Conference on Mathematics of Wave Phenomena

Maxwell's equations with hypersingularities at a conical plasmonic tip

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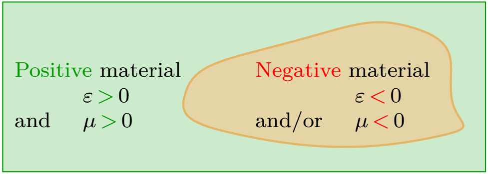
Goal and motivation

We study 3D time harmonic Maxwell's equations in presence of an inclusion of negative material:

$$
\left|\begin{array}{l}
\mathbf{curl}\, \boldsymbol{E} - i\omega\mu \boldsymbol{H} = 0 \text{ in } \Omega \\
\mathbf{curl}\, \boldsymbol{H} + i\omega\varepsilon \boldsymbol{E} = \boldsymbol{J} \text{ in } \Omega \\
+ \text{ PEC boundary cond.:} \\
\boldsymbol{E} \times \nu = 0 \text{ on } \partial\Omega \\
\mu \boldsymbol{H} \cdot \nu = 0 \text{ on } \partial\Omega
\end{array}\right.
$$



- For metals at optical frequencies, $\varepsilon < 0$ and $\mu > 0$.
- Artificial metamaterials have been realized which can be modelled for certain frequencies by $\varepsilon < 0$ and $\mu < 0$.

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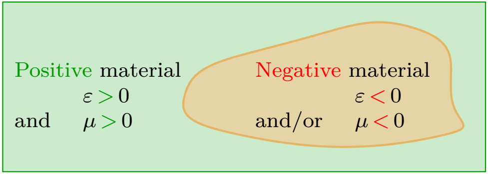

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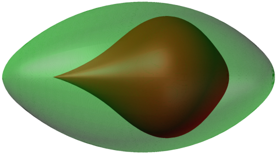
Particular motivation: non smooth gold nanoparticles.



Difficulty: usual results do not apply, singularities at the tip are amplified.

Outline of the talk

1. Positive coefficients
2. Sign-changing coefficients - non critical case
3. Scalar problems
4. Sign-changing coefficients - critical case

Classical case

- Let us first consider the classical case where $\varepsilon,\, \mu \geq c > 0$ in Ω.
- We focus our attention on the electric problem

$$(\mathscr{P})\left|\begin{array}{rcll} \mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\boldsymbol{E}-\omega^2\varepsilon\boldsymbol{E} & = & i\omega\boldsymbol{J} & \text{in } \Omega \\ \boldsymbol{E}\times\nu & = & 0 & \text{in } \partial\Omega \end{array}\right.$$

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Difficulty: $\nabla(\mathrm{H}^1_0) \subset \ker\mathbf{curl}\,\cdot$ and the embedding $\mathbf{H}_N(\mathbf{curl}\,) \subset \mathbf{L}^2(\Omega)$ is not compact which prevents using Fredholm alternative.

Classical case 2/3

Use the divergence free condition and work in the space

$$\mathbf{X}_N(\varepsilon) := \{\boldsymbol{u} \in \mathbf{H}_N(\mathbf{curl}\,) \,|\, \mathrm{div}\,(\varepsilon \boldsymbol{u}) = 0 \text{ in } \partial\Omega\}$$

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- This leads to the problem

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PROPOSITION: When ε, $\mu \geq c > 0$:

- the embedding $\mathbf{X}_N(\varepsilon) \subset \mathbf{L}^2(\Omega)$ is compact;

- $(\boldsymbol{u}, \boldsymbol{v}) \mapsto \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx$ is coercive in $\mathbf{X}_N(\varepsilon)$;

so that $(\mathscr{P}_{\mathbf{X}})$ satisfies the Fredholm alternative (uniqueness $\Rightarrow$ existence).

- Well-posedness of the initial problem comes from the following result:

Prop.: Assume that $\varepsilon \geq c > 0$. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

Proof. $\Rightarrow$ This implication is direct.

$\Leftarrow$ Assume that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$. For $\boldsymbol{E}' \in \mathbf{H}_N(\mathbf{curl}\,)$, let $\varphi \in \mathrm{H}^1_0(\Omega)$ be s.t.

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Then we have $\boldsymbol{E}' - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$ so that we can write

$$\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,(\boldsymbol{E}'-\nabla\varphi) - \omega^2\varepsilon\boldsymbol{E}\cdot(\boldsymbol{E}'-\nabla\varphi)\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot(\boldsymbol{E}'-\nabla\varphi)\,dx.$$

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This implies that $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$. $\square$

Sign-changing coefficients

- Now we allow for a possible change of sign of ε and/or μ in Ω.

Introduce the operator $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(A_\varepsilon \varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon \nabla\varphi \cdot \nabla\overline{\varphi'}\, dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_0(\Omega).$$

Working as above, one shows:

Proposition: Assume that A_ε is an isomorphism. Then $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{H}})$ iff $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}})$.

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PROPOSITION: Assume that A_ε is an isomorphism. Then we have

$$\|\boldsymbol{u}\|_\Omega \le C\,\|\mathbf{curl}\,\boldsymbol{u}\|_\Omega, \qquad \forall \boldsymbol{u} \in \mathbf{X}_N(\varepsilon).$$

Thus $\mathbf{X}_N(\varepsilon)$ endowed with $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

`Proof.` Write $\boldsymbol{u} = \nabla\varphi + \mathbf{curl}\,\boldsymbol{\psi}$ with $\varphi \in \mathrm{H}^1_0(\Omega)$ and $\boldsymbol{\psi} \in \mathbf{X}_T(1)$.
Then use that $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\psi} = \Delta\boldsymbol{\psi} = \mathbf{curl}\,\boldsymbol{u}$ and $A_\varepsilon\varphi = \mathrm{div}\,(\varepsilon\mathbf{curl}\,\boldsymbol{\psi})$. □

Sign-changing coefficients

How to study $(\mathscr{P}_{\mathbf{X}})$ now?

$$(\mathscr{P}_{\mathbf{X}}) \left| \begin{array}{l} \text{Find } \boldsymbol{E} \in \mathbf{X}_N(\varepsilon) \text{ such that for all } \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) : \\ \underbrace{\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{E}'}}_{a(\boldsymbol{E},\boldsymbol{E}')} - \omega^2 \underbrace{\int_\Omega \varepsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}'}}_{c(\boldsymbol{E},\boldsymbol{E}')} = \underbrace{\int_\Omega \boldsymbol{F} \cdot \overline{\boldsymbol{E}'}}_{\ell(\boldsymbol{E}')}, \end{array} \right.$$

When μ changes sign, $a(\cdot,\cdot)$ is not coercive.

When ε changes sign, is the embedding $\mathbf{X}_N(\varepsilon) \subset \mathbf{L}^2(\Omega)$ compact?

$\mathbb{T}$-coercivity in the vector case 1/2

If $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$, we have

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\begin{array}{rrcll}
& a(\boldsymbol{E},\boldsymbol{E}') - \omega^2 c(\boldsymbol{E},\boldsymbol{E}') & = & \ell(\boldsymbol{E}'), & \forall \boldsymbol{E}' \in \mathbf{X}_N(\varepsilon) \\
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The key idea is to construct $\mathbb{T} \in \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that $a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}) = \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,(\overline{\mathbb{T}\boldsymbol{E}'})$ is coercive in $\mathbf{X}_N(\varepsilon)$.

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To present the construction, set $\mathrm{H}^1_\#(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega) \,|\, \int_\Omega \varphi \, dx = 0\}$.

Introduce the operator $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ such that

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(A_\mu \varphi, \varphi')_{\mathrm{H}^1_\#(\Omega)} = \int_\Omega \mu \nabla\varphi \cdot \nabla\overline{\varphi'}\, dx, \qquad \forall \varphi, \varphi' \in \mathrm{H}^1_\#(\Omega).
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1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

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$$\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\varphi'\,dx = \int_\Omega \varepsilon\boldsymbol{u}\cdot\nabla\varphi'\,dx, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

☛ Ok when A_ε is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx$$

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

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2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

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$\mathbb{T}$-coercivity in the vector case

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

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$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1} \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{u}} \, dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2 \, dx.$$

Consider $\boldsymbol{E} \in \mathbf{X}_N(\varepsilon)$.

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Lemma. Suppose that

$$A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega) \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_{\#}(\Omega) \to \mathrm{H}^1_{\#}(\Omega) \text{ is an isomorphism.}$$

Then, there exists $\mathbb{T} : \mathbf{X}_N(\varepsilon) \to \mathbf{X}_N(\varepsilon)$ such that, for all $\boldsymbol{E}$, $\boldsymbol{E}'$

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}') = a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}'}\,dx$$

(this implies in particular that $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N(\varepsilon)$).

is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N(\varepsilon)$. There holds:

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Compact embedding and final result

Theorem. Assume that $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

`Proof.` 1) $\operatorname{div}(\varepsilon \boldsymbol{u}) = 0 \quad \Rightarrow \quad \varepsilon \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{\psi} \quad$ with $\boldsymbol{\psi} \in \mathbf{X}_T(1)$.

2) Then we get $\mathbf{curl}\,(\varepsilon^{-1}\mathbf{curl}\,\boldsymbol{\psi}) = \mathbf{curl}\,\boldsymbol{u}$.

3) When $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isom, there is $\mathbb{T} : \mathbf{X}_T(1) \to \mathbf{X}_T(1)$ s.t.

$$\|\mathbf{curl}\,\boldsymbol{\psi}\|^2_\Omega = \int_\Omega \varepsilon^{-1}\mathbf{curl}\,\boldsymbol{\psi} \cdot \mathbf{curl}\,(\mathbb{T}\boldsymbol{\psi})\,dx = \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot (\mathbb{T}\boldsymbol{\psi})\,dx. \quad \square$$

Compact embedding and final result

THEOREM. Assume that $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism. Then the embedding of $\mathbf{X}_N(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

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- This yields the final result (Bonnet-BenDhia, Chesnel, Ciarlet 14'):

THEOREM. Suppose that

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$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism.}$$

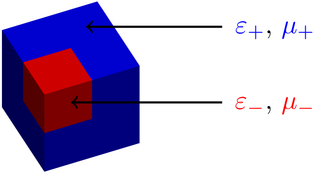
Then, the problem for the electric field is well-posed for all $\omega \in \mathbb{C}\setminus\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

Comments and example

- We have a similar result for the magnetic problem.
- These results extend to:

- situations where A_ε, A_μ are Fredholm of index zero with a non zero kernel;
- situations where Ω is not simply connected/$\partial\Omega$ is not connected.

Example of the Fichera's cube:



Proposition. Assume that

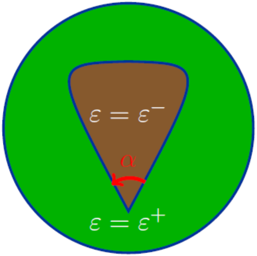
$$\frac{\varepsilon_-}{\varepsilon_+} \notin [-7; -\frac{1}{7}] \qquad \text{and} \qquad \frac{\mu_-}{\mu_+} \notin [-7; -\frac{1}{7}]\,. \quad *$$

Then, the problems for the electric and magnetic fields are well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

* *Note that 7 is the ratio of the blue volume over the red volume...*

2D scalar problem 1/2

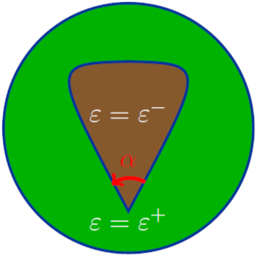
- Recall that $(A_\varepsilon\varphi,\varphi')_{\mathrm{H}^1_0(\Omega)} = \displaystyle\int_\Omega \varepsilon\nabla\varphi\cdot\nabla\overline{\varphi'}\,dx, \qquad \forall\varphi,\varphi'\in\mathrm{H}^1_0(\Omega).$



Features of A_ε depend on the angle α and on the contrast $\kappa := \varepsilon_-/\varepsilon_+$:

- If $\kappa \notin I_c := \left[\,-\frac{2\pi-\alpha}{\alpha};-\frac{\alpha}{2\pi-\alpha}\,\right]$, A_ε is Fredholm of index zero.
- If $\kappa \in I_c$, A_ε is not Fredholm (its range is not close in $\mathrm{H}^1_0(\Omega)$).

- Recall that $(A_\varepsilon\varphi,\varphi')_{\mathrm{H}^1_0(\Omega)} = \int_\Omega \varepsilon\nabla\varphi\cdot\nabla\overline{\varphi'}\,dx, \qquad \forall\varphi,\varphi'\in\mathrm{H}^1_0(\Omega).$



For $\alpha = \pi/2$,
$I_c = [-3; -1/3]$.

Features of A_ε depend on the angle α and on the contrast $\kappa := \varepsilon_-/\varepsilon_+$:

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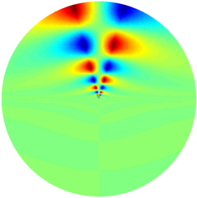
- For $\kappa \in I_c \setminus \{-1\}$, Fredholmness in $\mathrm{H}^1_0(\Omega)$ is lost due to the existence of propagating singularities:



$$\left| \begin{array}{l} s^{\pm}(x) = r^{\pm i\eta}\Phi(\theta), \quad \eta \in \mathbb{R} \setminus \{0\} \\ \mathrm{div}\,(\varepsilon \nabla s^{\pm}) = 0. \end{array} \right.$$

We have $s^{\pm} \in \mathrm{L}^2(\Omega)$ but $s^{\pm} \notin \mathrm{H}^1(\Omega)$.
Energy accumulates at the corner, $s^{\pm}$ are called black-hole singularities.

2D scalar problem 2/2

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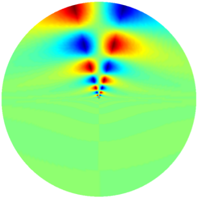

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Energy accumulates at the corner, $s^{\pm}$ are called black-hole singularities.

- To recover Fredholmness, we have to modify the functional framework and take into account these singularities:

- The corner is like infinity for scattering problem: it is necessary to select the outgoing behaviour s^{out}.

- Set $\mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega)$ where $\mathrm{V}^1_{-\beta}(\Omega)$ is a weighted Sobolev space of functions which decay at the corner and $\mathfrak{s}^{\mathrm{out}} := \chi s^{\mathrm{out}}$ (localization).

2D scalar problem

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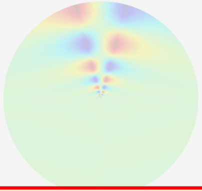

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the outgoing behaviour s^{out}.

THEOREM: Let $A^{\mathrm{out}}_{\varepsilon} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}^1_{\beta}(\Omega)^*$ be the operator such that

$$\langle A^{\mathrm{out}}_{\varepsilon}\varphi, \psi\rangle = \fint_{\Omega} \varepsilon \nabla\varphi \cdot \nabla\overline{\psi}\,dx := -\int_{\Omega} c_{\varphi}\mathrm{div}\,(\varepsilon\nabla\mathfrak{s}^{\mathrm{out}})\,\overline{\psi}\,dx + \int_{\Omega}\varepsilon\nabla\tilde{\varphi}\cdot\nabla\overline{\psi}\,dx$$

for all $\varphi = c_{\varphi}\,\mathfrak{s}^{\mathrm{out}} + \tilde{\varphi}$, $\psi \in \mathrm{V}^1_{\beta}(\Omega)$.

Then $A^{\mathrm{out}}_{\varepsilon}$ is Fredholm of index zero. (Bonnet-BenDhia, Chesnel, Claeys 13')

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3D scalar problem

- Let us consider the case of a conical tip, the simplest singular geometry in 3D. Now propagating singularities are of the form

$$s^{\pm}(x) = r^{\pm i\eta - 1/2}\Phi(\theta, \psi), \qquad \eta \in \mathbb{R} \setminus \{0\}$$

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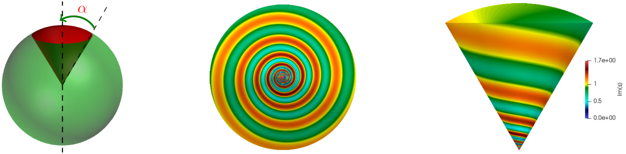

For the **circular conical tip**, they exist iff $\boldsymbol{\kappa \in (-1; -a_{\alpha})}$ (but not for $\kappa < -1$!) for a certain explicit a_{α}.

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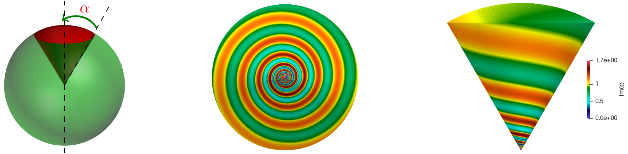

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- Contrary to 2D, in 3D we can have $N > 1$ singularities $s_1^{\pm}, \ldots, s_N^{\pm}$. Moreover $N \to +\infty$ when $\kappa \to -1^+$ or $\alpha \to 0^+$.

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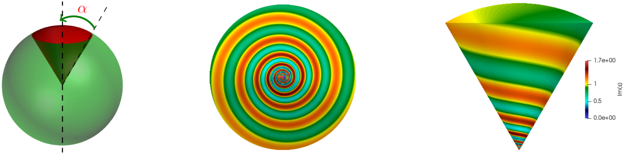

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The solution to $\operatorname{div}(\varepsilon\nabla\varphi) = f$ must be searched in

$$\left|\begin{array}{ll} \mathrm{H}^1_0(\Omega) & \text{when } \boldsymbol{\kappa} \notin [\boldsymbol{-1; -a_\alpha}]; \\ \mathrm{V}^{\mathrm{out}} := \mathrm{span}(\mathfrak{s}_1^{\mathrm{out}},\dots,\mathfrak{s}_N^{\mathrm{out}}) \oplus \mathrm{V}^1_{-\beta}(\Omega) & \text{when } \boldsymbol{\kappa} \in (\boldsymbol{-1; -a_\alpha}). \end{array}\right.$$

A new framework for electric fields

- We assume that the negative material has a conical tip and that there are N propagating singularities $\mathfrak{s}_1^{\text{out}}, \dots, \mathfrak{s}_N^{\text{out}}$ for the operator $\operatorname{div}(\varepsilon\nabla\cdot)$.

- We assume that μ is such that $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ is an isomorphism.

- Define the new space

$$\begin{aligned}\mathbf{X}_N^{\text{out}}(\varepsilon) := \{\boldsymbol{u} = \sum_{n=1}^{N} c_n \nabla\mathfrak{s}_n^{\text{out}} + \tilde{\boldsymbol{u}},\ c_n \in \mathbb{C},\ \tilde{\boldsymbol{u}} \in \mathbf{V}^0_{-\beta}(\Omega)\,|\\ \mathbf{curl}\,\boldsymbol{u} \in \mathbf{L}^2(\Omega), \operatorname{div}(\varepsilon\boldsymbol{u}) = 0 \text{ in } \Omega \text{ and } \boldsymbol{u}\times\nu = 0 \text{ on } \partial\Omega\}\end{aligned}$$

Here $\mathbf{V}^0_{-\beta}(\Omega) := \{\boldsymbol{u}\,|\,r^{-\beta}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\}$, $\beta > 0$.

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Here $\mathbf{V}^0_{-\beta}(\Omega) := \{\boldsymbol{u}\,|\, r^{-\beta}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\}$, $\beta > 0$.

- Note that $\mathbf{X}_N(\varepsilon) \subset \mathbf{X}_N^{\text{out}}(\varepsilon) \not\subset \mathbf{L}^2(\Omega)$ (infinite energy!).

PROPOSITION: When $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}^1_\beta(\Omega)^*$ is an isomorphism, we have

$$|c| + \|\tilde{\boldsymbol{u}}\|_{\mathbf{V}^0_{-\beta}(\Omega)} \le C\,\|\mathbf{curl}\,\boldsymbol{u}\|_\Omega, \qquad \forall \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon).$$

Thus $\mathbf{X}_N^{\text{out}}(\varepsilon)$ endowed with $(\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_\Omega$ is a Hilbert space.

A new functional framework

- Then we consider the problem

$$(\mathscr{P}_{\mathbf{X}^{\text{out}}}) \left| \begin{array}{l} \text{Find } \boldsymbol{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that for all } \boldsymbol{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1} \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, dx - \omega^2 \fint_\Omega \varepsilon \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, dx = i\omega \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}}\, dx \end{array} \right.$$

with $\displaystyle\fint_\Omega \varepsilon \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, dx = c_{\boldsymbol{u}} \overline{c_{\boldsymbol{v}}} \int_\Omega \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+\, dx + \int_\Omega \varepsilon \tilde{\boldsymbol{u}} \cdot \overline{\tilde{\boldsymbol{v}}}\, dx.$

A new functional framework

- Then we consider the problem

$$(\mathscr{P}_{\mathbf{X}^{\mathrm{out}}})\quad \left| \begin{array}{l} \text{Find } \boldsymbol{u} \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon) \text{ such that for all } \boldsymbol{v} \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon) \\ \displaystyle\int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,dx - \omega^2 \fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = i\omega\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,dx \end{array}\right.$$

with $\displaystyle\fint_\Omega \varepsilon\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,dx = c_{\boldsymbol{u}}\overline{c_{\boldsymbol{v}}}\int_\Omega \mathrm{div}\,(\varepsilon\nabla\overline{s^+})s^+\,dx + \int_\Omega \varepsilon\tilde{\boldsymbol{u}}\cdot\overline{\tilde{\boldsymbol{v}}}\,dx.$

Proposition: When $A_\varepsilon^{\mathrm{out}} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism, $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}^{\mathrm{out}}})$ iff $\boldsymbol{E}$ solves the initial problem.

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PROPOSITION: When $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}_\beta^1(\Omega)^*$ is an isomorphism, $\boldsymbol{E}$ solves $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$ iff $\boldsymbol{E}$ solves the initial problem.

- To study $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$, next we construct a $\mathbb{T}$-coercivity operator in $\mathbf{X}_N^{\text{out}}(\varepsilon)$.

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_{\#}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_{\#}(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\operatorname{div}(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

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2. Since $\operatorname{div}(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\text{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$.

1. Introduce $\psi \in \mathrm{H}^1_\#(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} - \nabla\psi \in \mathbf{X}_T(\mu)$. To proceed, solve

$$\int_\Omega \mu\nabla\psi\cdot\nabla\psi'\,dx = \int_\Omega \mu\mathbf{curl}\,\boldsymbol{E}\cdot\nabla\psi'\,dx, \quad \forall\psi' \in \mathrm{H}^1_\#(\Omega).$$

☛ Ok when A_μ is an isom.

2. Since $\mathrm{div}\,(\mu(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi)) = 0$, there is $\boldsymbol{u} \in \mathbf{X}_N(1)$ such that

$$\mathbf{curl}\,\boldsymbol{u} = \mu\,(\mathbf{curl}\,\boldsymbol{E} - \nabla\psi) \quad \text{in } \Omega.$$

Additionally, we can prove that $\boldsymbol{u} \in \mathbf{V}^0_{-\beta}(\Omega)$ for some $\beta > 0$.

3. Introduce $\varphi \in \mathrm{V}^{\text{out}}$ such that $\boldsymbol{u} - \nabla\varphi \in \mathbf{X}_N^{\text{out}}(\varepsilon)$. To proceed, solve

$$A_\varepsilon^{\text{out}}\varphi = -\mathrm{div}\,(\varepsilon\boldsymbol{u}).$$

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$$a(\boldsymbol{E},\mathbb{T}\boldsymbol{E}) = \int_\Omega \mu^{-1}\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\overline{\boldsymbol{u}}\,dx = \int_\Omega |\mathbf{curl}\,\boldsymbol{E}|^2\,dx.$$

$\mathbb{T}$-coercivity in $\mathbf{X}_N^{\mathrm{out}}(\varepsilon)$

Consider $\boldsymbol{E} \in \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$.

Lemma. When

$$A_\varepsilon^{\mathrm{out}} : \mathrm{V}^{\mathrm{out}} \to \mathrm{V}_\beta^1(\Omega)^* \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism,}$$

there exists $\mathbb{T} : \mathbf{X}_N^{\mathrm{out}}(\varepsilon) \to \mathbf{X}_N^{\mathrm{out}}(\varepsilon)$ such that, for all $\boldsymbol{E}$, $\boldsymbol{E}'$

$$a(\boldsymbol{E}, \mathbb{T}\boldsymbol{E}') = a(\mathbb{T}\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{E}'}\,dx$$

(this implies in particular that $\mathbb{T}$ is an isomorphism of $\mathbf{X}_N^{\mathrm{out}}(\varepsilon)$).

is an isom.

4. Finally, define $\mathbb{T}\boldsymbol{E} := \boldsymbol{u} - \nabla\varphi$. There holds:

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Compact embedding and final result

Theorem. Assume that $A_\varepsilon^{\rm out} : {\rm V}^{\rm out} \to {\rm V}^1_\beta(\Omega)^*$ is an isomorphism. Then the embedding of $\mathbf{X}_N^{\rm out}(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

Compact embedding and final result

THEOREM. Assume that $A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}^1_\beta(\Omega)^*$ is an isomorphism. Then the embedding of $\mathbf{X}_N^{\text{out}}(\varepsilon)$ in $\mathbf{L}^2(\Omega)$ is compact.

- This yields the final result (Bonnet-BenDhia, Chesnel, Rihani 22'):

THEOREM. Suppose that

$$A_\varepsilon^{\text{out}} : \mathrm{V}^{\text{out}} \to \mathrm{V}^1_\beta(\Omega)^* \text{ is an isomorphism}$$

$$A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega) \text{ is an isomorphism.}$$

Then, the problem $(\mathscr{P}_{\mathbf{X}^{\text{out}}})$ and the initial problem are well-posed for all $\omega \in \mathbb{C}\backslash\mathscr{S}$ where $\mathscr{S}$ is a discrete (or empty) set of $\mathbb{C}$.

Conclusion

What we obtained

1) When $A_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$, $A_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ are isomorphisms, the Maxwell's equations are well-posed in the usual spaces.

$\to$ For the circular conical tip, this corresponds to $\boldsymbol{\kappa_\varepsilon}$, $\boldsymbol{\kappa_\mu} \notin [\mathbf{-1}; \boldsymbol{-a_\alpha}]$.

2) For the circular conical tip with $\boldsymbol{\kappa_\varepsilon} \in (\mathbf{-1}; \boldsymbol{-a_\alpha})$, $\boldsymbol{\kappa_\mu} \notin [\mathbf{-1}; \boldsymbol{-a_\alpha}]$, the Maxwell's equations are well-posed only in a singular space.

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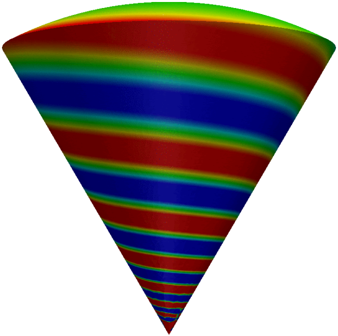
Comments and open questions

- In case 2), we also have a formulation for $\boldsymbol{H}$ (a bit more complex). Useful to study the case where both A_ε and A_μ are not Fredholm.

- In case 2), the problem $(\mathscr{P}_{\mathbf{X}})$ (in $\mathbf{X}_N(\varepsilon)$) is Fredholm but equivalence with the initial problem fails.

- Numerically, it is not clear how to compute the solution in case 2).

- How to study other 3D singular geometries, in particular with edges?



Thank you!

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