Applied Inverse Problems Conference 2019

Zero transmission via Fano resonance in non symmetric waveguides

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Collaboration with S. A. Nazarov[2].

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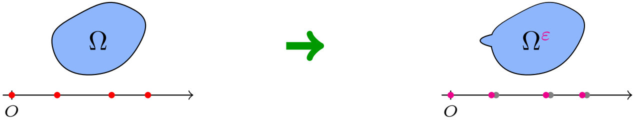
Introduction

- Consider the eigenvalue problem

$$\left|\begin{array}{rcll}\Delta v+\lambda v & = & 0 & \text{in } \Omega,\\ \partial_n v & = & 0 & \text{on } \partial\Omega.\end{array}\right.$$

- In bounded domains, small smooth perturbations of the geometry slightly shift the spectrum in $\mathbb{R}$ (eigenvalues remain eigenvalues).



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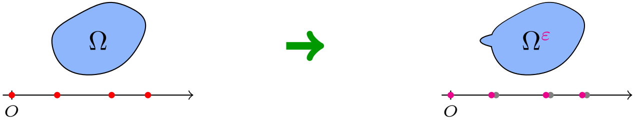

- In unbounded waveguides, small perturbations of the geom. transform eigenvalues embedded in the continuous spectrum into complex resonances.

→ See `Aslanyan, Parnovski, Vassiliev, Q. J. Mech. Appl. Math., 00`.

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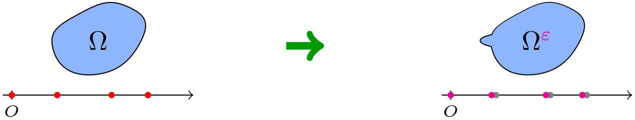

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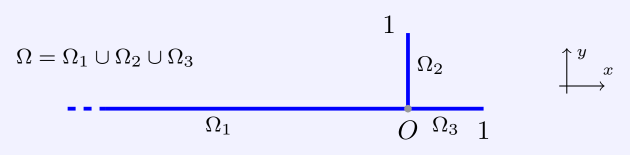
What is the influence of these resonances on the scattering properties ?

A 1D toy problem

- First, we consider a simple 1D problem.

A 1D toy problem



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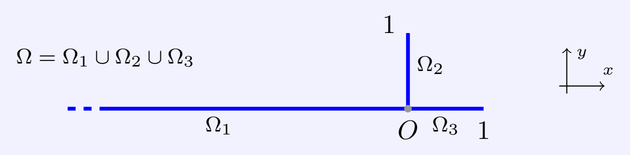

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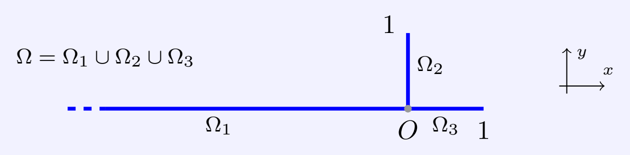

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- Well-posedness $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.

- Uniqueness $\Leftrightarrow$ $k \notin (2\mathbb{N}+1)\pi/2$. Existence for all $k \in \mathbb{R}$ ($F \in \ker \mathbb{M}^{\perp}$).

$$R = \frac{\cos(k) + 2i\sin(k)}{\cos(k) - 2i\sin(k)}$$

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- We perturb the geometry: $\Omega^\varepsilon = \Omega_1 \cup \Omega_2 \cup \Omega_3^\varepsilon$ with $\Omega_3^\varepsilon = (0; 1+\varepsilon)$.
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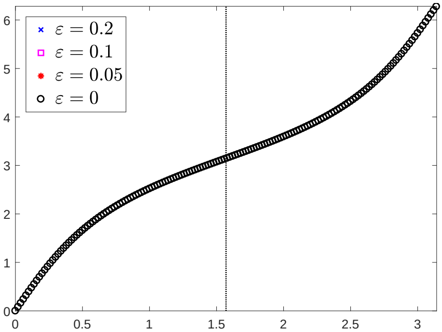

Figure: $k \mapsto \boldsymbol{\theta^\varepsilon(k)}$ for several ε (non uniqueness for $\varepsilon = 0$, $k = \pi/2$).

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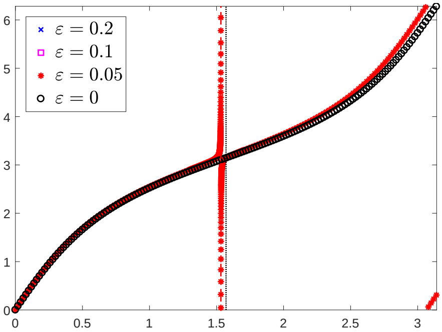

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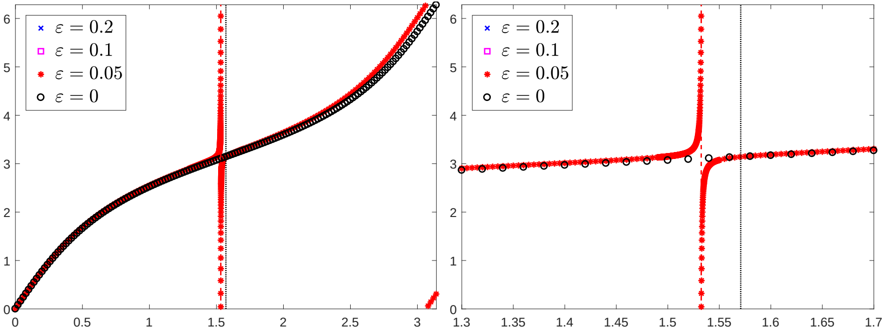

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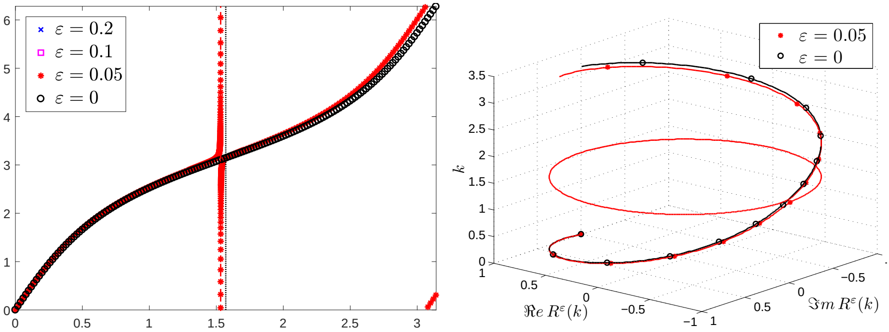

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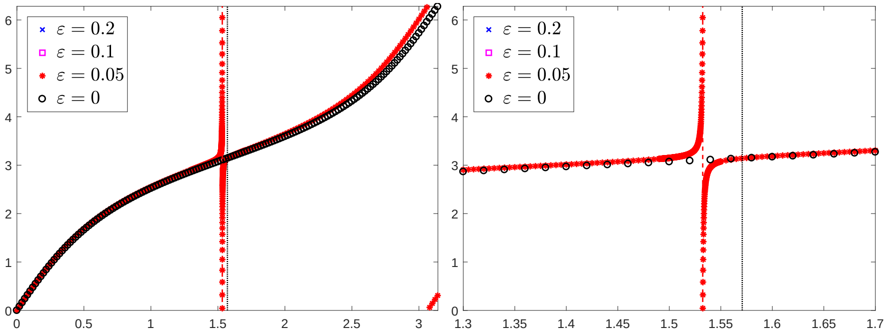

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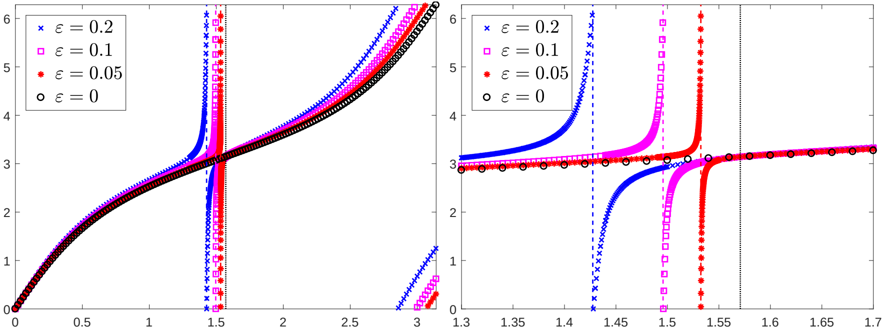

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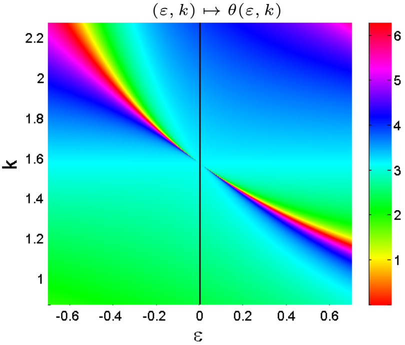
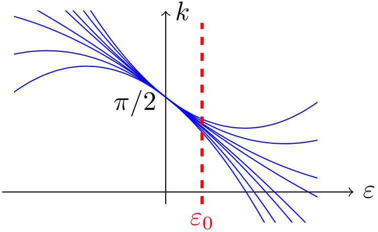
- Set $R(\varepsilon, k) = e^{i\theta(\varepsilon,k)}$ (functions of two variables).





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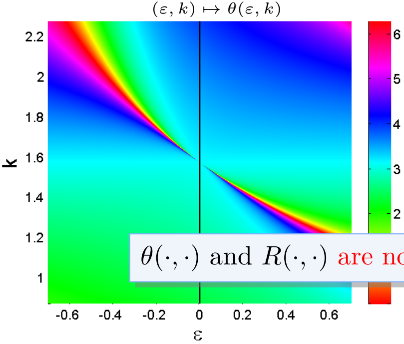

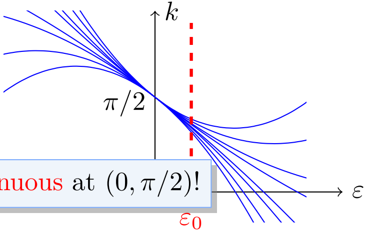

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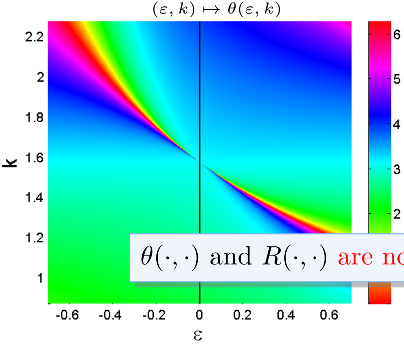

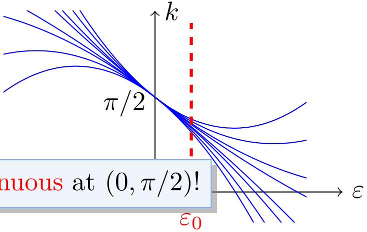

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Goals of the talk

1) Prove a similar Fano resonance phenomenon in waveguides.
2) Show that zero transmission always occurs during the phenomenon.

Outline of the talk

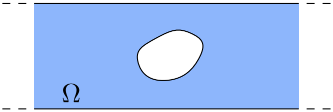
1. The Fano resonance in waveguides

2. Zero transmission

3. Numerical experiments

Setting

- Let Ω be a waveguide which coincides with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region. We consider the problem



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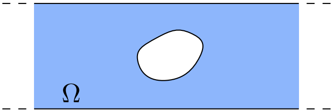

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- The scattering problem associated with $(*)$ writes

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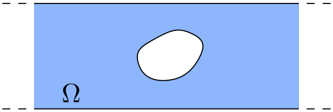

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- For this problem with $k := \sqrt{\lambda} \in (0;\pi)$, the modes are

$$\begin{array}{r|l} \text{Propagating} & w_\pm(x,y) = e^{\pm ikx}, \\ \text{Evanescent} & w_n^\pm(x,y) = e^{\mp\beta_n x}\cos(n\pi y), \quad \beta_n = \sqrt{n^2\pi^2 - \lambda},\ n \geq 1. \end{array}$$

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- For $\boldsymbol{v_i} = \boldsymbol{w_\pm}$, $(\mathscr{P})$ admits the scattering solutions (existence)

$$v_+ = \left| \begin{array}{r} w_+ + R_+ w_- + \dots \\ T \quad w_+ + \dots \end{array} \right. \qquad v_- = \left| \begin{array}{r} T \quad w_- + \dots \\ w_- + R_- w_+ + \dots \end{array} \right. \qquad \begin{array}{l} \text{for } x < 0 \\ \text{for } x > 0 \end{array}$$

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where $R_\pm, T \in \mathbb{C}$ and $\dots$ are exponentially decaying terms.

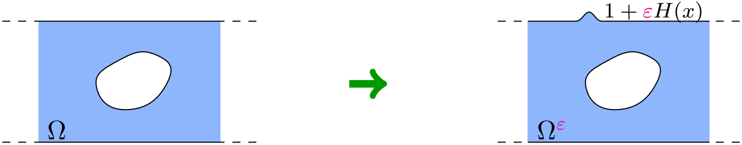
- The scattering matrix

$$\mathbb{S} = \begin{pmatrix} R_+ & T \\ T & R_- \end{pmatrix}$$

is uniquely defined (even for $\lambda = \lambda^0$), unitary ($\mathbb{S}\overline{\mathbb{S}}^\top = \mathrm{Id}$) and symmetric.

Small perturbation of the geometry 1/2

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry



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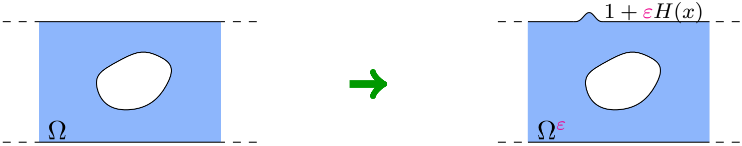

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The following theorem describes the behaviour of $(\varepsilon, \lambda) \mapsto \mathbb{S}(\varepsilon, \lambda)$ in a neighbourhood of $(0, \lambda^0)$ where trapped modes exist.

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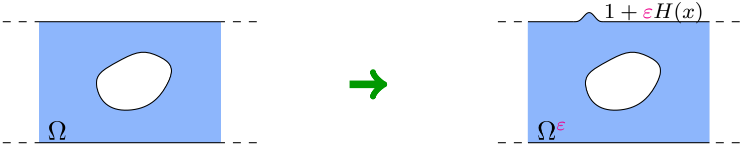

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$(\mathscr{H})$ | We assume that λ^0 is a simple eigenvalue for $(*)$ and that the eigenfunctions do not decay faster than $C\, e^{-\beta_1 |x|}$ as $|x| \to +\infty$.

THEOREM: Set $\mathbb{S}^0 = \mathbb{S}(0, \lambda^0)$. There is $\lambda'_p \in \mathbb{R}$ such that when $\varepsilon \to 0$,

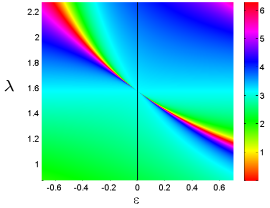
$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda') = \mathbb{S}^{\mathbf{0}} + O(\varepsilon) \qquad \text{for } \lambda' \neq \lambda'_p,$$

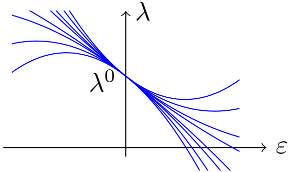
and, for any $\mu \in \mathbb{R}$,

$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) = \mathbb{S}^{\mathbf{0}} + \frac{\boldsymbol{\tau}^\top\boldsymbol{\tau}}{\boldsymbol{i\tilde{\mu}} - |\boldsymbol{\tau}|^2/2} + O(\varepsilon).$$

Here $\tau = (a, b) \in \mathbb{C} \times \mathbb{C}$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some unessential real constants A, B with $A \neq 0$.

- Similar to the 1D picture:





Small perturbation of the geometry 2/2

THEOREM: Set $\mathbb{S}^0 = \mathbb{S}(0, \lambda^0)$. There is $\lambda'_p \in \mathbb{R}$ such that when $\varepsilon \to 0$,

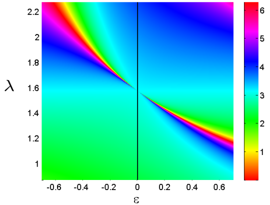
$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda') = \mathbb{S}^{\mathbf{0}} + O(\varepsilon) \qquad \text{for } \lambda' \neq \lambda'_p,$$

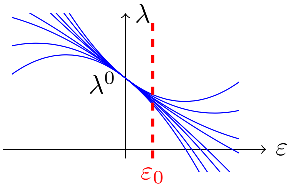
and, for any $\mu \in \mathbb{R}$,

$$\mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) = \mathbb{S}^{\mathbf{0}} + \frac{\boldsymbol{\tau}^\top \boldsymbol{\tau}}{\boldsymbol{i\tilde{\mu}} - |\boldsymbol{\tau}|^2/2} + O(\varepsilon).$$

Here $\tau = (a, b) \in \mathbb{C} \times \mathbb{C}$ depends only on Ω and $\tilde{\mu} = A\mu + B$ for some unessential real constants A, B with $A \neq 0$.

- Similar to the 1D picture:





COMMENTS:

- $\mathbb{S}(\cdot,\cdot)$ is not continuous at $(0,\lambda^0)$.

- For a small given ε_0, the map $\lambda \mapsto \mathbb{S}(\varepsilon_0,\lambda)$ varies quickly at $\lambda^0+\varepsilon^0\lambda'_p$.

- Under certain conditions on H, the variation can be even quicker...

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Small perturbation of the geometry 2/2

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- For a small given ε_0, the map $\lambda \mapsto \mathbb{S}(\varepsilon_0,\lambda)$ varies quickly at $\lambda^0+\varepsilon^0\lambda'_p$.

- Under certain conditions on H, the variation can be even quicker...

Ingredients of the proof:

- Use weighted Sobolev spaces with detached asymptotics to define scattering solutions with non standard radiation conditions.

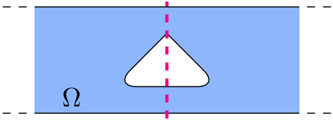
- Define an augmented scattering matrix $\mathfrak{S}$ (Nazarov, Plamenevsky, 94).

- Compute an asymptotic expansion of $\mathfrak{S}$ which is smooth at $(0,\lambda^0)$ because uniqueness holds for the problem with non standard radiation conditions.

- Use the connection existing between $\mathbb{S}$ and $\mathfrak{S}$ to get an expansion for $\mathbb{S}$.

Symmetric waveguide 1/2

- We assume that Ω is symmetric with respect to the (Oy) axis.



$$\left|\begin{array}{rcll} \Delta v + \lambda v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega. \end{array}\right.$$

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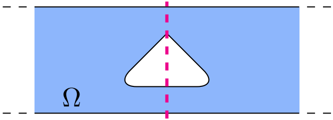

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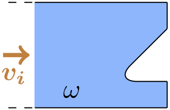
- Introduce the two half-waveguide problems



$$\left|\begin{array}{rl}\Delta u+\lambda u=0 & \text{in } \omega\\ \partial_n u=0 & \text{on } \partial\omega\end{array}\right.$$

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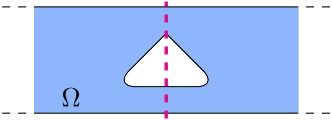

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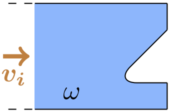

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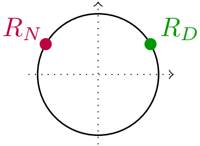
- They admit the solutions

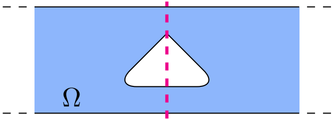
$$\left|\begin{array}{rcl}u & = & e^{ikx} + R_N\, e^{-ikx} + \dots \\ U & = & e^{ikx} + R_D\, e^{-ikx} + \dots\end{array}\right.$$



with $|R_N| = |R_D| = 1$ (conservation of energy).

Symmetric waveguide 1/2

- We assume that Ω is symmetric with respect to the (Oy) axis.



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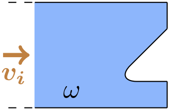

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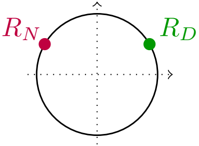

$$\left|\begin{array}{rcl} u &=& e^{ikx} + R_N\, e^{-ikx} + \dots \\ U &=& e^{ikx} + R_D\, e^{-ikx} + \dots \end{array}\right.$$

with $|R_N| = |R_D| = 1$ (conservation of energy).

- One can prove that $\boxed{R_\pm = \dfrac{R_N + R_D}{2}}$ and $\boxed{T = \dfrac{R_N - R_D}{2}.}$

Symmetric waveguide 2/2

$$\boxed{R_{\pm} = \frac{R_N + R_D}{2}} \qquad \boxed{T = \frac{R_N - R_D}{2}}$$

- To set ideas, we assume that eigenfunctions are symmetric w.r.t. (Oy).
  $\Rightarrow$ They are eigenfunctions for the pb with Neumann B.Cs.

i) λ^0 is not an eigenvalue for the pb with Dirichlet condition. This implies

$$|R_D(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) - R_D(0, \lambda^0)| \leq C\,\varepsilon, \qquad \forall\varepsilon \in (0; \varepsilon_0],\ \mu \in [-c\varepsilon^{-1}; c\varepsilon].$$

ii) $\mu \mapsto R_N(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$ rushes on the unit circle for $\mu \in [-c\varepsilon^{-1}; c\varepsilon]$.

Symmetric waveguide 2/2

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Proposition:

$\exists \lambda^\varepsilon$, with $\lambda^\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $R_\pm(\varepsilon, \lambda^\varepsilon) = 0$ (zero reflection).

$\exists \tilde\lambda^\varepsilon$, with $\tilde\lambda^\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $T(\varepsilon, \tilde\lambda^\varepsilon) = 0$ (zero transmission).

Symmetric waveguide 2/2

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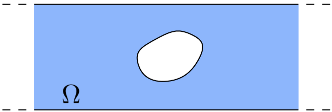
$\rightarrow$ Similar results in `Shipman and Tu, SIAM Appl. Math, 2012`. We use a different approach and consider a perturbation of the geometry.

Non symmetric waveguide 1/2



- We can not work as before but we can still prove the following result.

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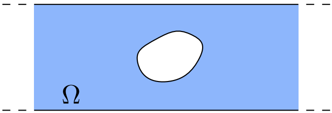

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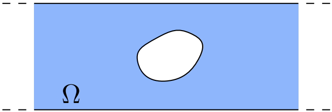
`Proof. 1)` Set $T^\varepsilon(\mu) = T(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$. The expansion of $\mathbb{S}$ yields

$$|T^\varepsilon(\mu) - T^{\text{asy}}(\mu)| \le C\,\varepsilon \qquad \text{with} \qquad T^{\text{asy}}(\mu) = T^0 + \frac{ab}{i\tilde{\mu} - (|a|^2 + |b|^2)/2},$$

where a, b are some constants.

Non symmetric waveguide 1/2



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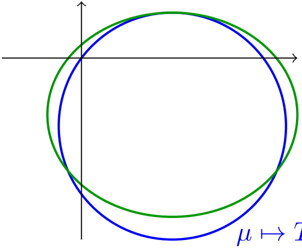
where a, b are some constants.

`2)` Properties of the Möbius transform and the unitarity of $\mathbb{S}^0$ guarantee that $\{T^{\text{asy}}(\mu)\,|\,\mu \in \mathbb{R}\}$ is a circle passing through zero.

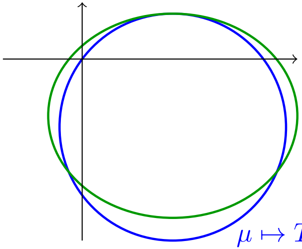
Non symmetric waveguide 2/2



3) If $\mu \mapsto T^{\varepsilon}(\mu)$ does not pass through zero, $\mu \mapsto 2\,\text{phase}(T^{\varepsilon}(\mu))$ varies quickly. One can show that this contradicts the identity

$$T^{\varepsilon}(\mu)/\overline{T^{\varepsilon}(\mu)} = -R^{\varepsilon}_{+}(\mu)/\overline{R^{\varepsilon}_{-}(\mu)}$$

which is a consequence of the unitarity of $\mathbb{S}^{\varepsilon}(\mu)$. $\square$



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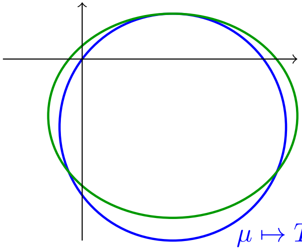
The unitarity structure of $\mathbb{S}$ is the key to conclude.

Non symmetric waveguide



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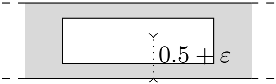
→ Similar idea in `Lee, Phys. Rev. Lett., 99` using a perturbation argument.

Symmetric waveguide

- Numerics using FE methods (`Freefem++`) with DtN maps or PMLs.

Symmetric waveguide

- Left: waveguide. Right: eigenfunction for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.42$.



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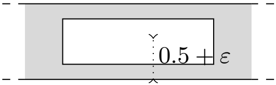
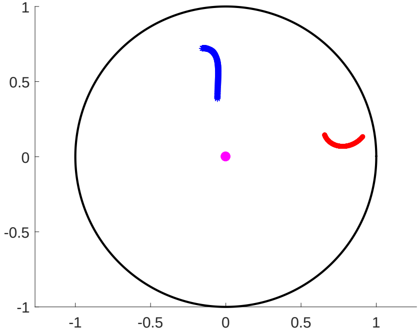


- **Scattering coefficients** for $k \in (2.2; 2.7)$.

No shift ($\boldsymbol{\varepsilon = 0}$)



$k \mapsto R_{\pm}(0,k)$ $k \mapsto T(0,k)$

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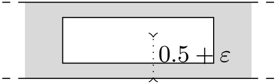

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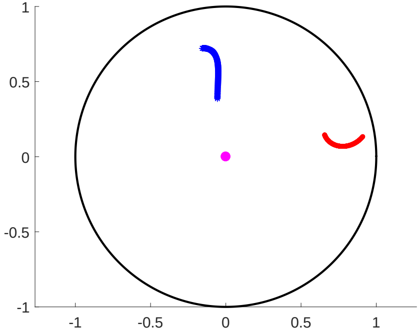
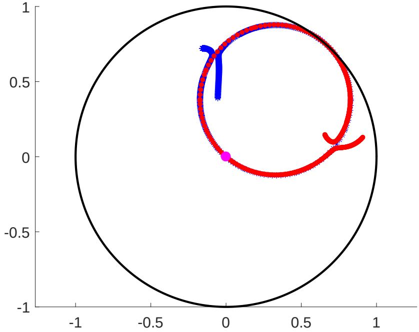


$k \mapsto R_{\pm}(0, k)$ $k \mapsto T(0, k)$

Small shift $(\boldsymbol{\varepsilon} > \mathbf{0})$

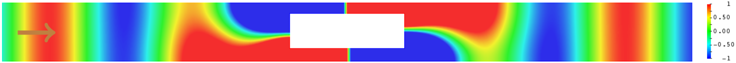


$k \mapsto R_{\pm}(0.02, k)$ $k \mapsto T(0.02, k)$

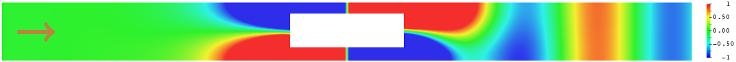
Symmetric waveguide

- Example of setting where $R_{\pm}(\varepsilon, \lambda^{\varepsilon}) = 0$ (zero reflection).

$\Re e\, v_+$



$\Re e\,(v_+ - w_+)$



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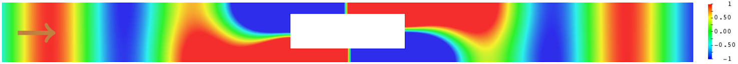

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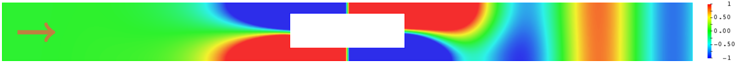
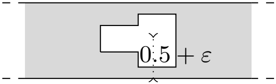


- Example of setting where $T(\varepsilon, \tilde{\lambda}^{\varepsilon}) = 0$ (zero transmission).

$\Re e\, v_+$

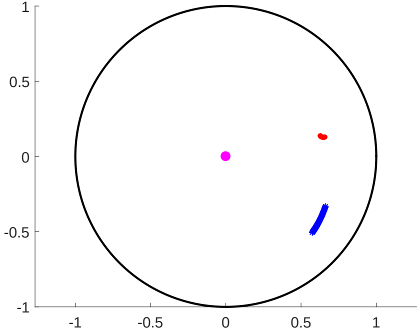
Non symmetric waveguide

- Left: waveguide. Right: eigenfunction for $\varepsilon = 0$ and $k^0 := \sqrt{\lambda^0} \approx 2.03$.



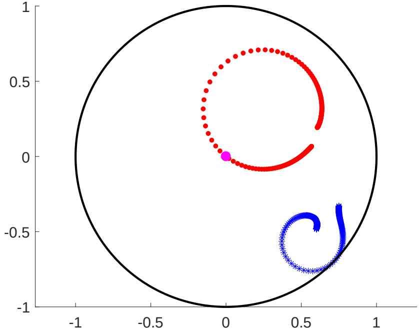
- **Scattering coefficients** for $k \in (1.8; 2.2)$.

No shift $(\boldsymbol{\varepsilon = 0})$



$k \mapsto R_+(0,k)$ $k \mapsto T(0,k)$

Small shift $(\boldsymbol{\varepsilon > 0})$



$k \mapsto R_+(0.1,k)$ $k \mapsto T(0.1,k)$

Non symmetric waveguide

- Example of setting where $\boxed{T(\varepsilon, \lambda^{\varepsilon}) = 0}$ (zero transmission).

$\Re e\, v_+$

Frequency behaviour

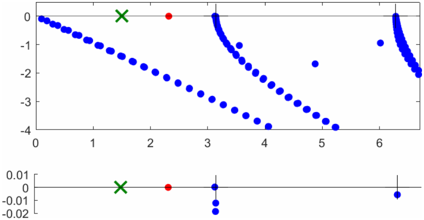
No shift $(\varepsilon = \mathbf{0})$ | Small shift $(\varepsilon > \mathbf{0})$

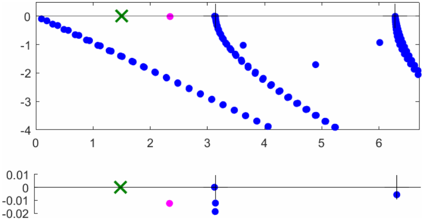
- $k \mapsto \Re e\, v_+(k)$

- **Complex spectrum** computed with PMLs (we zoom at the real axis).

• Trapped mode

• Complex resonance





Conclusion

What we did

- We proved the Fano resonance phenomenon in a 2D waveguide.

  If trapped modes exist for $(\varepsilon, \lambda) = (0, \lambda^0)$*, then for* $\varepsilon > 0$ *small,* $\lambda \mapsto \mathbb{S}(\varepsilon, \lambda)$ *has a quick variation at* λ^0*.* `Symmetry is not needed.`

- If Ω `symmetric w.r.t.` (Oy), zero reflection, zero transmission occur.
  If Ω `not symmetric`, zero transmission occurs.

- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dimension.

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- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dimension.

Future work

1) Is there zero reflection/zero transmission for $k > \pi$ (monomode regime was essential in the mechanism)?
2) What happens if λ^0 is not a simple eigenvalue?

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Thank you for your attention!