A new complex spectrum related to invisibility in waveguides

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General setting

- We are interested in the propagation of waves in acoustic waveguides.

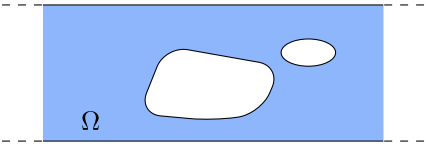
- In this talk, we study questions of invisibility.

> Can we find situations where waves
> go through like if there were no defect ?

- One can wish to have good energy transmission through the structure.
- One can wish to hide objects.

Scattering problem in a waveguide

- Scattering in time-harmonic regime of an incident wave in the acoustic waveguide Ω coinciding with $\{(x,y)\in\mathbb{R}\times(0;1)\}$ outside a compact region.



Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} \Delta v + k^2 v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \\ & & v_s \text{ is outgoing.} & \end{array}$$

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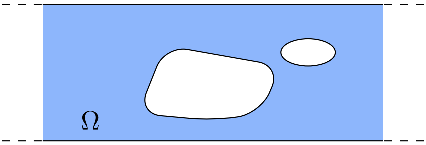

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- For this problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$, the modes are

Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

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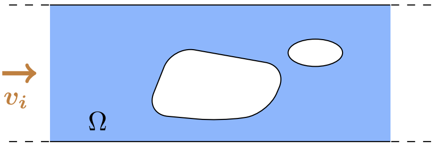

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- Set $v_i = \displaystyle\sum_{n=0}^{N-1} \alpha_n w_n^+$ (propagating) for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

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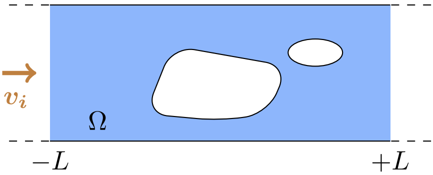

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- v_s is outgoing $\Leftrightarrow$ $\boxed{v_s = \sum_{n=0}^{+\infty} \gamma_n^\pm\, w_n^\pm \quad \text{for } \pm x \geq L, \text{ with } (\gamma_n^\pm) \in \mathbb{C}^{\mathbb{N}}.}$

Goal of the talk

DEFINITION: v is a non reflecting mode if v_s is expo. decaying for $x \leq -L$
$\Leftrightarrow \quad \gamma_n^- = 0,\ n \in [\![0, N-1]\!] \quad \Leftrightarrow \quad$ energy is completely transmitted.

GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

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GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

$\rightarrow$ Note that non reflection occurs for **particular** $\boldsymbol{v_i}$ to be computed.

Outline of the talk

1. Introduction

2. Classical complex scaling

   We recall how to use classical complex scaling to compute trapped modes and complex resonances.

3. Conjugated complex scaling

   We explain how to use conjugated complex scaling to compute non reflecting modes.

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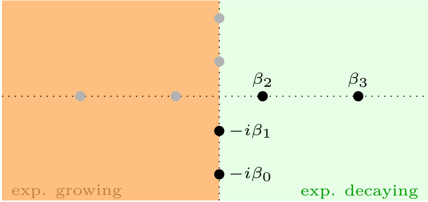
Classical complex scaling to compute v_s 1/2

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \ \pm x \geq L.$



Modal exponents for v_s $(x \leq -L)$

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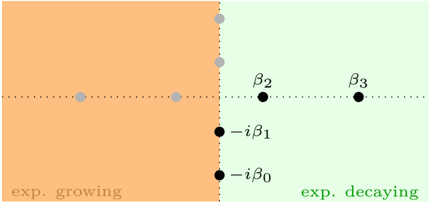

Modal exponents for v_s ($x \leq -L$)

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{I}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\, e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\, e^{i\theta} & \text{for } x \geq L. \end{array} \right.$$

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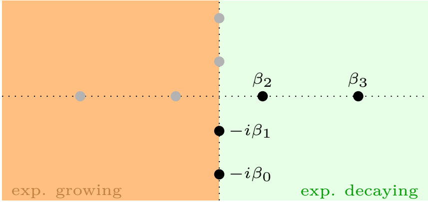

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- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y)$.

1) $v_\theta = v_s$ for $|x| < L$.
2) v_θ is exp. decaying at infinity.

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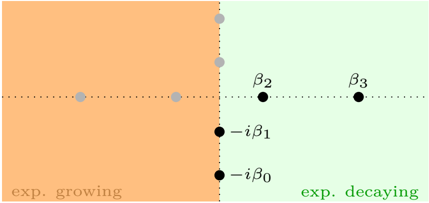

Modal exponents for v_s ($x \leq -L$)

$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^{\pm} e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^{\pm} e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

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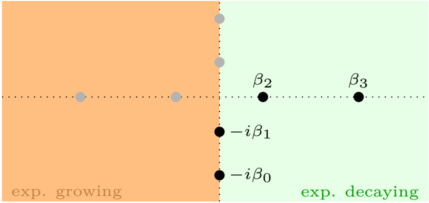

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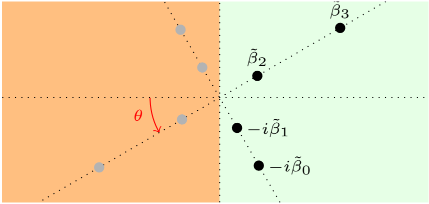

Modal exponents for v_θ $(x \leq -L)$

$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^{\pm} e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^{\pm} e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y)$.

1) $v_\theta = v_s$ for $|x| < L$.
2) v_θ is exp. decaying at infinity.

Classical complex scaling to compute v_s 2/2

- v_θ solves $(*)\left|\begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$

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$$\alpha_\theta(x) = 1 \text{ for } |x| < L \qquad \alpha_\theta(x) = e^{-i\theta} \text{ for } |x| \geq L$$

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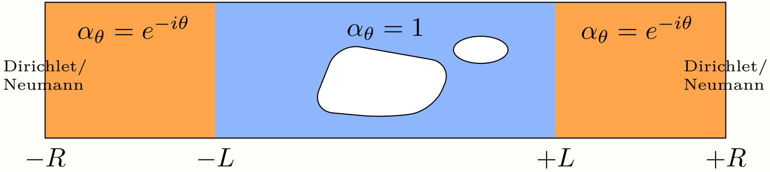
$\alpha_\theta(x) = 1$ for $|x| < L$ $\qquad$ $\alpha_\theta(x) = e^{-i\theta}$ for $|x| \geq L$

- Numerically we solve $(*)$ in the truncated domain



$\Rightarrow$ We obtain a good approximation of v_s for $|x| < L$.

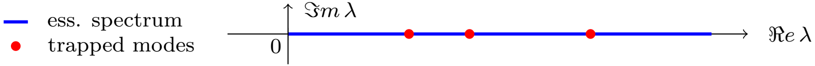
- This is the method of Perfectly Matched Layers (PMLs).

Spectral analysis

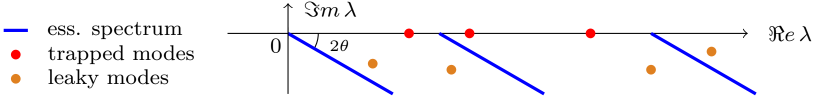
- Define the operators A, A_θ of $\mathrm{L}^2(\Omega)$ such that

$$A v = -\Delta v, \qquad A_\theta v = -\left(\alpha_\theta \frac{\partial}{\partial x}\left(\alpha_\theta \frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial y^2}\right) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\text{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.

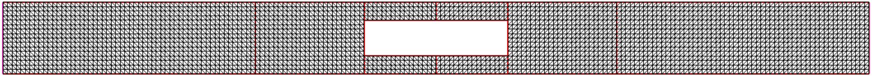


- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
- $\sigma_{\text{ess}}(A_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\}$.
- real eigenvalues of A_θ = real eigenvalues of A.



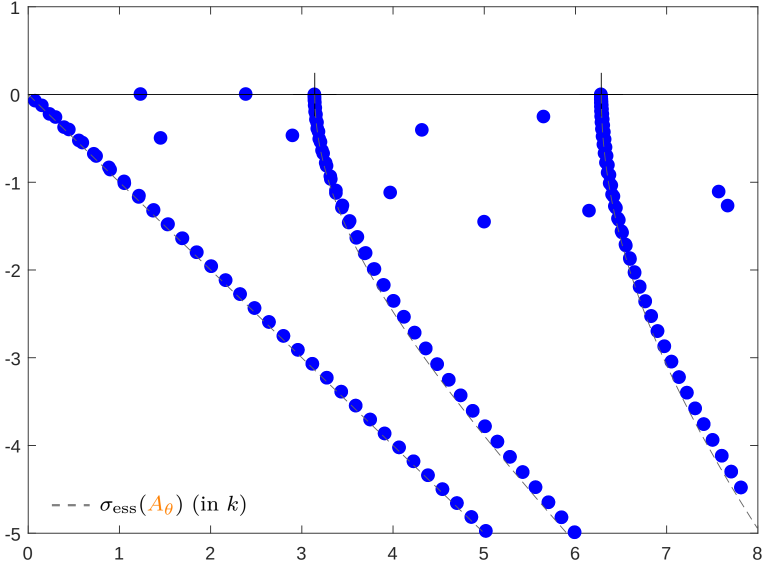
Numerical results

- We work in the geometry



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



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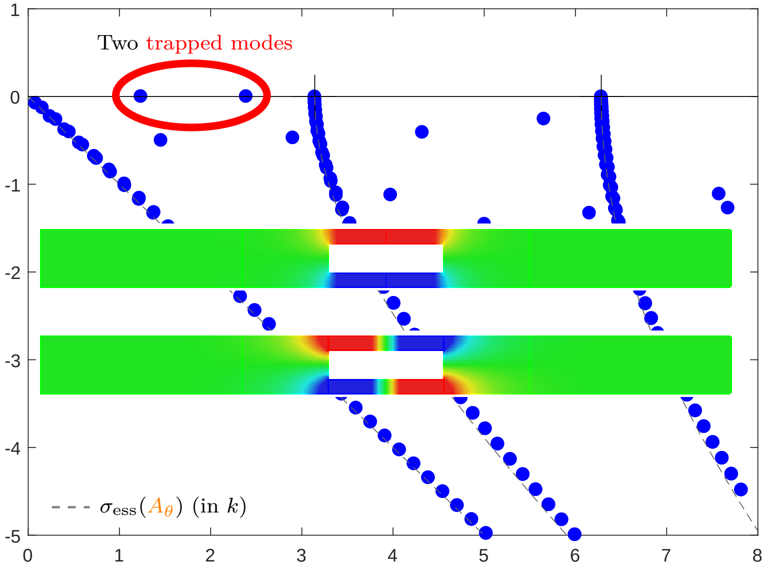

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We recall how to use classical complex scaling to compute trapped modes and complex resonances.

3. Conjugated complex scaling

We explain how to use conjugated complex scaling to compute non reflecting modes.

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

Important remark: **general** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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IMPORTANT REMARK: **non reflecting** v decompose as

$$v = \sum_{n=0}^{N-1} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

- In other words, **non reflecting** v are

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- In other words, **non reflecting** v are

ingoing at $-\infty$ and outgoing at $+\infty$.

Let us **change the sign** of the complex scaling at $-\infty$!

A new complex spectrum for non reflecting v

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{J}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\ e^{-i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\ e^{+i\theta} & \text{for } x \geq L. \end{array} \right.$$

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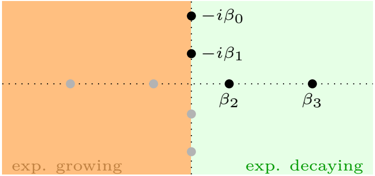
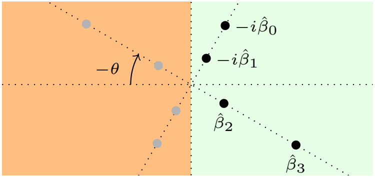
- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

2) u_θ is exp. decaying at infinity.



Modal exponents for v $(x \leq -L)$



Modal exponents for u_θ $(x \leq -L)$

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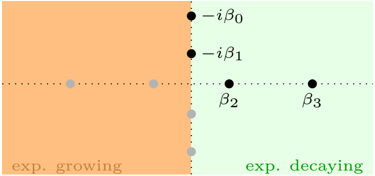
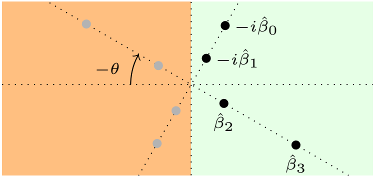


Modal exponents for v ($x \leq -L$)



Modal exponents for u_θ ($x \leq -L$)

- u_θ solves

$$(*) \left| \begin{array}{rl} \beta_\theta \dfrac{\partial}{\partial x}\left(\beta_\theta \dfrac{\partial u_\theta}{\partial x}\right) + \dfrac{\partial^2 u_\theta}{\partial y^2} + k^2 u_\theta = 0 & \text{in } \Omega \\ \partial_n u_\theta = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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- For $\theta \in (0;\pi/2)$, consider the complex change of variables

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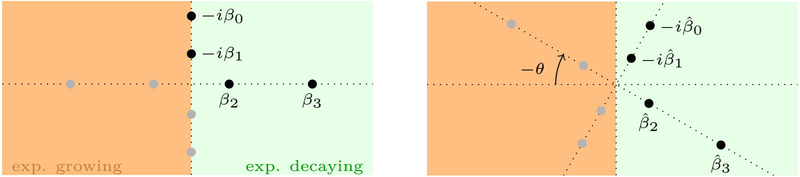

Modal exponents for v $(x \le -L)$ Modal exponents for u_θ $(x \le -L)$

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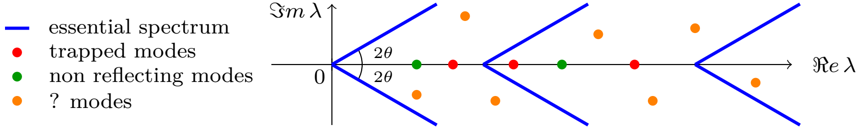
$\beta_\theta(x) = 1$ for $|x| < L$, $\beta_\theta(x) = e^{i\theta}$ for $x \le -L$, $\beta_\theta(x) = e^{-i\theta}$ for $x \ge L$

Spectral analysis

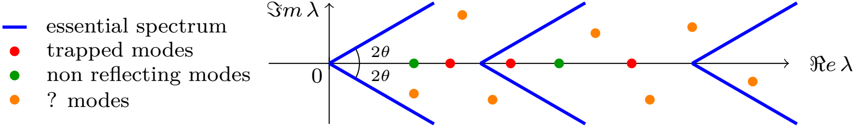
- Define the operator B_θ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta v = -\Big(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- B_θ is not selfadjoint. $\sigma(B_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0, \gamma \in [-2\theta; 2\theta]\}$.
- $\sigma_{\text{ess}}(B_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\} \cup \{n^2\pi^2 + t\, e^{2i\theta},\ t \geq 0\}$.
- real eigenvalues of B_θ = real eigenvalues of A+non reflecting k^2.



Remarks



1) **? modes** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

Different from **leaky modes** which are exp. growing both at $\pm\infty$ ...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

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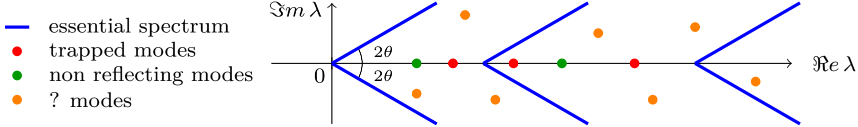

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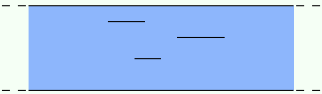
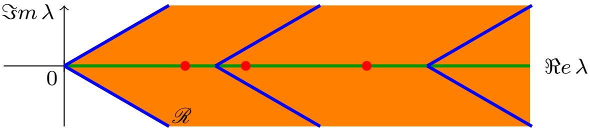


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Remarks

- essential spectrum (blue)
- trapped modes (red)
- non reflecting modes (green)
- ? modes (orange)



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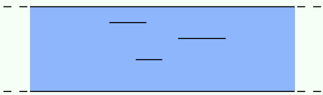

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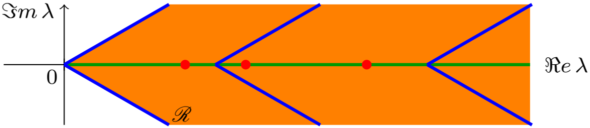

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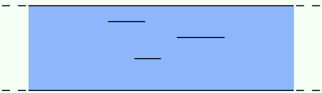

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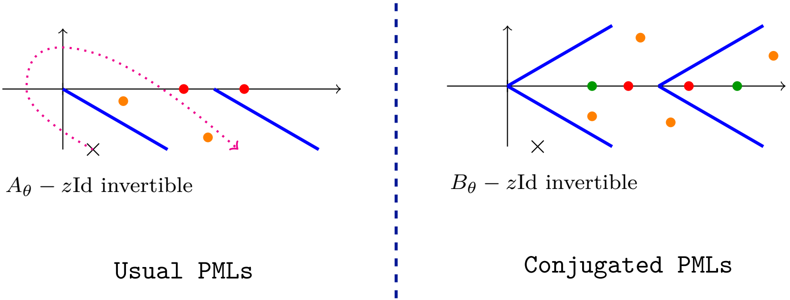

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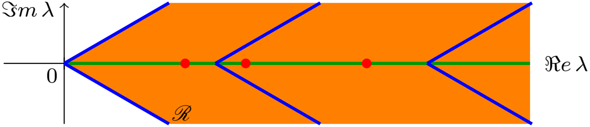

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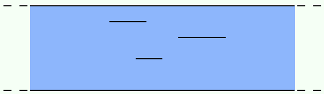

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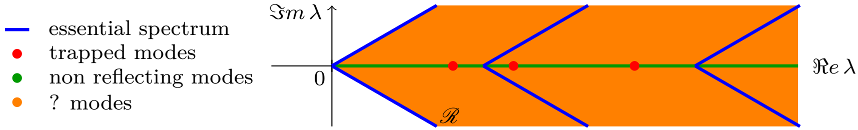

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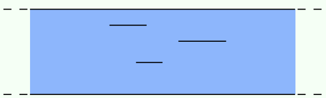

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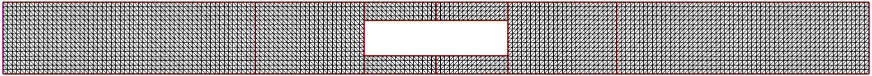
$\rightarrow$ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

$\rightarrow$ A compact perturbation can change drastically the spectrum (B_θ is not selfadjoint).

Numerical consequences?

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

PROP.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y)\in\Omega\}$, B_θ is $\mathcal{PT}$ symmetric:

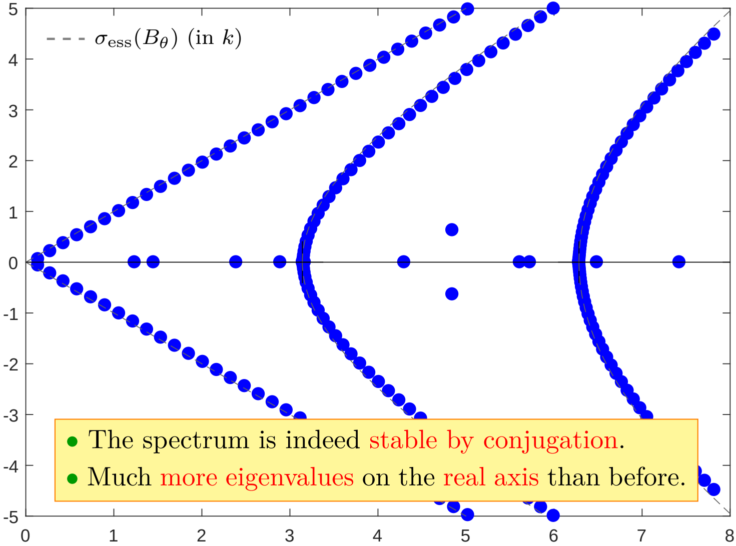
$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If λ is an "isolated" eigenvalue located close to the real axis, then $\lambda \in \mathbb{R}$!

Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

Numerical results

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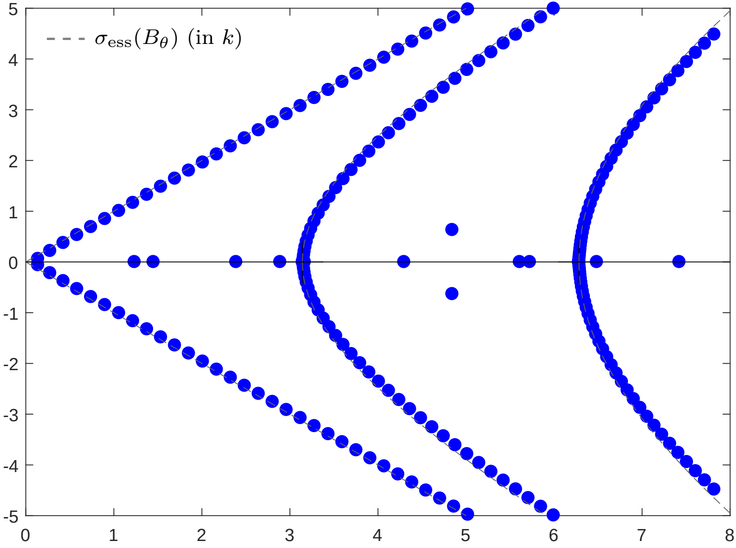

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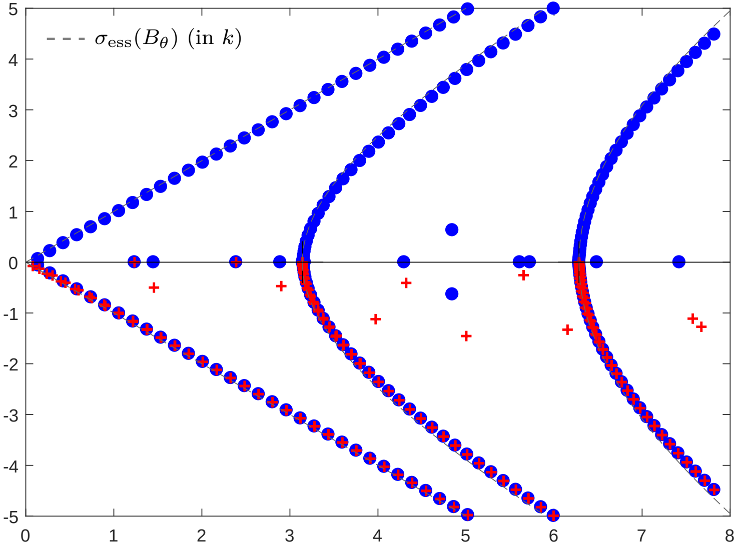
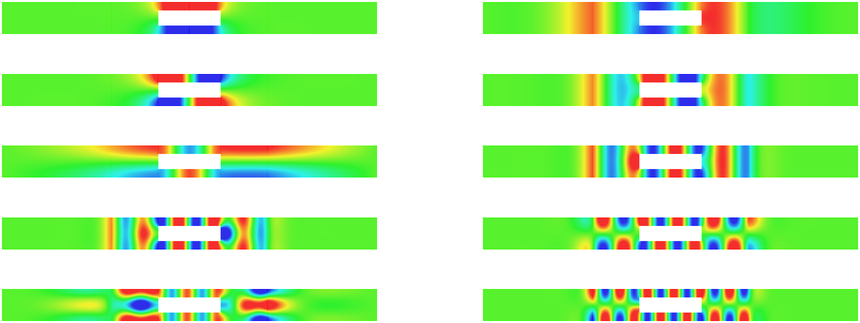


● PMLs with different signs ✚ Classical PMLs

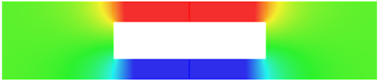
Numerical results

► We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



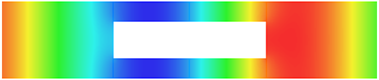
Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

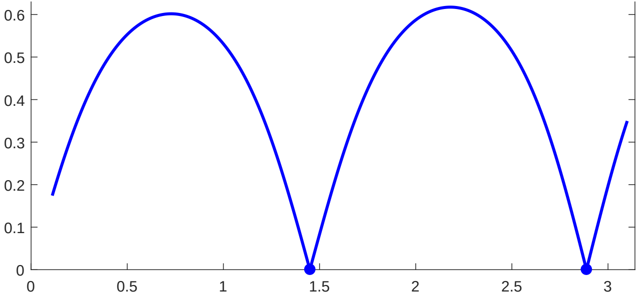


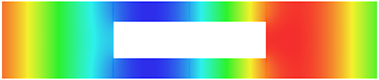
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





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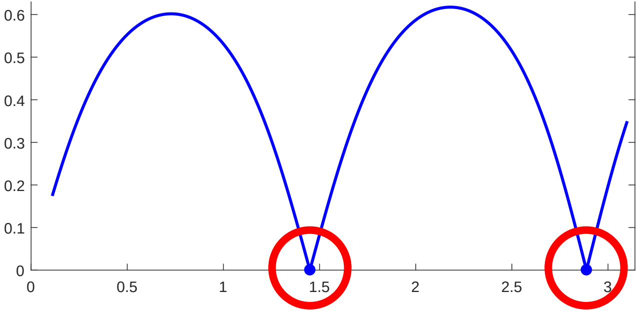

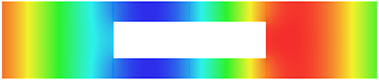

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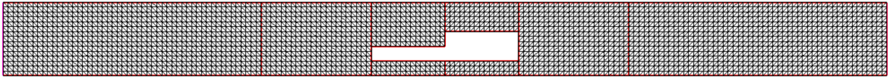
There is perfect agreement!

Numerical results

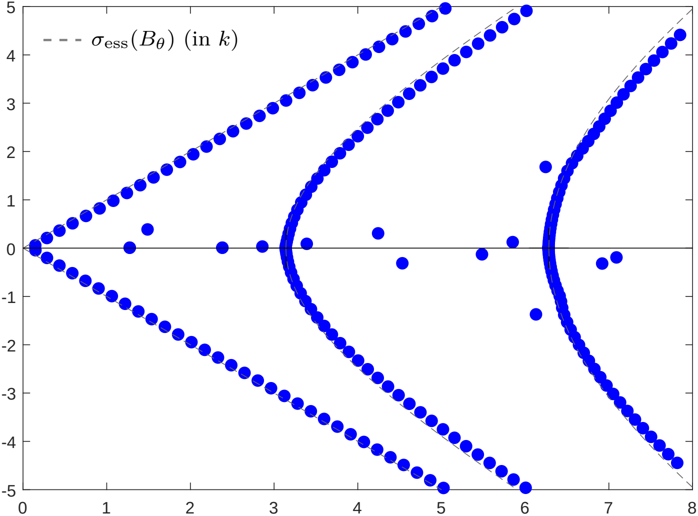
- Now the geometry is not symmetric in x nor in y:



- The operator B_θ is no longer $\mathcal{PT}$-symmetric and we expect:
  - No trapped modes
  - No invariance of the spectrum by complex conjugation.

Numerical results

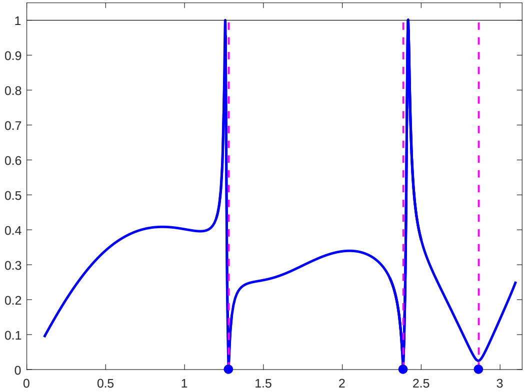
- Discretized spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is not symmetric w.r.t. the real axis.

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

Numerical results

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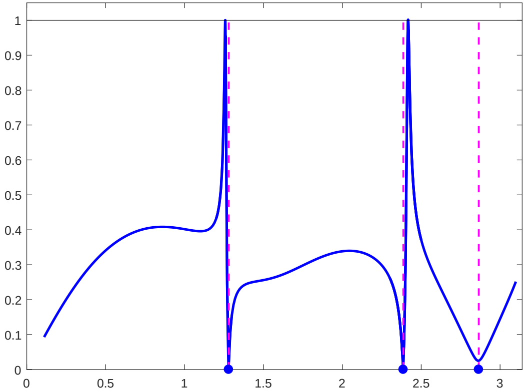

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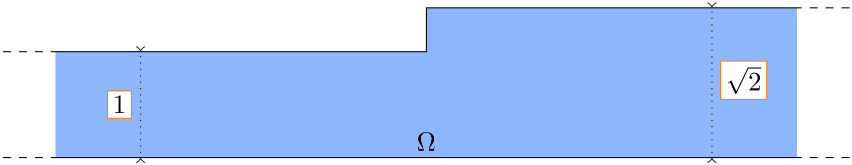
Complex eigenvalues also contain information on almost no reflection.

Remark

- For the Dirichlet problem

$$\left|\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } v = v_i + v_s \text{ s. t.}} \\ \Delta v + k^2 v & = & 0 & \text{in } \Omega, \\ v & = & 0 & \text{on } \partial\Omega, \\ \multicolumn{4}{c}{v_s \text{ is outgoing}} \end{array}\right.$$

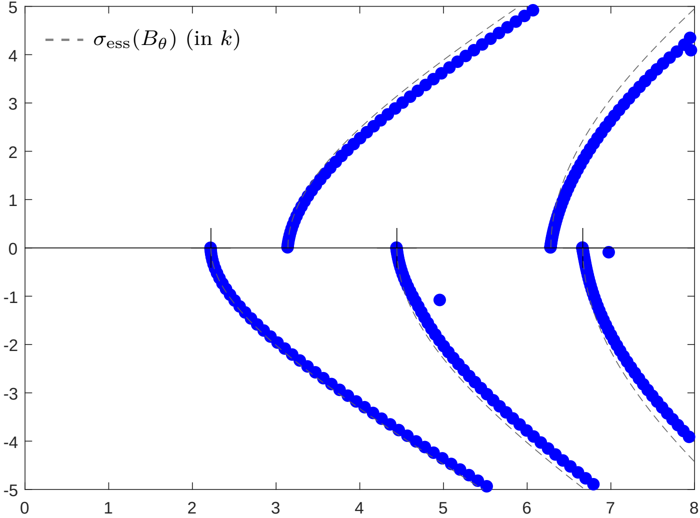
in the junction of waveguides



the set $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is connected. The sets of threshold frequencies are $\{n^2\pi^2,\, n \in \mathbb{N}^*\}$ and $\{m^2\pi^2/2,\, m \in \mathbb{N}^*\}$.

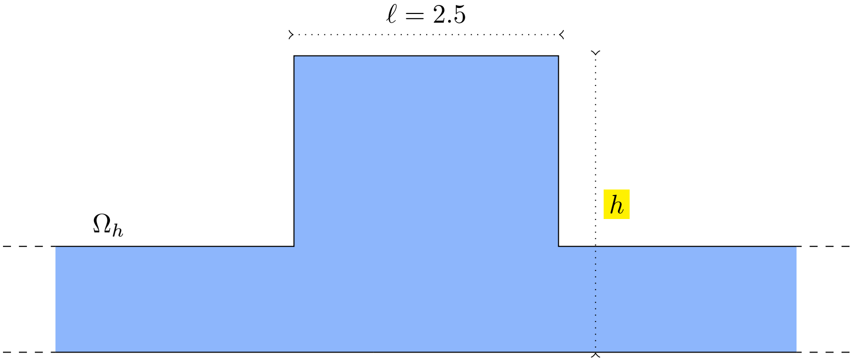
Remark

- Discretized spectrum of B_θ (Dirichlet) in k (not in k^2) with $\theta = \pi/4$.



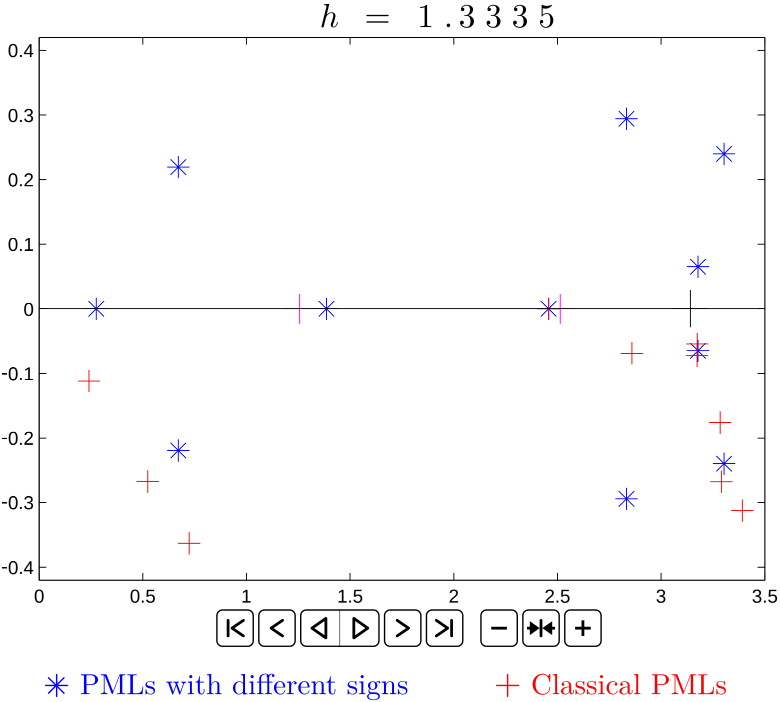
Spectra for a changing geometry

- Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.



- The magenta marks on the real axis correspond to $k = \pi/\ell$ & $k = 2\pi/\ell$. For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain h.

- We zoom at the region $0 < \Re e\, k < \pi$.



$h = 1.3335$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
PMLs with different signs
Classical PMLs

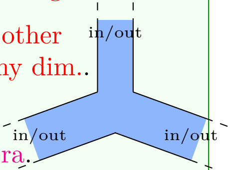
Outline of the talk

Conclusion

What we did

- We presented an approach to compute non reflecting k (values of k s.t. there is a v_i whose v_s is exp. decaying at $-\infty$) for a given Ω.
- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dim..

  With N leads, 2^N in/out spectra:
  1 purely in, 1 purely out, $2^N - 2$ non reflecting spectra.



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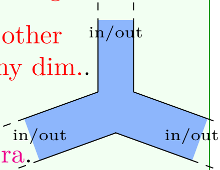
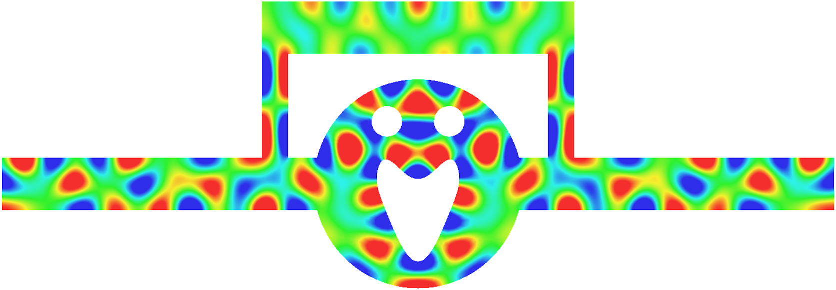


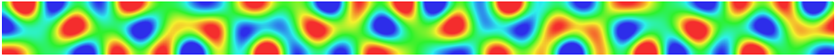
Future work

1) How to justify the numerics? Absence of spectral pollution?
2) Can we find a spectral approach to compute completely reflecting or completely invisible k for a given geometry?
3) Can we find a spectral approach to identify modal conversion?
4) Can we prove existence of non reflecting k for the $\mathcal{PT}$-symmetric pb?

v



v_i



Thank you for your attention!