbations when $\sigma^0 \neq 1$?

3) Can we reiterate the process to construct larger invisible perturbations of the reference conductivity?

4) Can we construct invisible conductivity perturbations for other models (Complete Electrode Model)?
Kiitos!