Acoustic passive cloaking using thin outer resonators

Lucas Chesnel

Coll. with J. Heleine[1], S.A. Nazarov[2].

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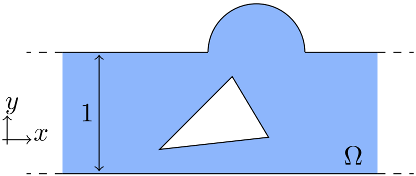
Introduction 1/2

- We consider the propagation of waves in a 2D acoustic waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).



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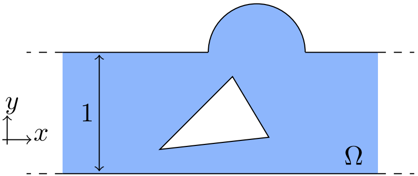

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- We fix $k \in (0;\pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.

- The scattering of these waves leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

$$u_+ = \left| \begin{array}{l} e^{ikx} + R_+ \, e^{-ikx} + \dots \\ \qquad T \quad e^{+ikx} + \dots \end{array} \right. \qquad u_- = \left| \begin{array}{l} \qquad T \quad e^{-ikx} + \dots \\ e^{-ikx} + R_- \, e^{+ikx} + \dots \end{array} \right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

$R_\pm$, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

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We wish to slightly perturb the walls of the guide to obtain $R_\pm = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ cloaking at “infinity”.

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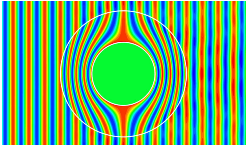


REMARK: Different from the usual cloaking picture (Pendry *et al.* 06, Leonhardt 06, Greenleaf *et al.* 09).
$\rightarrow$ Less ambitious but doable without fancy materials (and relevant in practice).

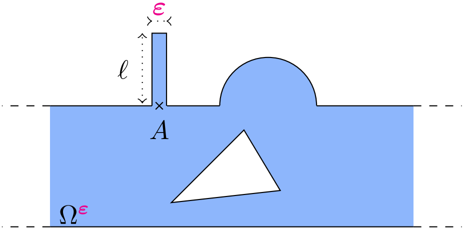
Difficulty: the scattering coefficients have a not explicit and not linear dependence wrt the geometry.

Outline of the talk

1. Asymptotic analysis in presence of thin resonators
2. Almost zero reflection
3. Cloaking

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^{\varepsilon}) \left| \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^{\varepsilon}, \\ \partial_n u = 0 & \text{on } \partial\Omega^{\varepsilon} \end{array} \right.$$

- In this geometry, we have the scattering solutions

$$u_+^{\varepsilon} = \left| \begin{array}{r} e^{ikx} + R_+^{\varepsilon}\, e^{-ikx} + \dots \\ T^{\varepsilon}\, e^{+ikx} + \dots \end{array} \right. \qquad u_-^{\varepsilon} = \left| \begin{array}{rl} T^{\varepsilon}\, e^{-ikx} + \dots & x \to -\infty \\ e^{-ikx} + R_-^{\varepsilon}\, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

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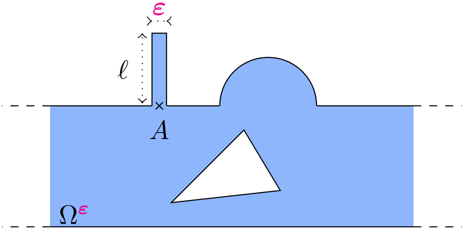

$$(\mathscr{P}^\varepsilon)\ \left|\ \begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } \Omega^\varepsilon, \\ \partial_n u &= 0 \quad \text{on } \partial\Omega^\varepsilon \end{aligned}\right.$$

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Next we compute an asymptotic expansion of $u_\pm^\varepsilon$, $R_\pm^\varepsilon$, T^ε as $\varepsilon \to 0$.
To proceed we use techniques of matched asymptotic expansions
(see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00...).

Limit problem in the resonator

► Considering the restriction of $(\mathscr{P}^{\varepsilon})$ to the thin resonator, when ε tends to zero, we meet the 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

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- At the **resonance lengths**,

$$\ell = \pi(m+1/2)/k, \qquad m \in \mathbb{N},$$

$(\mathscr{P}_{1\mathrm{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \pi(m+1/2)/k$, $m \in \mathbb{N}$. Then when $\varepsilon \to 0$, we get

$$\begin{aligned}
&u^{\varepsilon}_{\pm}(x,y) = u_{\pm} + o(1) && \text{in } \Omega,\\
&u^{\varepsilon}_{\pm}(x,y) = u_{\pm}(A)\, v_0(y) + o(1) && \text{in the resonator},\\
&R^{\varepsilon}_{\pm} = R_{\pm} + o(1), && T^{\varepsilon} = T + o(1).
\end{aligned}$$

Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

$\to$ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R_\pm^\varepsilon = 0 + \dots \\ T^\varepsilon = 1 + \dots \end{array} \right.$

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$$u_+^\varepsilon(x, y) = u_+(x, y) + a(\eta)k\gamma(x, y) + o(1) \quad \text{in } \Omega,$$

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$$R_+^\varepsilon = R_+ + ia(\eta)u_+(A)/2 + o(1), \qquad T^\varepsilon = T + ia(\eta)u_-(A)/2 + o(1).$$

– γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega \end{array} \right.$

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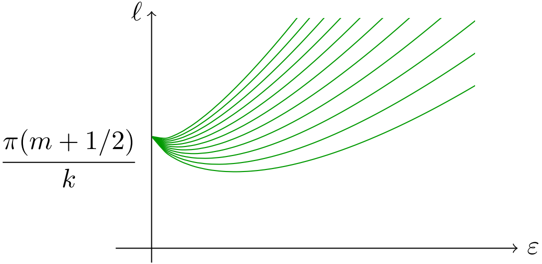
– $a(\eta)$ is a constant with an explicit dependence with respect to η.

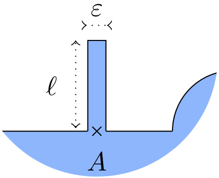
This time the thin resonator has an influence at order ε^0 which depends on the choice of η.

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \pi(m+1/2)/k + \varepsilon(\eta - \pi^{-1}|\ln\varepsilon|)),\ \varepsilon > 0\} \subset \mathbb{R}^2.$$





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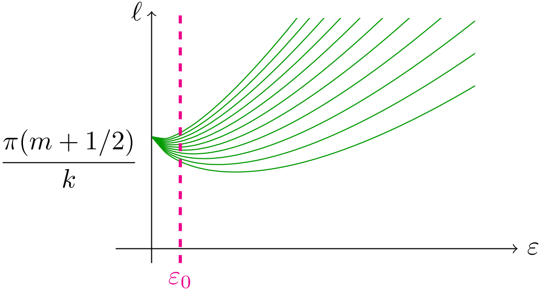

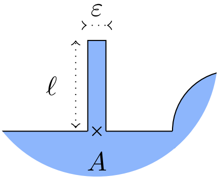

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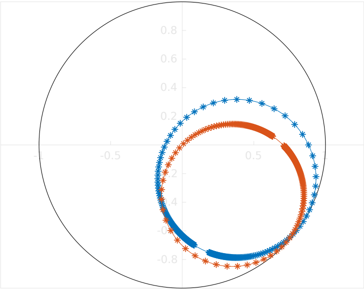
- For a fixed small ε_0, the scattering coefficients have a rapid variation for ℓ varying in a neighbourhood of the resonance length.

Almost zero reflection

- We got $\left|\begin{array}{rcl} R_+^\varepsilon & = & R_+^0(\eta) + o(1) \\ T^\varepsilon & = & T^0(\eta) + o(1) \end{array}\right.$ with $\left|\begin{array}{rcl} R_+^0(\eta) & := & R_+ + ia(\eta)\, u_\pm(A)\,/2 \\ T^0(\eta) & := & T + ia(\eta)\, u_\pm(A)\,/2. \end{array}\right.$

- One finds that the sets $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$, $\{T^0(\eta)\,|\,\eta\in\mathbb{R}\}$ are circles in $\mathbb{C}$.



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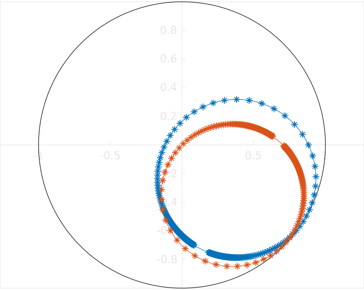

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**.

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- We got $\left|\begin{array}{rcl} R_+^\varepsilon & = & {\color{blue}R_+^0(\eta)} + o(1) \\ T^\varepsilon & = & {\color{red}T^0(\eta)} + o(1) \end{array}\right.$ with $\left|\begin{array}{rcl} {\color{blue}R_+^0(\eta)} & := & R_+ + ia(\eta)\, u_\pm(A)\, /2 \\ {\color{red}T^0(\eta)} & := & T + ia(\eta)\, u_\pm(A)\, /2. \end{array}\right.$

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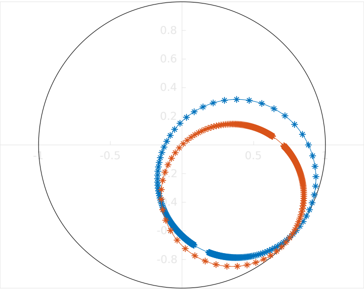

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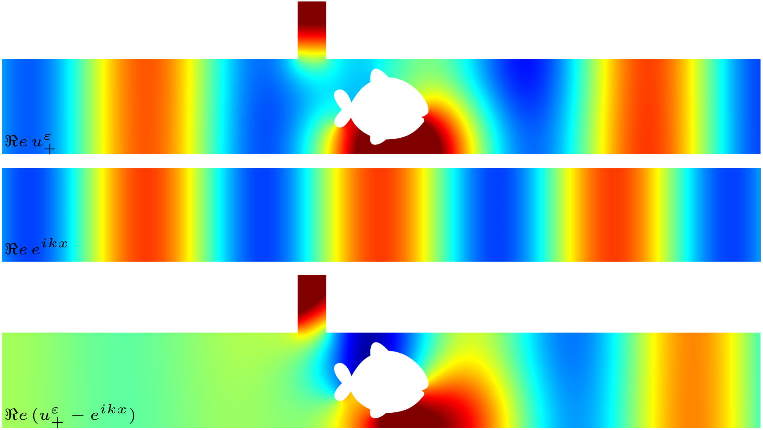
Proposition: There are **positions of the resonator A** such that the circle $\{{\color{blue}R_+^0(\eta)}\,|\,\eta\in\mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

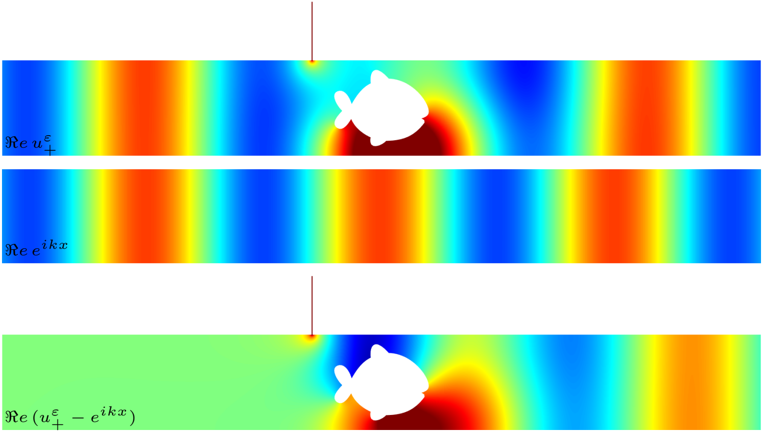
- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.3}$).



$\rightarrow$ Simulations realized with the `Freefem++` library.

Almost zero reflection

- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.01}$).



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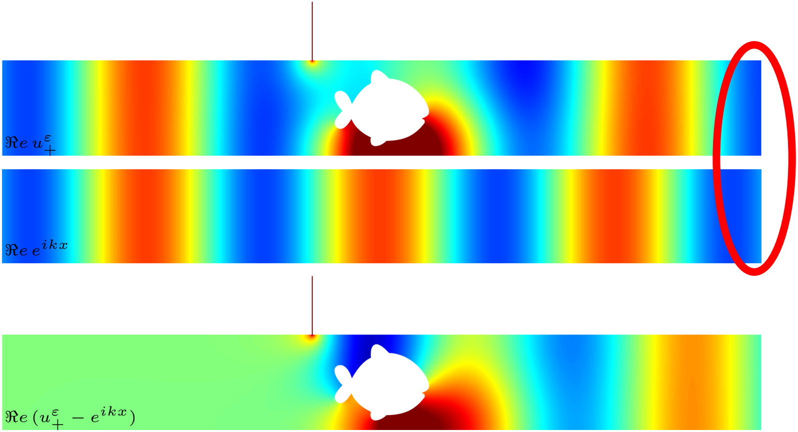

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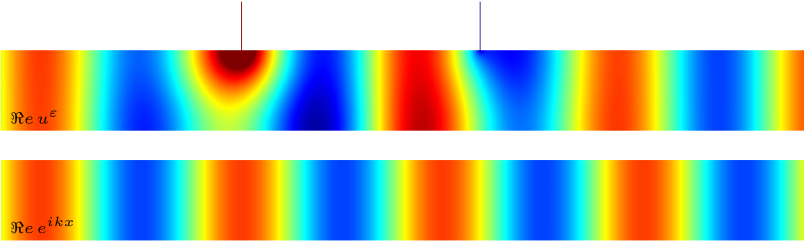
To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

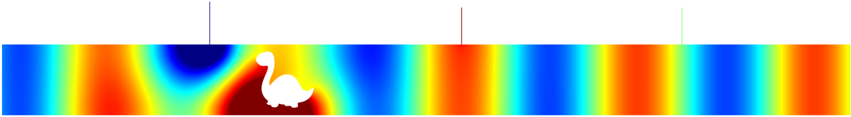


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

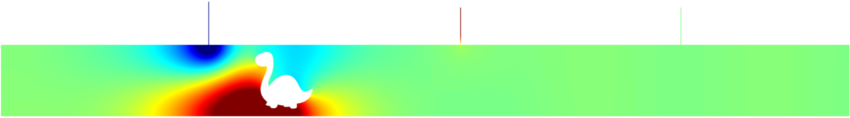
Cloaking with three resonators

- Gathering the two previous results, we can cloak any object with three resonators.

$\Re e\, u_+$



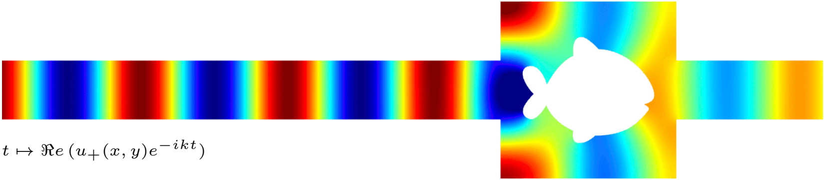
$\Re e\, u_+^\varepsilon$



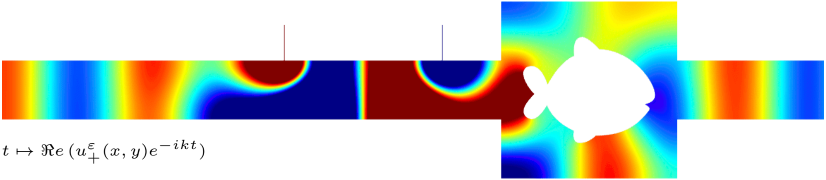
$\Re e\, (u_+^\varepsilon - e^{ikx})$

Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$



$t \mapsto \Re e\,(u_+^{\varepsilon}(x,y)e^{-ikt})$

$t \mapsto \Re e\,(e^{ik(x-t)})$

Conclusion

What we did

- ♠ We explained how to approximately cloak any object in monomode regime using thin resonators. Two main ingredients:
  - Around resonant lengths, effects of order ε^0 with perturb. of width ε.
  - The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

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♠ We explained how to approximately cloak any object in monomode regime using thin resonators. Two main ingredients:

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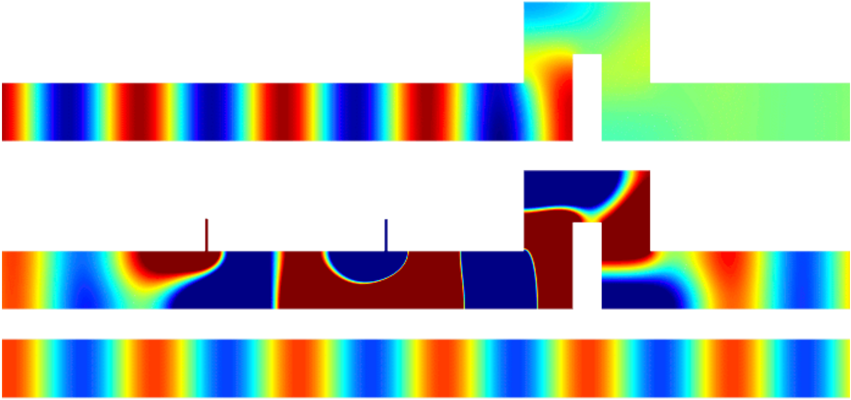
Possible extensions and open questions

1) We can similarly hide penetrable obstacles or work in 3D.
2) We can do cloaking at a finite number of wavenumbers (thin structures are resonant at one wavenumber otherwise act at order ε).
3) With Dirichlet BCs, other ideas must be found.
4) Can we realize exact cloaking ($T = 1$ exactly)? This question is also related to robustness of the device.



Thank you for your attention!

L. Chesnel, J. Heleine and S.A. Nazarov. Acoustic passive cloaking using thin outer resonators. submitted, arXiv:2105.00922, 2021.