Séminaire EDP/physique mathématique

Invisibility and complete reflectivity in acoustic waveguides

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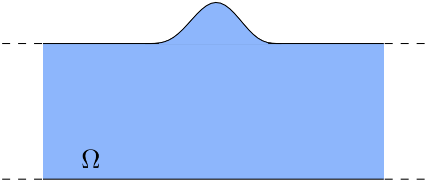
Institut de Mathématiques de Bordeaux, 19/09/2017

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.



Find $v = v_i + v_s$ s. t.

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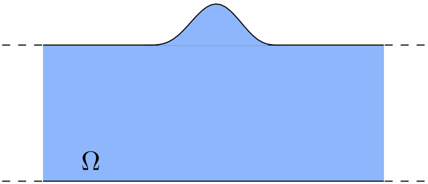

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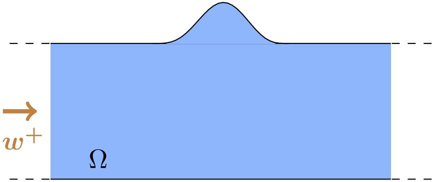

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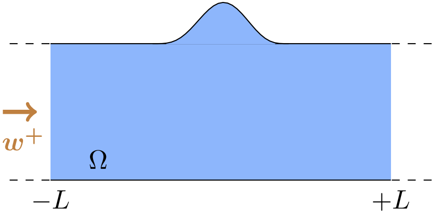

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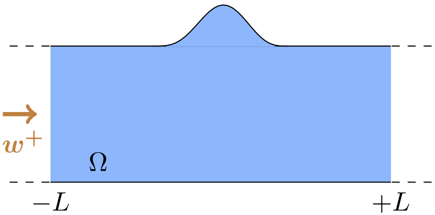

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DEFINITION: v_i = incident field, v = total field, v_s = scattered field.

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

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GOAL

We explain how to find waveguides such that

$$R = 0 \; (|T| = 1), \;\; T = 1 \; (R = 0) \;\; \text{or} \;\; T = 0 \; (|R| = 1).$$

Outline of the talk

1. First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3. A spectral approach to determine non reflecting wavenumbers

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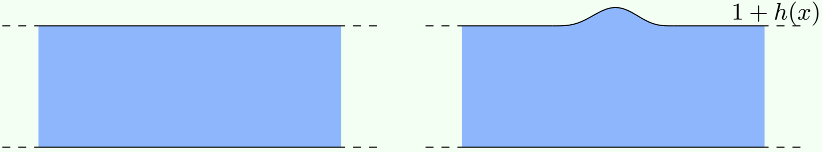
General picture

- Perturbative technique: we construct small non reflecting defects using variants of the implicit functions theorem.

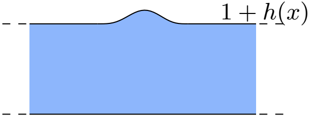


$R = 0$ $\qquad$ $R = 0$

- The idea was used in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, set $R = R(h) \in \mathbb{C}$.



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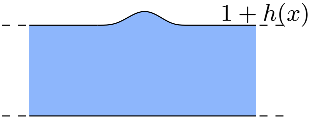

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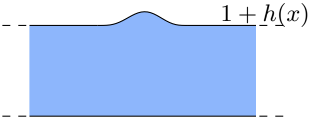

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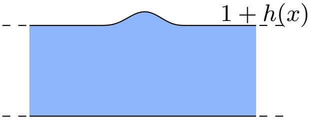

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- We look for small perturbations of the reference medium: $h = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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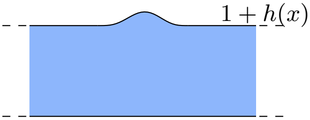

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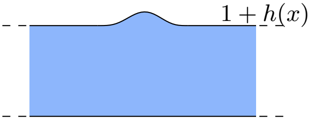

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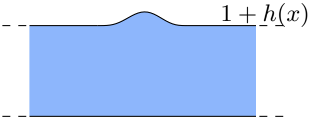

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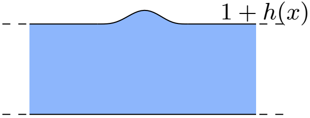

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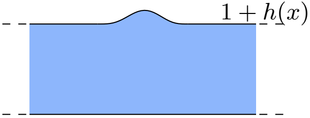

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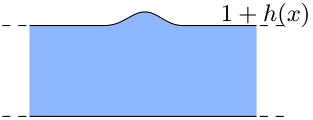

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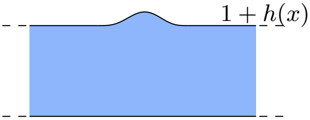

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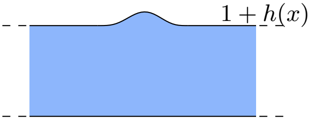

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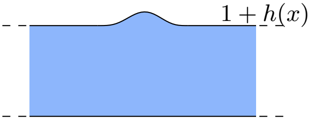

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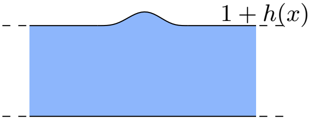

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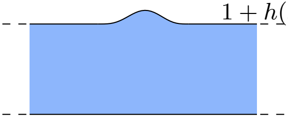

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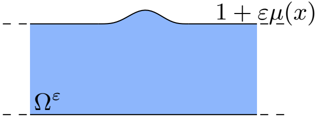
Set $h^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $R(h^{\mathrm{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential



- Using classical results of asymptotic analysis, we obtain

$$R(\varepsilon\mu) = 0 + \varepsilon \left(-\frac{1}{2} \int_{-\ell}^{\ell} \partial_x \mu(x) (w^+(x,1))^2 \, dx \right) + O(\varepsilon^2).$$

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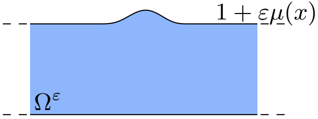

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$dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto $\Rightarrow$ we can get non trivial Ω s.t. $R = 0$.

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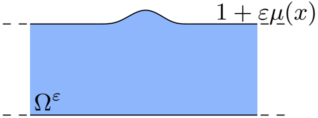

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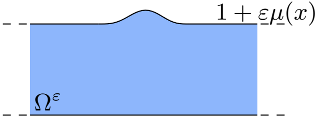

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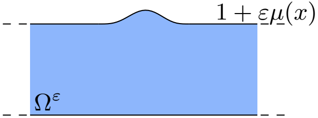

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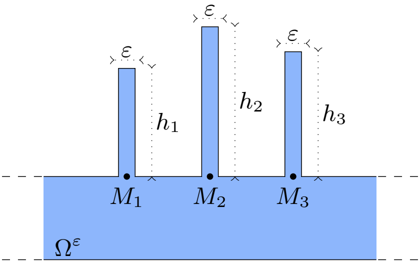
$dT(0)$ is not onto $\Rightarrow$ the approach fails to impose $T = 1$.

A perturbative method to get $T = 1$

- We study the same problem in the geometry Ω^ε



- We obtain

$$\begin{aligned} R &= 0 + \varepsilon \left(ik \textstyle\sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2) \\ T &= 1 + \varepsilon \left(i/2 \textstyle\sum_{n=1}^{3} \tan(kh_n) \right) + O(\varepsilon^2) \end{aligned}$$

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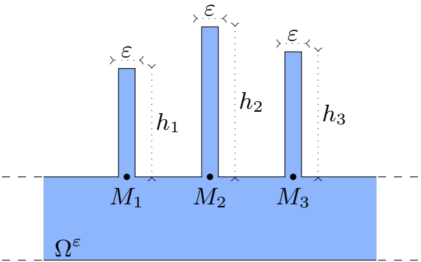

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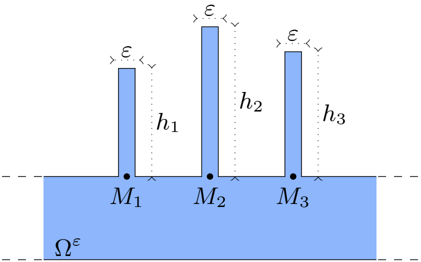

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2) Then changing h_n into $h_n + \tau_n$, and choosing a good $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ (fixed point), we can get $R = 0$ and $\Im m\, T = 0$.

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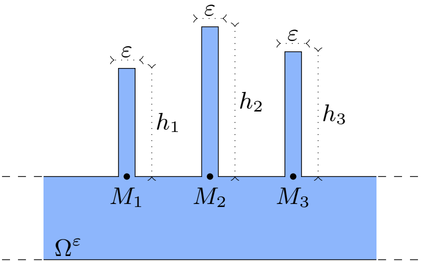

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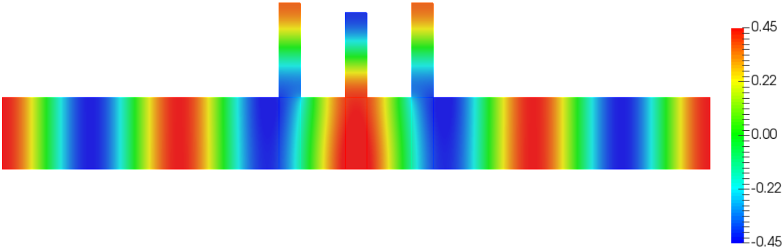
3) Energy conservation + $[T = 1 + O(\varepsilon)] \quad \Rightarrow \quad T = 1$.

Numerical results

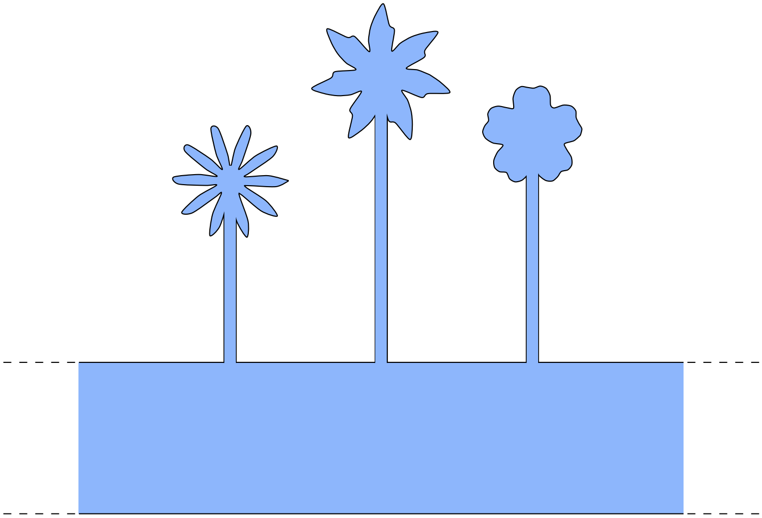
- Perturbed waveguide ($\Re e\,(v(x,y)e^{-i\omega t})$)



- Reference waveguide ($\Re e\,(v_i(x,y)e^{-i\omega t})$)

Remark

- We could also have worked with gardens of flowers!



Outline of the talk

1. First constructive method

   k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

   k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

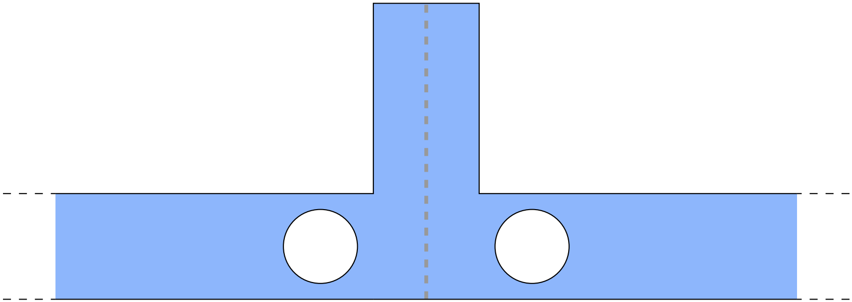
3. A spe

   For a gi k solving a spectral problem.

First approach was perturbative.
How to get large invisible defects ?

Geometrical setting

- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



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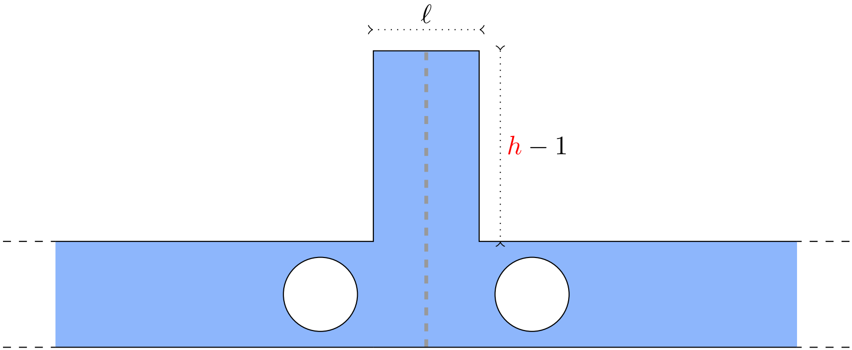

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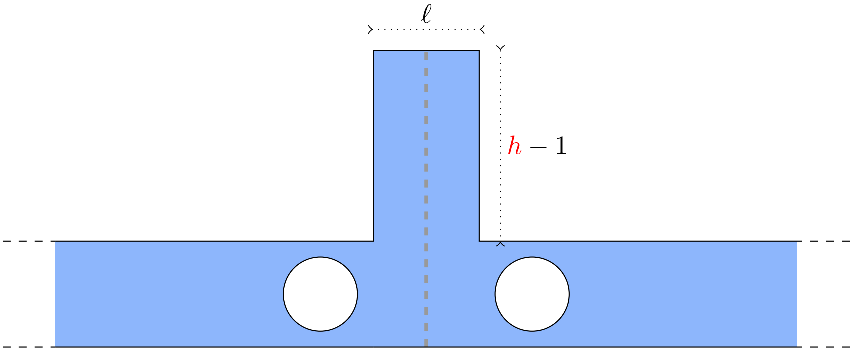

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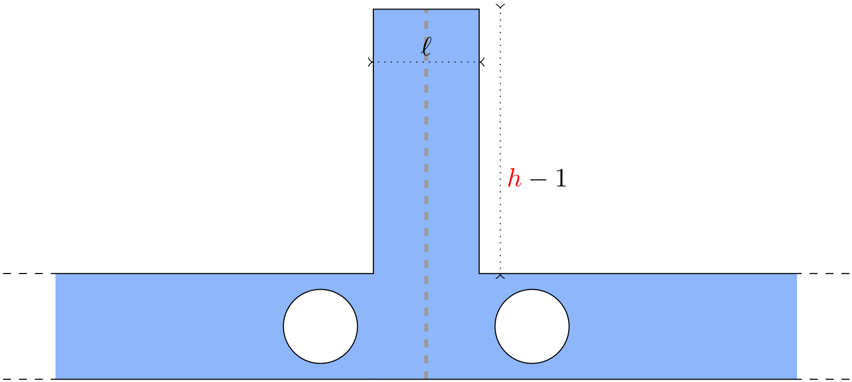

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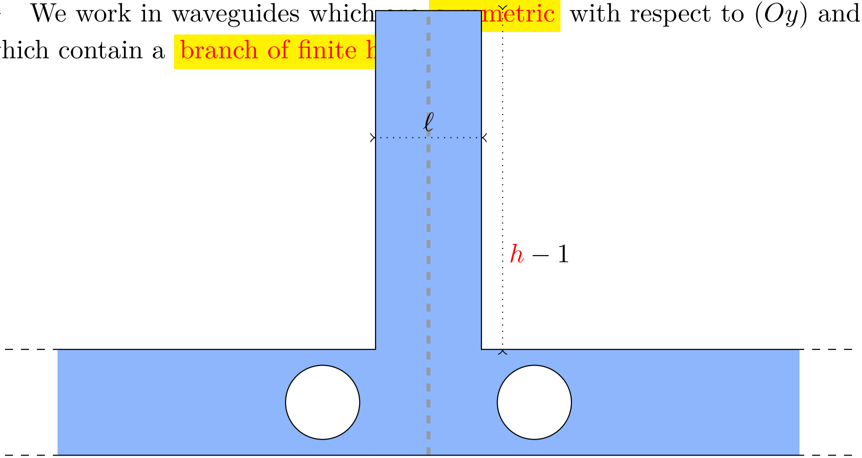

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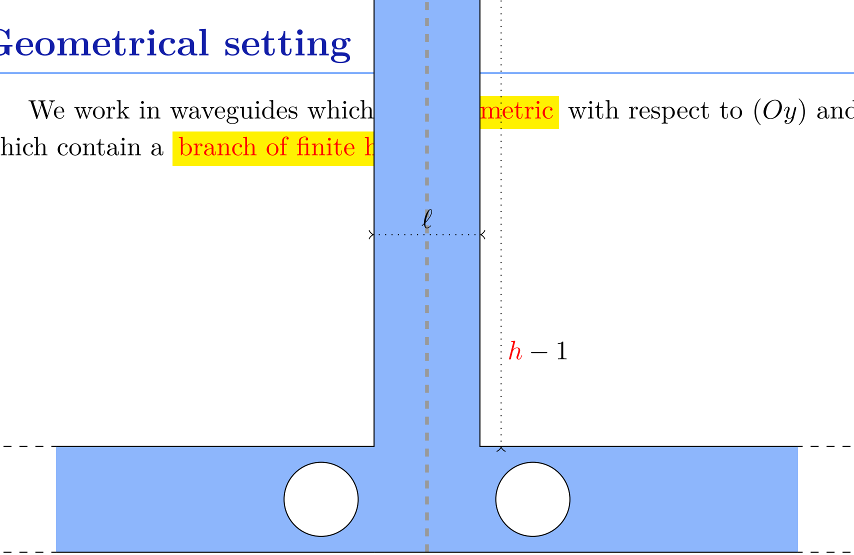


$\rightarrow$ We will study the behaviour of the coefficients R, $T \in \mathbb{C}$ as $h \rightarrow +\infty$.

1 First constructive method

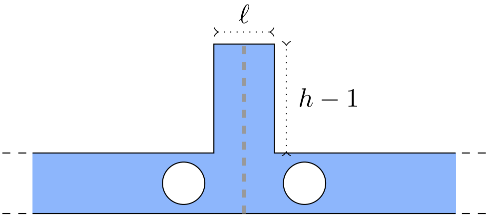
2 Second constructive method

- Main analysis
- Numerical results
- Variants and extensions

3 A spectral approach to determine non reflecting wavenumbers

Half-waveguide problems

- Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



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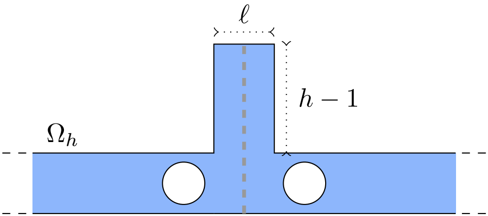

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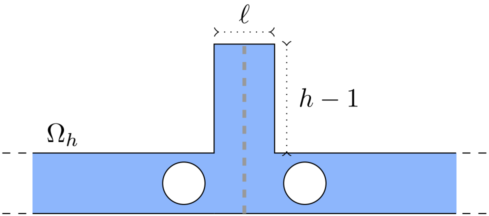

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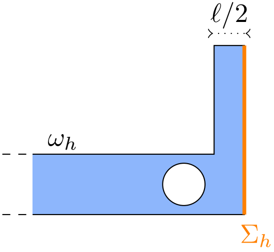
- Introduce the two half-waveguide problems



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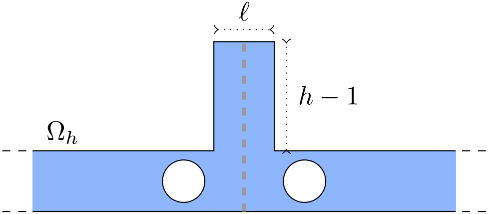

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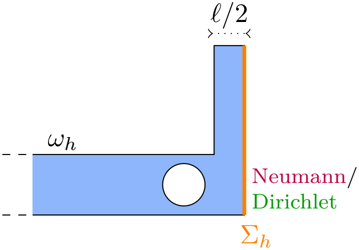

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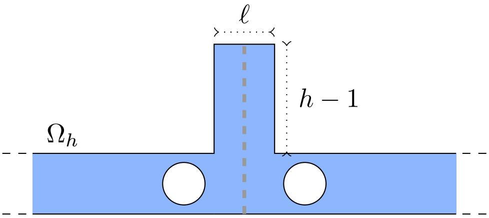

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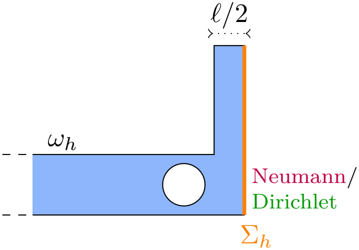

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Neumann B.C.

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Mixed B.C.

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$
\begin{aligned}
u &= w^+ + R^N w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\
U &= w^+ + R^D w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h).
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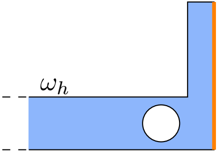

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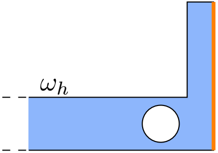

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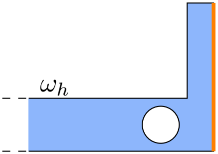

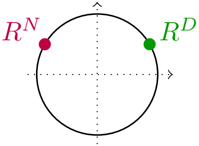

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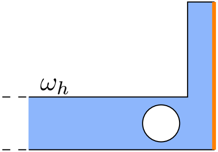

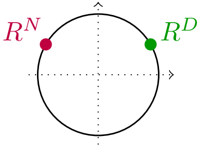

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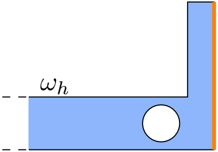

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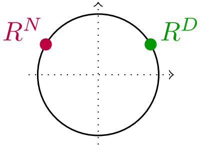

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Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

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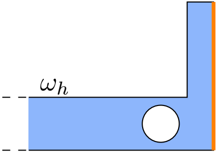

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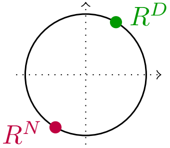

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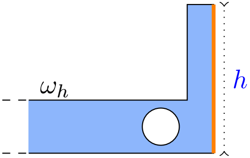

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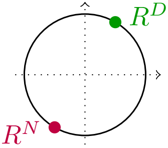


- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

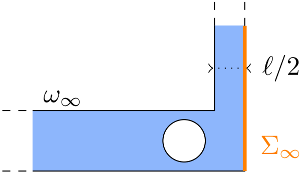
we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

$\rightarrow$ Now, we study the behaviour of $R^N = R^N(h)$, $R^D = R^D(h)$ as $h \rightarrow +\infty$.

Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

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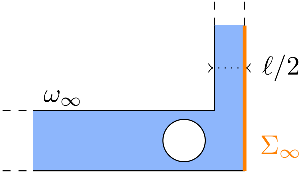

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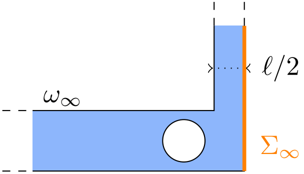
- Analysis for R^D

- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\;\left|\;\begin{array}{rcll} -\Delta\varphi &=& k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi &=& 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

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▶ Analysis for R^D

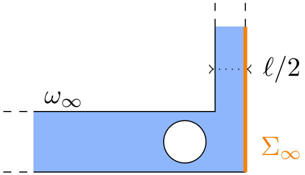
- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R_\infty^D\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R_\infty^D| = 1.$$

Asymptotics of R^N, R^D 1/2

Depend on the nb. of propagating modes in the vertical branch of ω_∞



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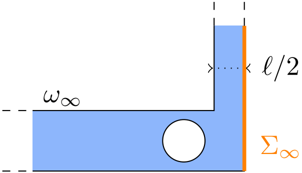
($w_1^\pm = \chi_l\, w^\mp$ where χ_l is a cut-off function s.t. $\chi_l = 1$ for $x \leq -2\ell$, $\chi_l = 0$ for $x \geq -\ell$)

Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



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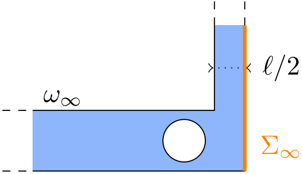
Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



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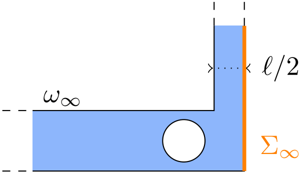
- As $h \to +\infty$, we have $U = U_\infty + \dots$ which implies $|R^D - R_\infty^D| \le C\, e^{-\beta h}$.

Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



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- Analysis for R^D

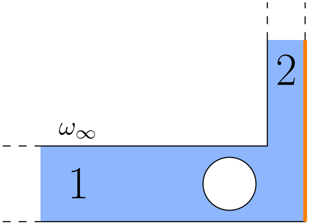
For $\boldsymbol{\ell} \in (\mathbf{0}; \boldsymbol{\pi}/\boldsymbol{k})$, $h \mapsto R^D(h)$ tends to a constant on $\mathscr{C} := \{z \in \mathbb{C}, |z| = 1\}$.

Asymptotic of R^N, R^D 2/2

▶ Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$



Asymptotic of R^N, R^D 2/2

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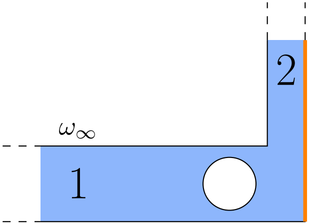
(χ_t is a cut-off function such that $\chi_t = 1$ for $y \geq 2$, $\chi_t = 0$ for $y \leq 1$)



Asymptotic of R^N, R^D

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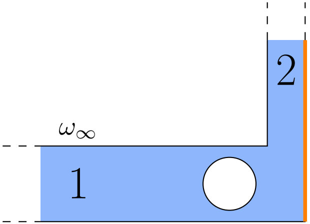

Asymptotic of R^N, R^D

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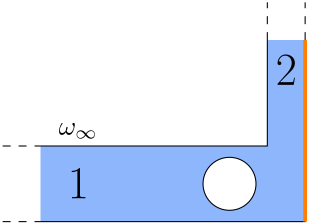
- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, \qquad && \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, \qquad && \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$



The scattering matrix

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$ is unitary.

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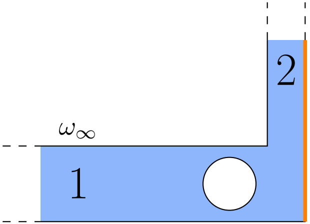
- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.



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Asymptotic of R^N, R^D

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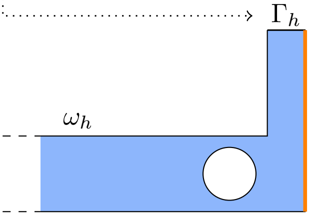
On Γ_h $\qquad 0 = \partial_n u = C\,(s_{12}e^{ikh} + a(h)\,(-e^{-ikh} + s_{22}e^{ikh})) + \dots$.



► $\boxed{\text{Analysis for } R^N}$

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Asymptotic of R^N, R^D

- Analysis for R^N

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- This gives $a(h)$ and implies, as $h \to +\infty$,

$$\boxed{|R^N - R^N_{\text{asy}}(h)| \leq C\, e^{-\beta h} \quad \text{with} \quad R^N_{\text{asy}}(h) = s_{11} + \frac{s_{12}\, s_{21}}{e^{-2ikh} - s_{22}}.}$$

Asymptotic of R^N, R^D

- Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

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- Unitarity of $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \Rightarrow h \mapsto R^N_{\mathrm{asy}}(h)$ runs periodically on $\mathscr{C}$.

Asymptotic of R^N, R^D 2/2

- Analysis for R^N

For $\boldsymbol{\ell} \in \boldsymbol{(0; 2\pi/k)}$, $h \mapsto R^N(h)$ runs continuously and almost period. on $\mathscr{C}$.

Conclusions for $\ell \in (0;\pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

PROPOSITION: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

Conclusions for $\ell \in (0; \pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Proposition: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

Proposition: There is an unbounded sequence (h_n) such that for $h = h_n$, $R^N = -R^D$ and so $R = 0$ (non reflectivity).

Conclusions for $\ell \in (0; \pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Proposition: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

Proposition: There is an unbounded sequence (h_n) such that for $h = h_n$, $R^N = -R^D$ and so $R = 0$ (non reflectivity).

Proposition: There is an unbounded sequence $(\mathcal{H}_n)$ such that for $h = \mathcal{H}_n$, $R^N = R^D$ and so $T = 0$ (complete reflectivity).

- Sequences (h_n) and $(\mathcal{H}_n)$ are almost periodic. As $n \to +\infty$, we have

$$h_{n+1} - h_n = \pi/k + \dots \qquad \text{and} \qquad \mathcal{H}_{n+1} - \mathcal{H}_n = \pi/k + \dots .$$

1 First constructive method

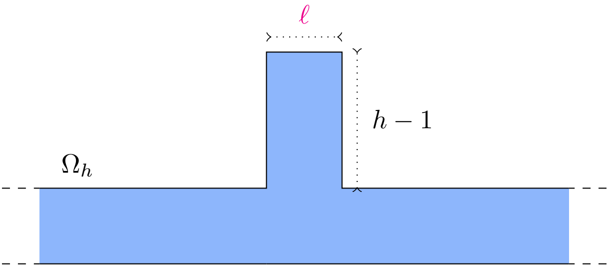
2 Second constructive method

- Main analysis
- Numerical results
- Variants and extensions

3 A spectral approach to determine non reflecting wavenumbers

Setting

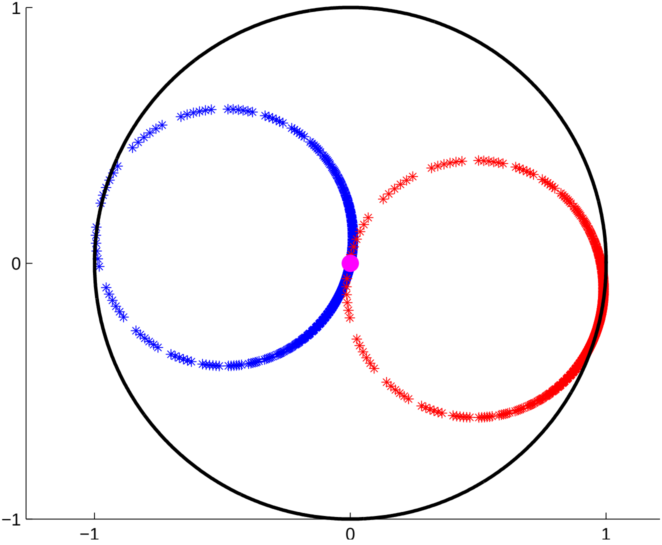
- We compute numerically R, T for $h \in (2; 10)$ in the geometry Ω_h



- We use a P2 finite element method with Dirichlet-to-Neumann maps.
- We set $k = 0.8\pi$ and $\ell = 1 \in (0; \pi/k)$.

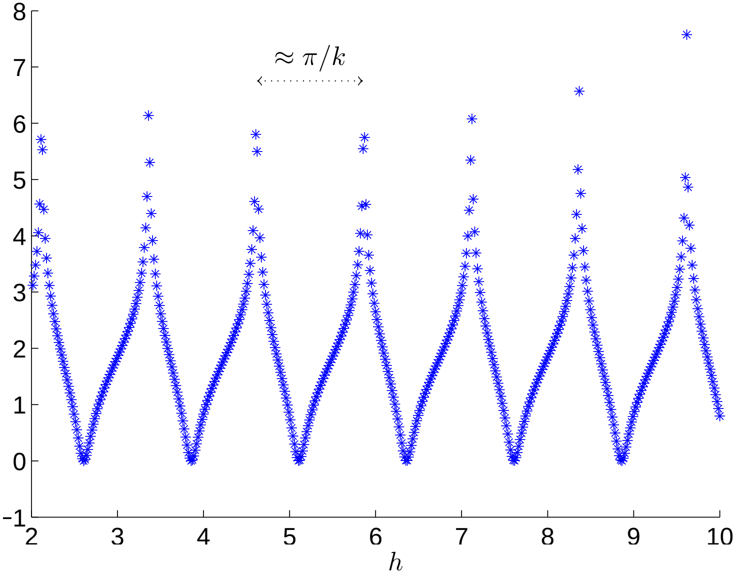
Numerical results

- Reflection coefficient R and transmission coefficient T for $h \in (2; 10)$.



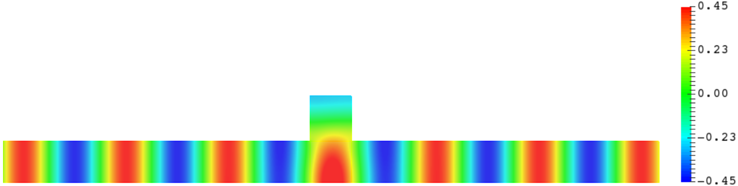
Non reflectivity

- Curve $h \mapsto -\ln|R|$. Peaks correspond to non reflectivity.

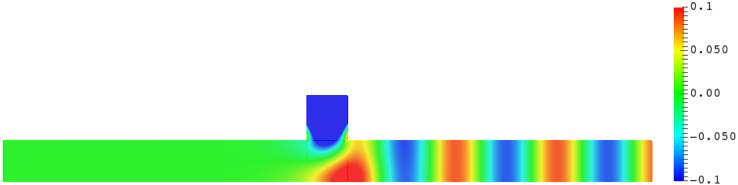


Non reflectivity

- Total field v for h such that $R = 0$.



- Scattered field v_s.



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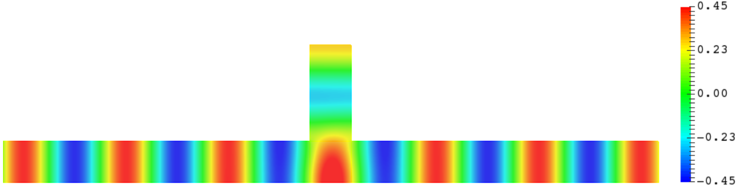

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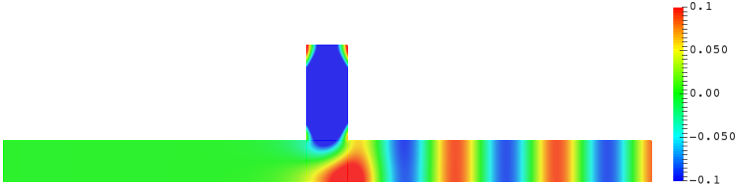

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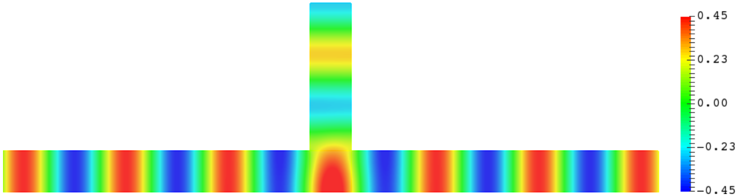

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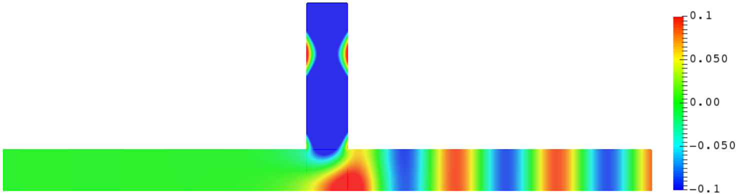
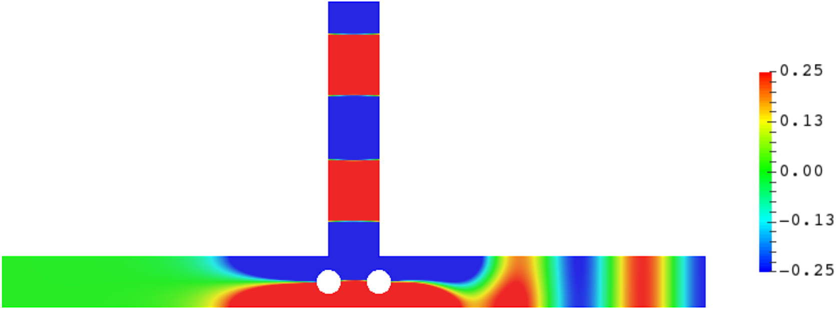


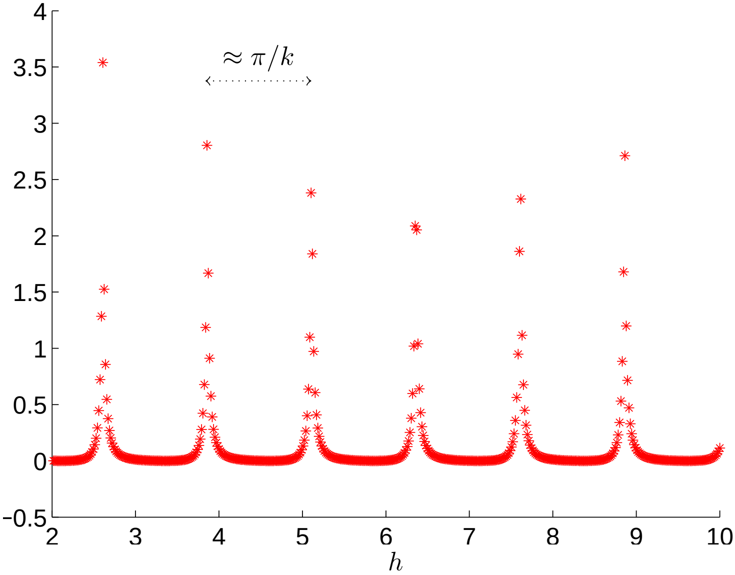
Other non reflecting geometry

- Scattered field v_s.



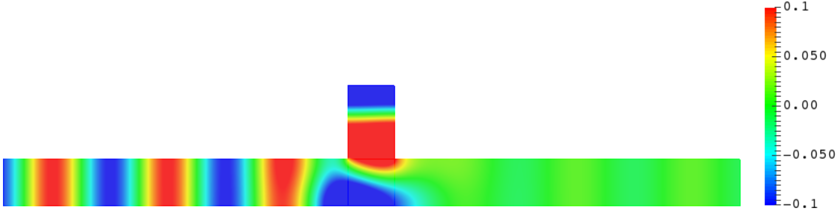
Complete reflectivity

- Curve $h \mapsto -\ln|T|$. Peaks correspond to complete reflectivity.



Complete reflectivity

- Total field v for h such that $T = 0$.



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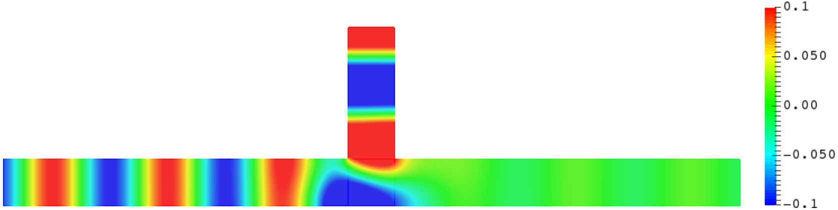
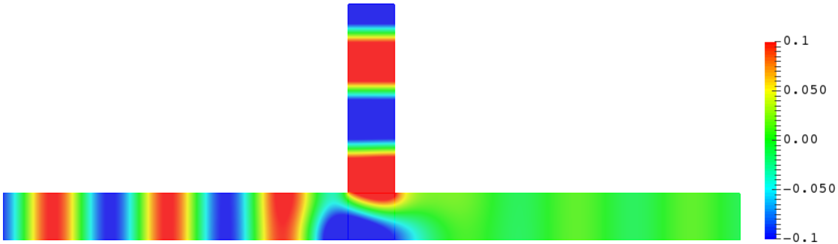


Complete reflectivity

- Total field v for h such that $\color{red}{T} = 0$.



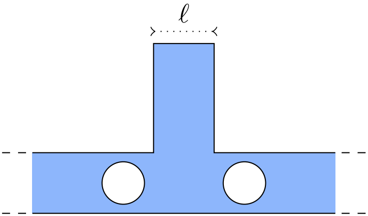
1 First constructive method

2 Second constructive method

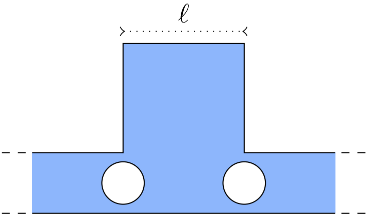
- Main analysis
- Numerical results
- Variants and extensions

3 A spectral approach to determine non reflecting wavenumbers

Analysis for $\ell \in (\pi/k; 2\pi/k)$



We did $\ell \in (0; \pi/k)$



Now $\ell \in (\pi/k; 2\pi/k)$

Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

- Now 2 prop. modes exist in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

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- As before, we can show, with $\alpha = \sqrt{k^2 - (\pi/\ell)^2}$,

$$\boxed{|R^D - R^D_{\text{asy}}(h)| \le C\, e^{-\beta h} \quad \text{with} \quad R^D_{\text{asy}}(h) = S_{11} + \frac{S_{12}\, S_{21}}{e^{-2i\alpha h} - S_{22}}.}$$

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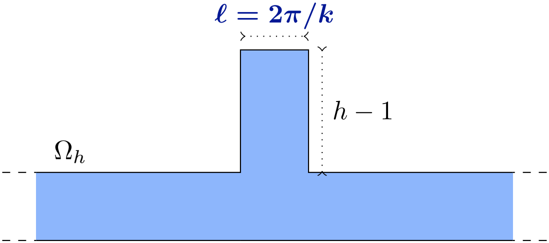
⋆ The curves $h \mapsto R(h)$, $T(h)$ still pass through zero an infinite nb. of times.

⋆ Behaviours of $h \mapsto R(h)$, $T(h)$ can be much more complex than before...

The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



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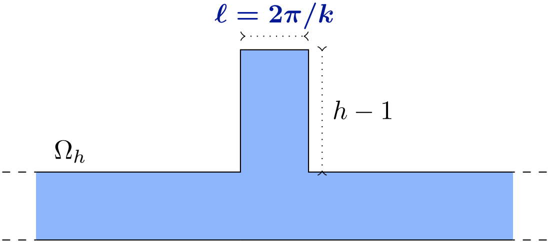

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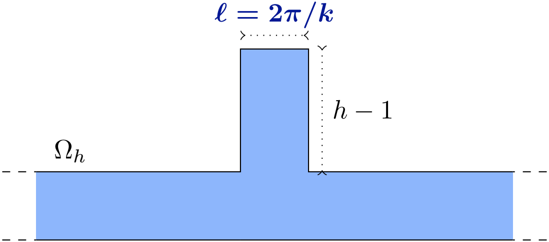

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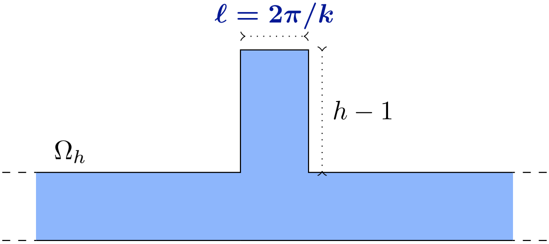

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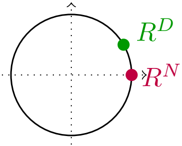

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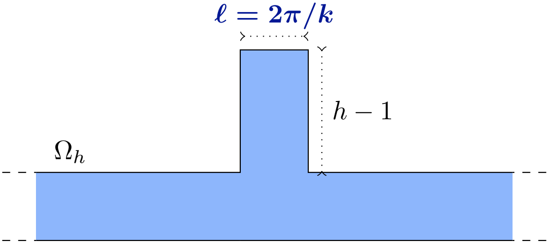

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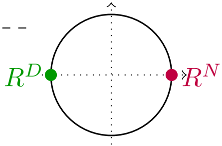

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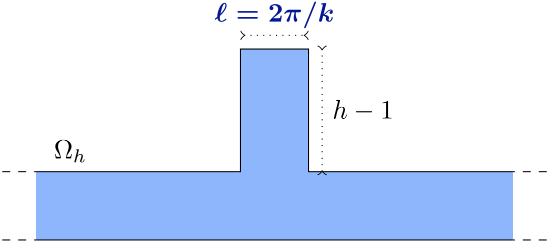

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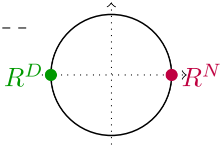

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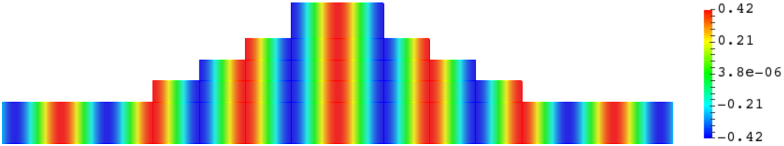
$\star$ $h \mapsto R^D(h)$ still runs on the unit circle and goes through -1.

There is a sequence (h_n) such that $T = 1$ (perfect invisibility)

The special case $\ell = 2\pi/k$ - perfect invisibility

- Works also in the geometry below (h is the height of the central branch).

- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



- Reference waveguide ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^\pm = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ and $\boxed{w_2^\pm = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$

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LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and* $u_2 \in \mathrm{H}^1(\omega_h)$ *is a trapped mode.*

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$$\Rightarrow \mathfrak{s}_{11} = 1 \qquad \Rightarrow |\mathfrak{s}_{22}| = 1, \qquad \forall h > 1.$$

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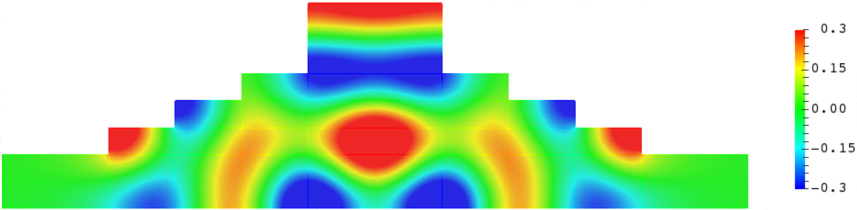
There is a sequence (h_n) such that trapped modes exist in ω_h.

The special case $\ell = 2\pi/k$ - trapped modes

- Symmetry argument w.r.t. $(Oy) \Rightarrow$ existence of trapped modes in Ω_h. It works also in the geometry below (h is the height of the central branch).



Non zero $v \in \mathrm{H}^1(\Omega_h)$ satisfying $\Delta v + k^2 v = 0$ in Ω_h, $\partial_n v = 0$ on $\partial\Omega_h$.

Outline of the talk

1 First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2 Second constructive method

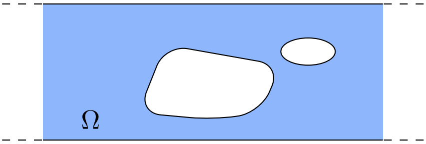
k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3 A spectral approach to determine non reflecting wavenumbers

For a given geometry, we explain how to find non reflecting k solving a spectral problem.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \\ & & v_s \text{ is outgoing.} & \end{array}$$

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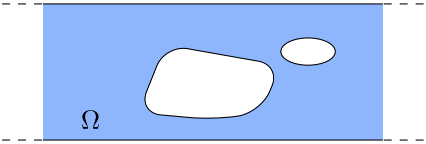

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- For this problem, the modes are

$$\begin{array}{l|l} \text{Propagating} & w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!] \\ \text{Evanescent} & w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N. \end{array}$$

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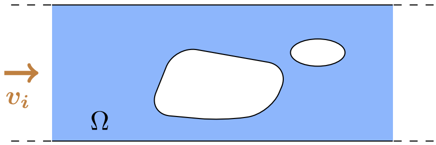

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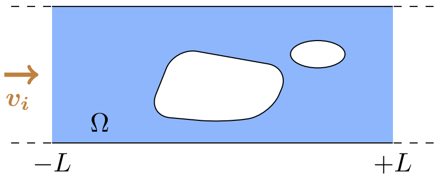
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



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Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N.$

- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

- v_s is outgoing $\Leftrightarrow$ $\boxed{v_s = \sum_{n=0}^{+\infty} \gamma_n^{\pm} w_n^{\pm} \quad \text{for } \pm x \geq L, \text{ with } (\gamma_n^{\pm}) \in \mathbb{C}^{\mathbb{N}}.}$

Analytic dilation to compute v_s

- Consider the complex change of variables $\mathcal{I}_\theta : \Omega \to \mathbb{C} \times (0;1)$ such that

$$\mathcal{I}_\theta(x,y) = \left| \begin{array}{rl} (-L + (x+L)\, e^{i\theta}, y) & \text{for } x \leq -L \\ (x,y) & \text{for } |x| < L \\ (+L + (x-L)\, e^{i\theta}, y) & \text{for } x \geq L. \end{array} \right. \qquad \text{with } \theta \in (0;\pi/2).$$

- Since $\Re e\,(ie^{i\theta}) < 0$ and $\Re e\,(-e^{i\theta}) < 0$, the functions $w_n^\pm \circ \mathcal{I}_\theta$ are exponentially decaying at $\pm\infty$.

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1) $v_\theta = v_s$ for $|x| < L$.

2) v_θ is exp. decaying at infinity.

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- v_θ solves

$$(*) \left| \begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\Big(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\Big) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array} \right.$$

Analytic dilation to compute v_s

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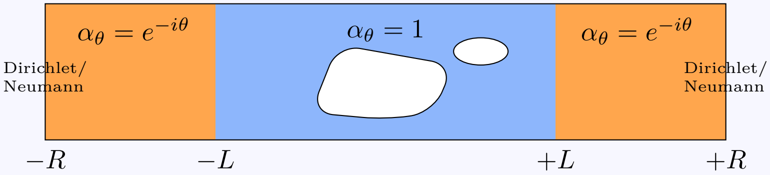
$\alpha_\theta(x) = 1$ for $|x| < L$ $\qquad$ $\alpha_\theta(x) = e^{-i\theta}$ for $|x| \ge L$

Analytic dilation to compute v_s

- Numerically we solve $(*)$ in the truncated domain



$\Rightarrow$ We obtain a good approximation of v_s for $|x| < L$.

- This is the method of Perfectly Matched Layers (PMLs).

► v_θ solves $(*)$
$$\left|\begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$$

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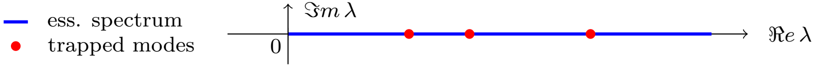
Spectral analysis

- Define the operators A, A_θ of $\mathrm{L}^2(\Omega)$ such that

$$A v = -\Delta v, \qquad A_\theta v = -\Big(\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\mathrm{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.



- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
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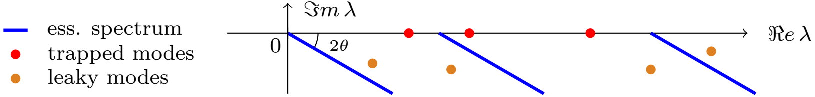
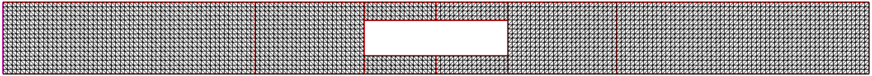
- real eigenvalues of A_θ = real eigenvalues of A.



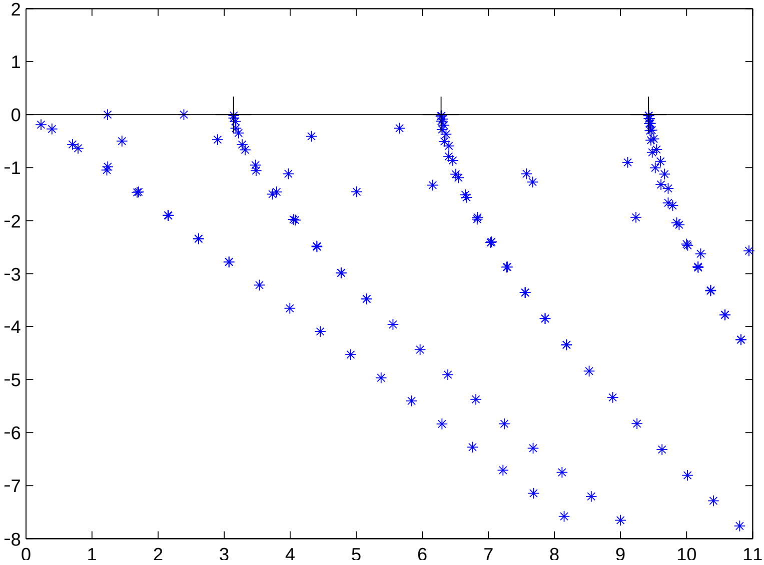
Numerical results

- We work in the geometry



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



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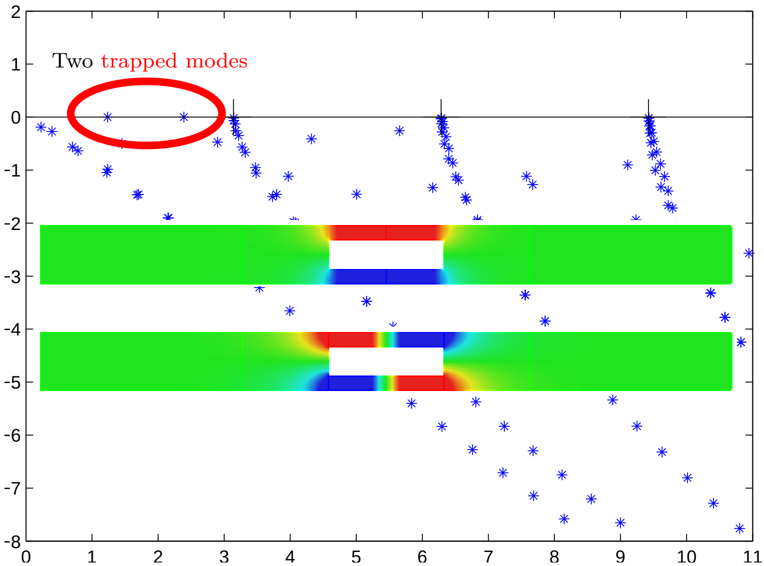

A new complex spectrum for non reflecting k

- Usual complex stretching selects solutions which are

$$\text{outgoing at } -\infty \qquad \text{and} \qquad \text{outgoing at } +\infty.$$

IMPORTANT REMARK: for **general** k, the total field has the form

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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IMPORTANT REMARK: for **non reflecting** k, the total field has the form

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Important remark: for k, the total field has the form

$$v = \sum_{n=0}^{N-1} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

- In other words, v is

ingoing at $-\infty$ and outgoing at $+\infty$.

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- In other words, v is

ingoing at $-\infty$ and outgoing at $+\infty$.

Let us change the sign of the complex stretching at $-\infty$!

A new complex spectrum for non reflecting k

- Consider the complex change of variables $\mathcal{J}_\theta : \Omega \to \mathbb{C} \times (0;1)$ such that

$$\mathcal{J}_\theta(x,y) = \left| \begin{array}{rl} (-L + (x+L)\, e^{-i\theta}, y) & \text{for } x \leq -L \\ (x,y) & \text{for } |x| < L \\ (+L + (x-L)\, e^{i\theta}, y) & \text{for } x \geq L. \end{array} \right. \qquad \text{with } \theta \in (0; \pi/2).$$

- One can check that the functions $w_n^+ \circ \mathcal{J}_\theta$, $n \in [\![0, N-1]\!]$ and $w_n^\pm \circ \mathcal{J}_\theta$, $n \geq N$, are exponentially decaying at $\pm\infty$.

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- u_θ solves

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Spectral analysis

- Define the operators B_θ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta v = -\Big(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

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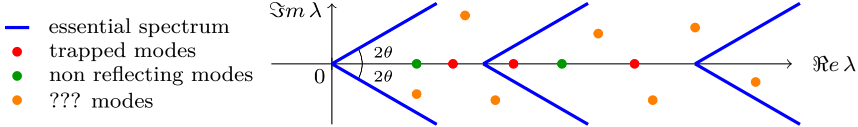
- real eigenvalues of B_θ = real eigenvalues of A+non reflecting k^2.



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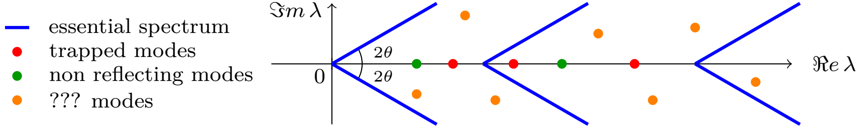

Remark: Not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

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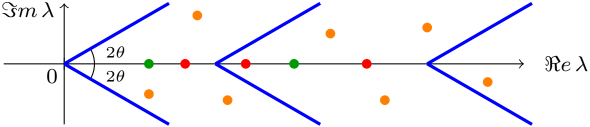
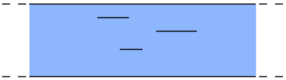
essential spectrum
trapped modes
non reflecting modes
??? modes



REMARK: Not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

$\rightarrow$ **Not true in general.**



$w_0^+ \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

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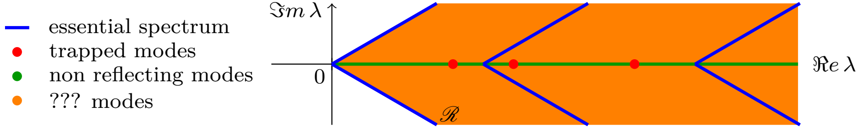

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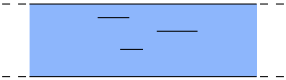
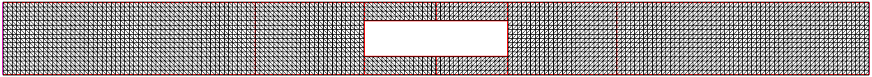


$w_0^+ \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

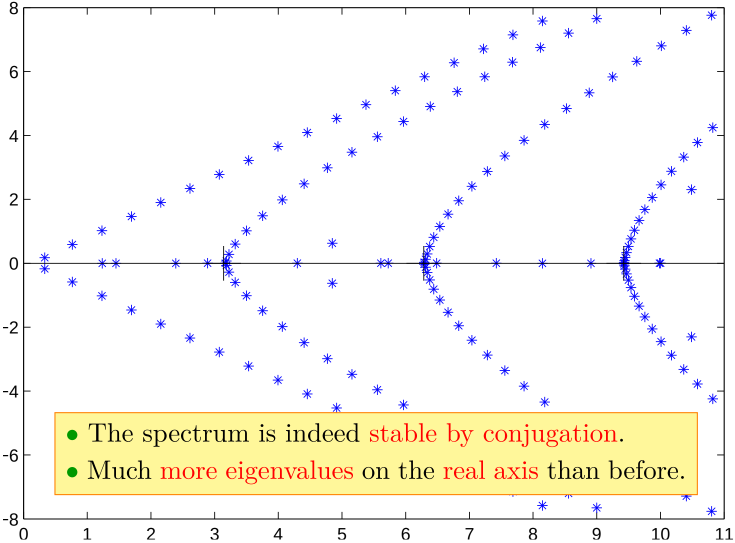
PROP.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y) \in \Omega\}$, B_θ is $\mathcal{PT}$ symmetric:

$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

Numerical results

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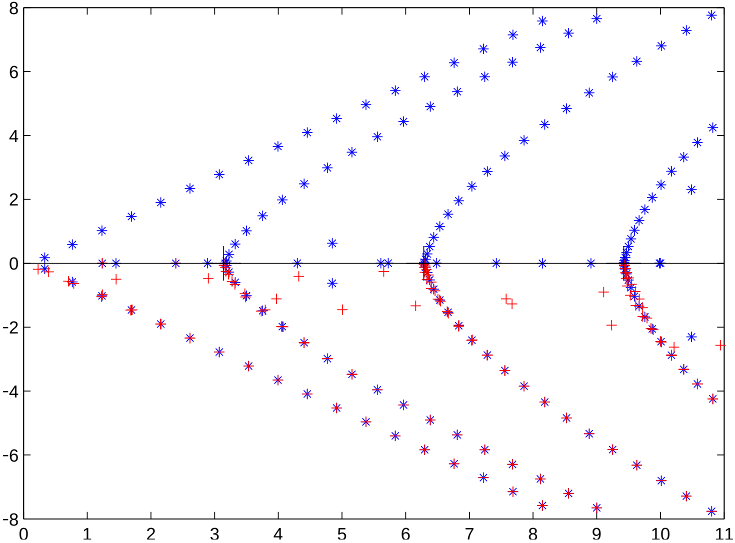
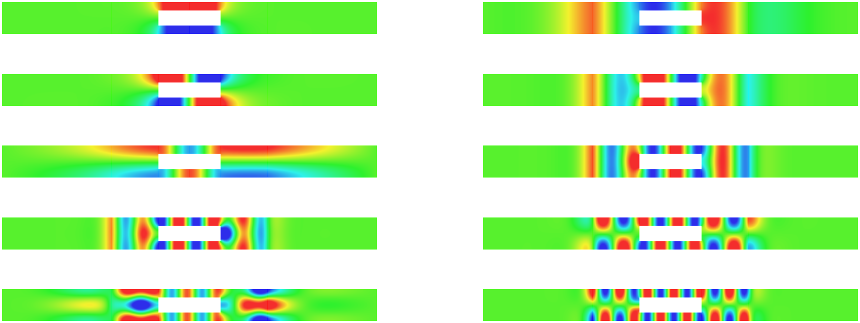


$\ast$ PMLs with different signs $\quad$ $+$ Classical PMLs

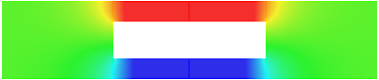
Numerical results

- We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



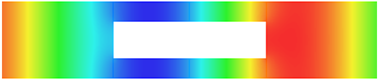
Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

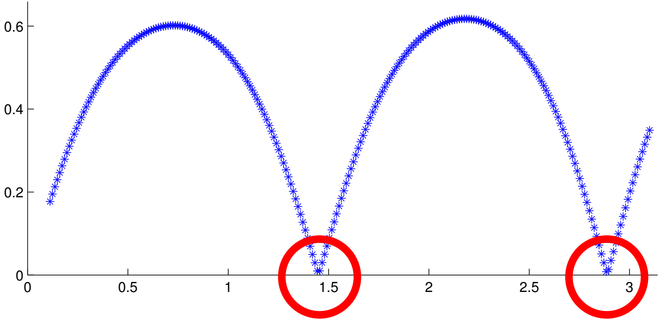


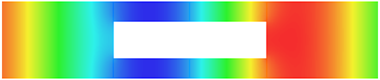
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





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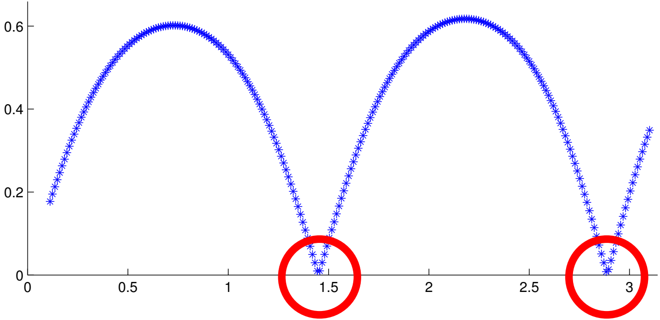
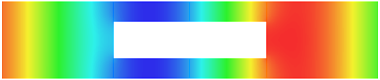
- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





First non reflecting mode

$k = 1.4513...$

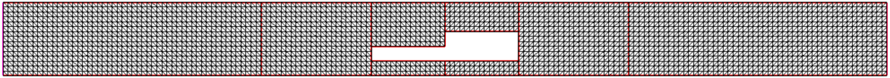
Second non reflecting mode

$k = 2.8896...$

There is perfect agreement!

Numerical results

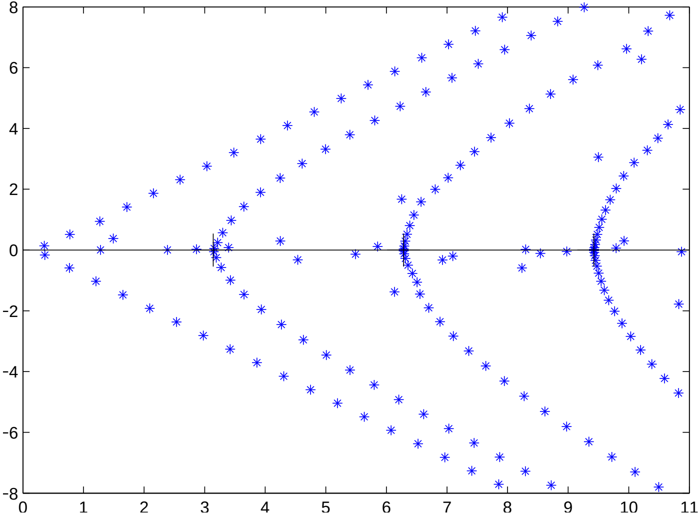
- Now the geometry is not symmetric in x nor in y:



- The operator B_θ is no longer $\mathcal{PT}$-symmetric and we expect:
    - No trapped modes
    - No invariance of the spectrum by complex conjugation.

Numerical results

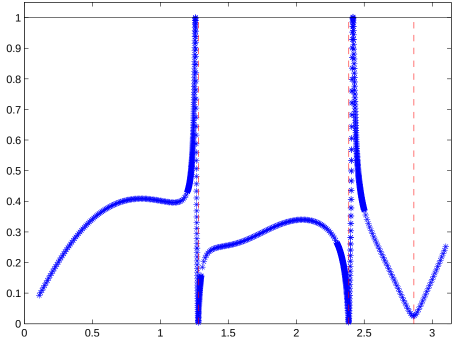
- Discretized spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is not symmetric w.r.t. the real axis.

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



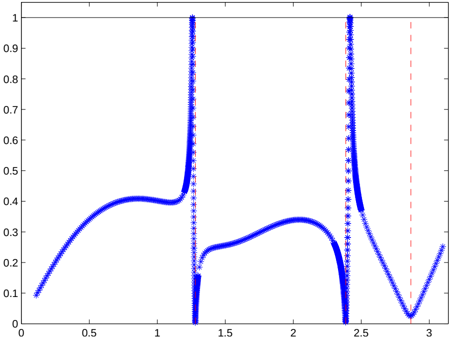
$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$k = 1.28 + 0.0003i$

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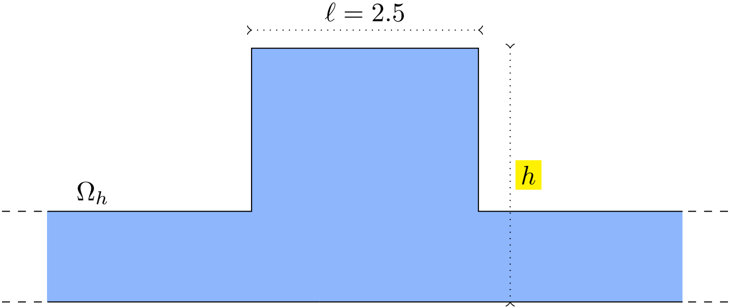
Complex eigenvalues also contain information on almost no reflection.

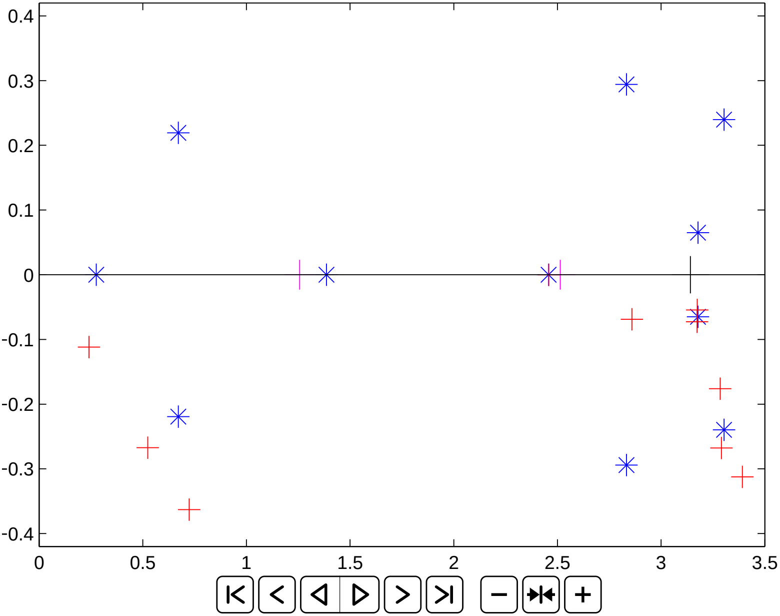
Spectra for a changing geometry

- Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.



- The magenta marks on the real axis correspond to the particular frequencies $k = \pi/\ell$ and $k = 2\pi/\ell$.
- For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain h.
- We zoom at the region $0 < \Re e\, k < \pi$.

$h = 1.3335$



PMLs with different signs   Classical PMLs

Outline of the talk

1. First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3. A spectral approach to determine non reflecting wavenumbers

For a given geometry, we explain how to find non reflecting k solving a spectral problem.

Conclusion

What we did

- ♠ We presented two methods to construct geometries such that $R = 0$, $T = 0$, $T = 1$ at a given frequency $k \in (0; \pi)$.
- ♠ We proposed a spectral approach to compute non reflecting k $(R = 0)$ for a given geometry.

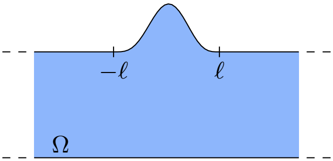
Future work

1) How to construct invisible or completely reflecting defects for a given $k > \pi$ (several propagating modes)?
2) Can we find a spectral approach to compute completely reflecting or completely invisible k for a given geometry?
3) Can we prove existence of non reflecting k for the $\mathcal{PT}$-symmetric pb?
4) Can we work in free space with a finite number of directions? on other equations (electromagnetism, elasticity,...)?

Thank you for your attention!

Can we get $T = 1$?

- More generally, for any Neumann waveguide, one can show that $T = 1$ implies
$$\int_{\Omega} |\nabla v_s|^2 - k^2 |v_s|^2 \, d\boldsymbol{x} = 0.$$



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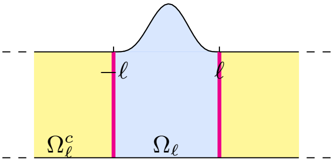

- Decomposing in Fourier series, one finds

$$\int_{\Omega_\ell^c} |\nabla v_s|^2 - k^2 |v_s|^2 \, d\boldsymbol{x} \geq 0.$$

- Note that $T = 1 \Rightarrow v_s \in \mathrm{Y} := \{\varphi \in \mathrm{H}^1(\Omega_\ell) \,|\, \int_{x=\pm\ell} \varphi \, d\sigma = 0\}$. Define

$$\lambda_\dagger := \inf_{\varphi \in \mathrm{Y}\setminus\{0\}} \left(\int_{\Omega_\ell} |\nabla \varphi|^2 \, d\boldsymbol{x} \right) \Big/ \left(\int_{\Omega_\ell} |\varphi|^2 \, d\boldsymbol{x} \right) > 0.$$

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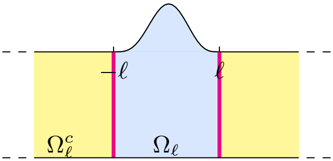

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Proposition: For a given shape, $T = 1$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

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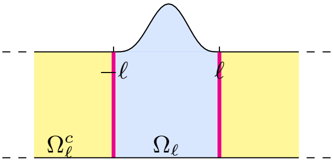

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Proposition: For a given shape, $T = 1$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

$\rightarrow$ For a small smooth perturbation of amplitude εh, one finds $|\lambda_\dagger - \pi^2| \leq C\,\varepsilon$.

Can we get $T = 1$?

- More generally, for any Neumann waveguide, one can show that $T = 1$ implies

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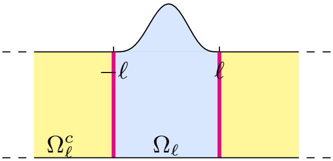

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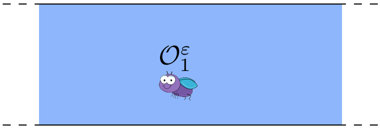
PROPOSITION: For a given shape, $T = 1$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

$\rightarrow$ To impose invisibility at low frequency, we need to work with special shapes.

Small Dirichlet obstacle

Can one hide a small Dirichlet obstacle centered at M_1 ?



Find $v = v_i + v_s$ s. t.

$$\begin{aligned} -\Delta v &= k^2 v \quad \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ v &= 0 \quad \text{on } \partial\Omega^\varepsilon, \\ &v_s \text{ is outgoing.} \end{aligned}$$

- With Dirichlet B.C., the modes are not the same as previously but this not important.

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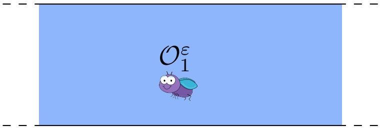

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- In 3D, we obtain

$$\begin{array}{rcl} R & = & 0 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O}) w^+(M_1)^2)\ + O(\varepsilon^2) \\ T & = & 1 + \varepsilon\ (4i\pi\,\mathrm{cap}(\mathcal{O}) |w^+(M_1)|^2)\ + O(\varepsilon^2). \end{array}$$

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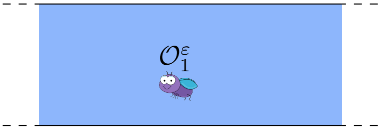

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Non zero terms!
$(\mathrm{cap}(\mathcal{O}) > 0)$

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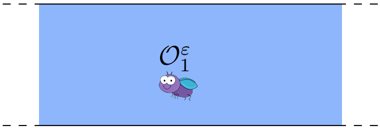

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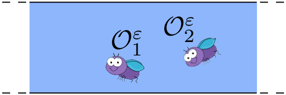
Non zero terms!
($\mathrm{cap}(\mathcal{O}) > 0$)

$\Rightarrow$ One single small obstacle cannot even be non reflecting.

Small Dirichlet obstacles



- Let us try with **TWO** small Dirichlet obstacles at M_1, M_2.

- We obtain
$$\begin{aligned} R &= 0 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} w^+(M_n)^2) + O(\varepsilon^2) \\ T &= 1 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} |w^+(M_n)|^2) + O(\varepsilon^2). \end{aligned}$$

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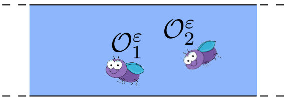

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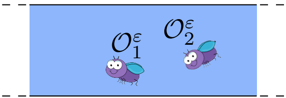

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We can find M_1, M_2 such that $R = O(\varepsilon^2)$.

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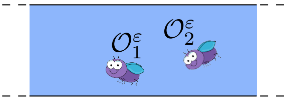

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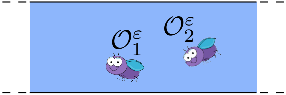
We can find M_1, M_2 such that $R = O(\varepsilon^2)$. Then moving $\mathcal{O}_1^\varepsilon$ from M_1 to $M_1 + \varepsilon\tau$, and choosing a good $\tau \in \mathbb{R}^3$ (fixed point), we can get $R = 0$.

Small Dirichlet obstacles



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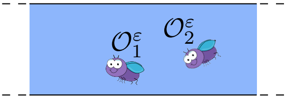
COMMENTS:

→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $T = 1$ with this strategy.

→ When there are more propagative waves, we need more obstacles.

Small Dirichlet obstacles



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Comments:

→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $T = 1$ with this strategy.

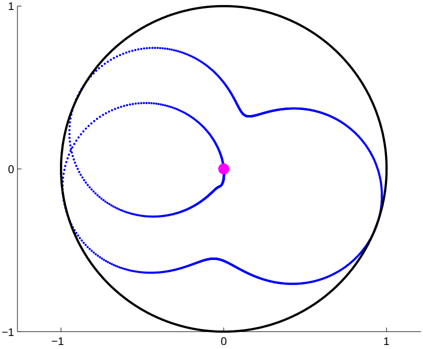
→ When there are more propagative waves, we need more obstacles.

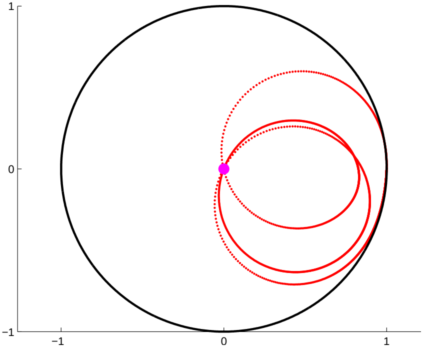
Acting as a team, flies can become invisible!

Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; +\infty)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} = 2.$$





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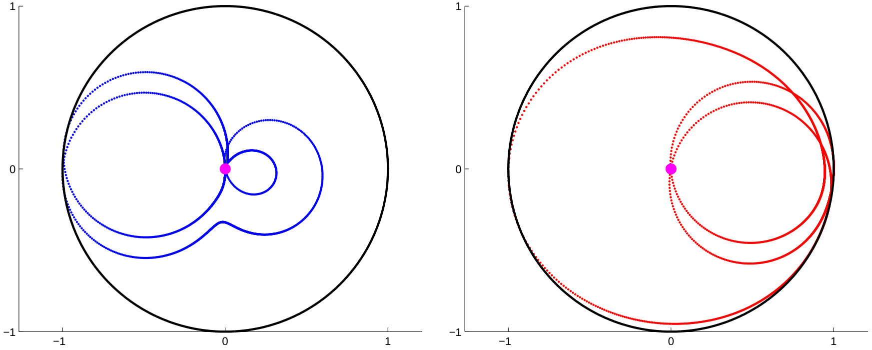

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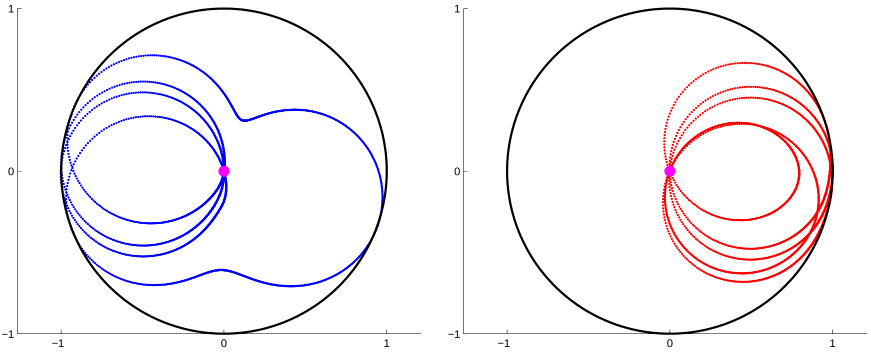

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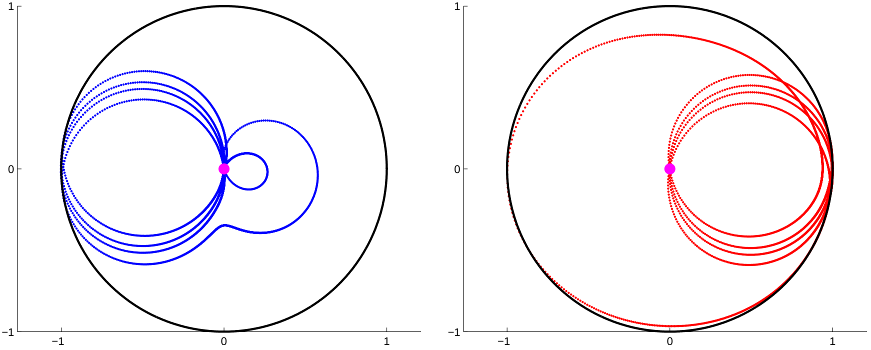
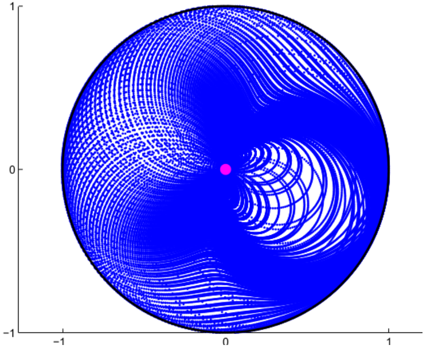
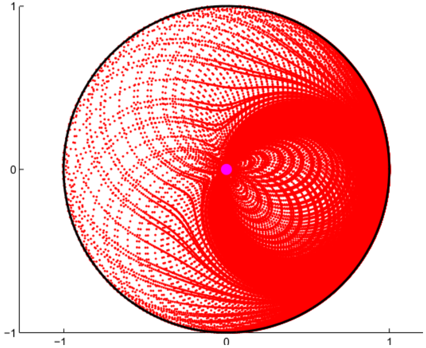


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; 100)$ and ℓ such that

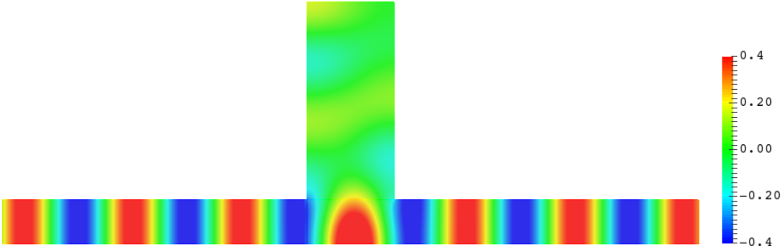
$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} \notin \mathbb{Q}.$$





Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Non reflecting geometry ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



- Completely reflecting geometry ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).