Séminaire d'analyse de l'EPFL

Invisibility in acoustic waveguides

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Coll. with A. Bera[2], A.-S. Bonnet-Ben Dhia[2], S.A. Nazarov[3] and V. Pagneux[4].

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[4]LAUM, Université du Maine, France

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General setting

- We are interested in the propagation of waves in acoustic waveguides.

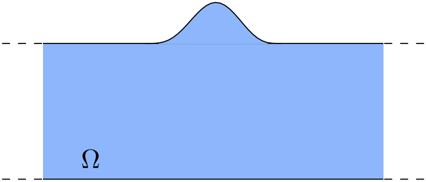
- In this talk, we study questions of invisibility.

> Can we find situations where waves
> go through like if there were no defect ?

- One can wish to have good energy transmission through the structure.
- One can wish to hide objects.

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} \Delta v + k^2 v &=& 0 & \text{in } \Omega, \\ \partial_n v &=& 0 & \text{on } \partial\Omega, \\ & & v_s \text{ is outgoing.} & \end{array}$$

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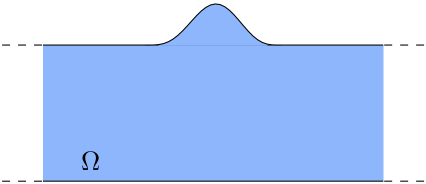

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- For this problem, the modes are

Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N.$

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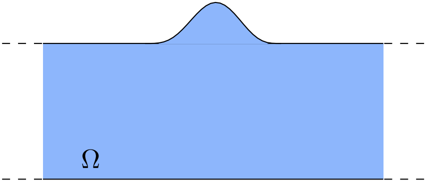

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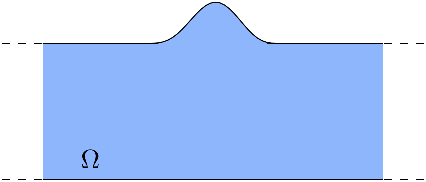

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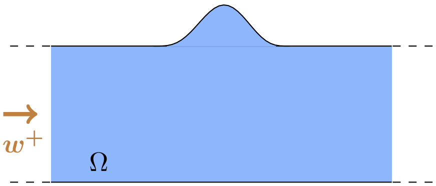

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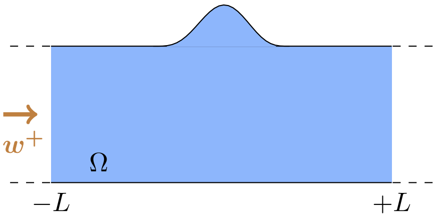

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- v_s is outgoing $\quad \Leftrightarrow \quad \boxed{v_s = s^{\pm} w^{\pm} + \tilde{v}_s \quad \text{for } \pm x \geq L,}$

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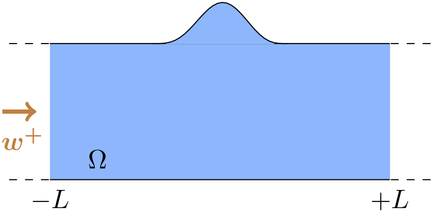

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DEFINITION:	v_i = incident field v = total field v_s = scattered field.

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

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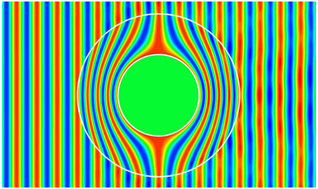
Remark: less ambitious than usual cloaking and therefore, more accessible. Also relevant for applications.



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GOAL

We explain how to find waveguides such that

$$R = 0 \ (|T| = 1), \ T = 1 \ (R = 0) \ \text{ or } \ T = 0 \ (|R| = 1).$$

Outline of the talk

1. First constructive method

   k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

   k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3. A spectral approach to determine non reflecting wavenumbers

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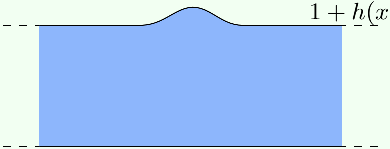
General picture

► Perturbative technique: we construct small non reflecting defects using variants of the implicit functions theorem.

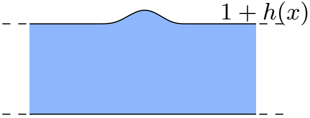


$R = 0$ $R = 0$

► The idea was used in `Nazarov 11` to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, set $R = R(h) \in \mathbb{C}$.



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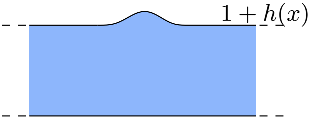

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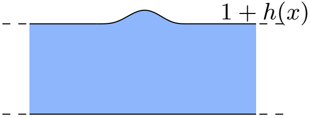

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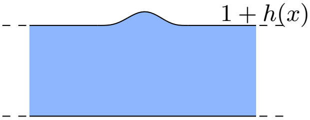

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- We look for small perturbations of the reference medium: $h = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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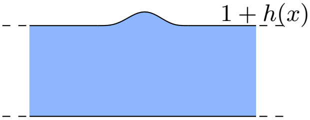

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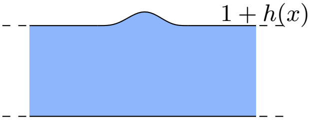

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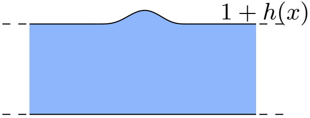

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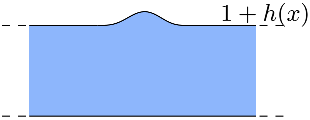

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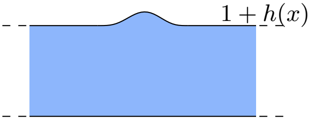

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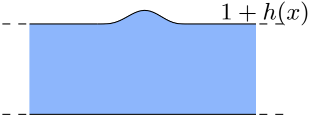

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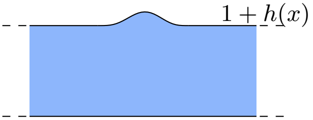

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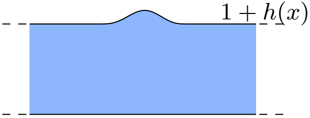

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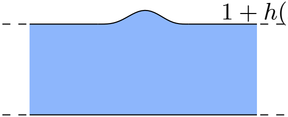

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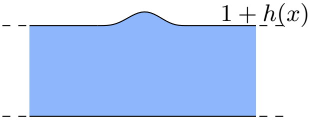

Our goal: to find $h \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $R(h) = 0$ (with $h \not\equiv 0$).

- Taylor: $R(\varepsilon\mu) = \varepsilon dR(0)(\mu) + \varepsilon^2 \tilde{R}^\varepsilon(\mu)$.

Assume that $dR(0) : \mathscr{C}_0^\infty(\mathbb{R}) \to \mathbb{C}$ is onto.

$\exists \mu_0, \mu_1, \mu_2 \in \mathscr{C}_0^\infty(\mathbb{R})$ s.t. $dR(0)(\mu_0) = 0$, $dR(0)(\mu_1) = 1$ and $dR(0)(\mu_2) = i$.

- Take $\mu = \mu_0 + \tau_1\mu_1 + \tau_2\mu_2$ where the τ_n are real parameters to set:

$$0 = R(\varepsilon\mu) \quad \Leftrightarrow \quad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})} \quad \text{where} \quad \left| \begin{array}{l} \vec{\tau} = (\tau_1, \tau_2)^\top \\ G^\varepsilon(\vec{\tau}) = -\varepsilon(\Re e\, \tilde{R}^\varepsilon(\mu), \Im m\, \tilde{R}^\varepsilon(\mu))^\top. \end{array} \right.$$

If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, set $R = R(h) \in \mathbb{C}$.

  > Note that $R(0) = 0$
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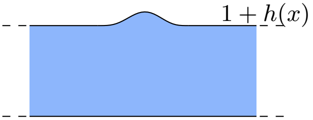

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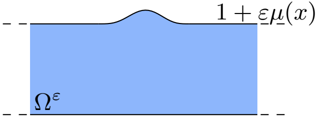
If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.
Set $h^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $R(h^{\mathrm{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential



- Using classical results of asymptotic analysis, we obtain

$$R(\varepsilon\mu) = 0 + \varepsilon\left(-\frac{1}{2}\int_{-\ell}^{\ell} \partial_x\mu(x)(w^+(x,1))^2\,dx\right) + O(\varepsilon^2).$$

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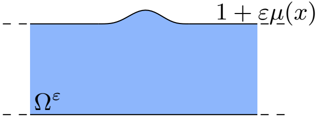

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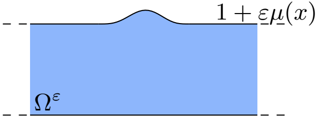

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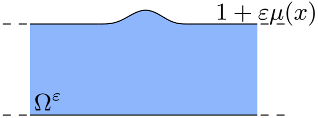

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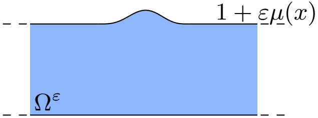

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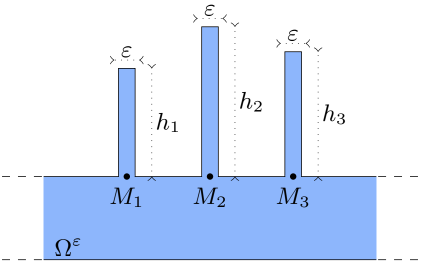
$$T(\varepsilon\mu) - 1 = 0 + \varepsilon\,\boxed{0} + O(\varepsilon^2).$$

$dT(0)$ is not onto $\Rightarrow$ the approach fails to impose $T = 1$.

A perturbative method to get $T = 1$

- We study the same problem in the geometry Ω^ε



- We obtain $$\begin{aligned} R &= 0 + \varepsilon \left(ik \textstyle\sum_{n=1}^{3} (w^+(M_n))^2 \tan(kh_n) \right) + O(\varepsilon^2) \\ T &= 1 + \varepsilon \left(i/2 \textstyle\sum_{n=1}^{3} \tan(kh_n) \right) + O(\varepsilon^2) \end{aligned}$$

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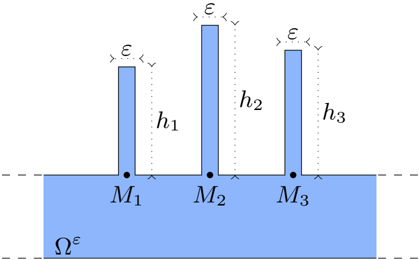

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1) We can find M_n, h_n such that $R = O(\varepsilon^2)$ *and* $T = 1 + O(\varepsilon^2)$.

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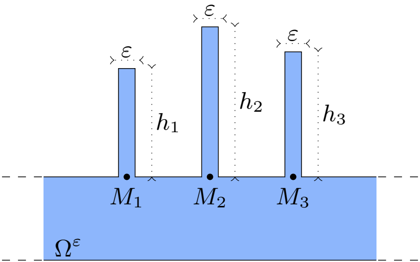

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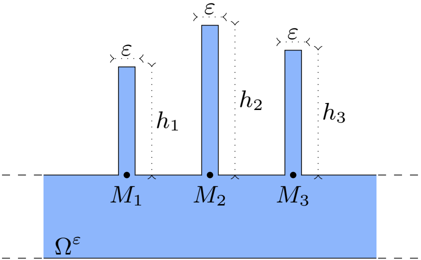

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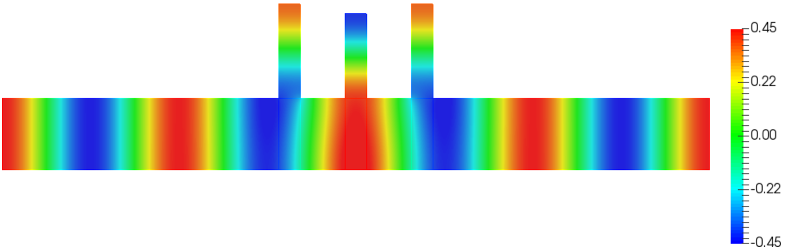
3) Energy conservation + $[T = 1 + O(\varepsilon)]$ $\Rightarrow$ $T = 1$.

Numerical results

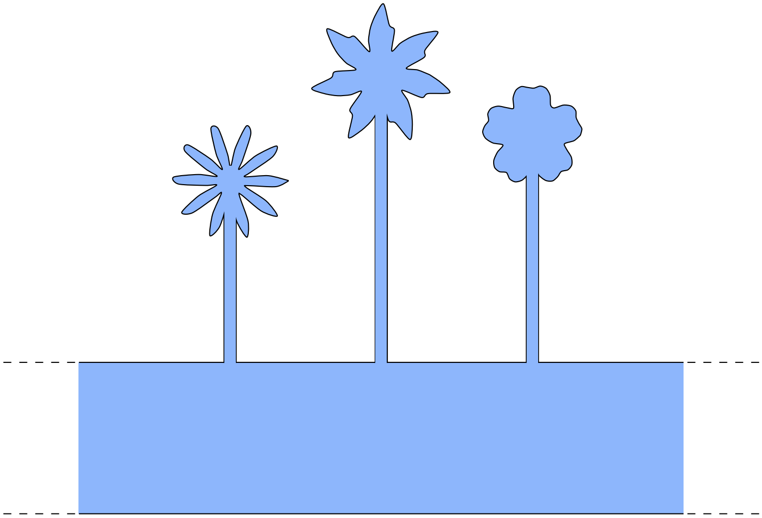
- Perturbed waveguide ($\Re e\,(v(x,y)e^{-i\omega t})$)



- Reference waveguide ($\Re e\,(v_i(x,y)e^{-i\omega t})$)

Remark

- We could also have worked with gardens of flowers!



Outline of the talk

1 First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2 Second constructive method

k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

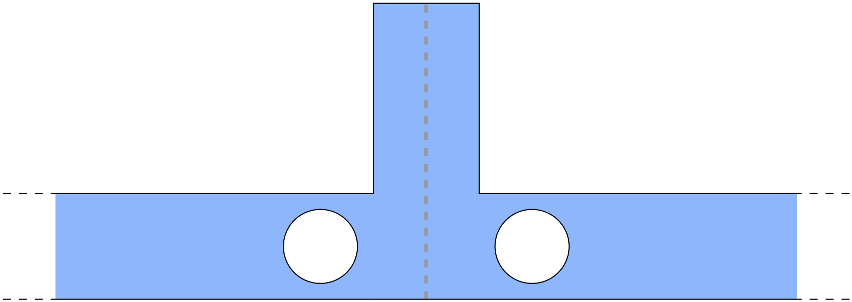
3 A spe

For a gi ... k solving a spectral problem.

First approach was perturbative.
How to get large invisible defects ?

Geometrical setting

- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



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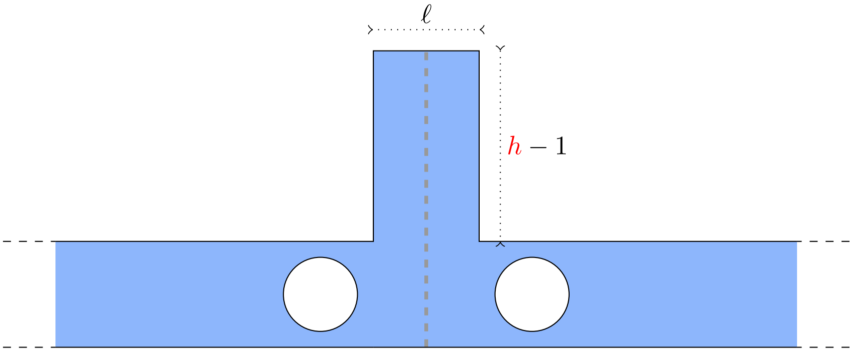

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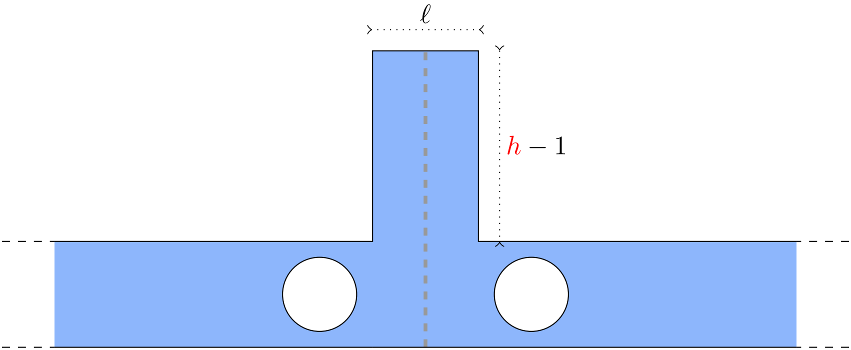

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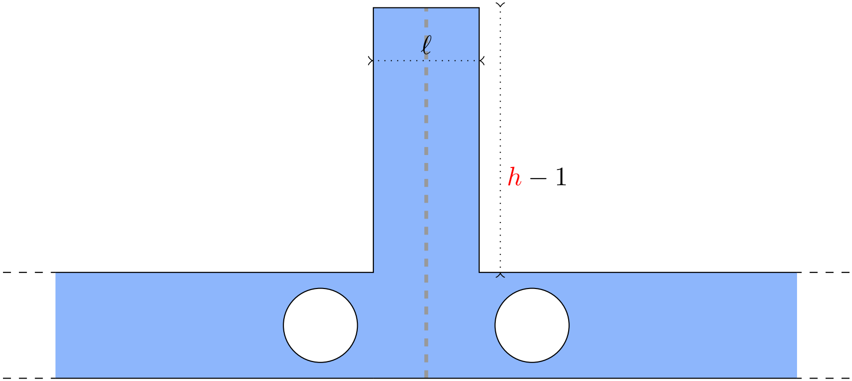

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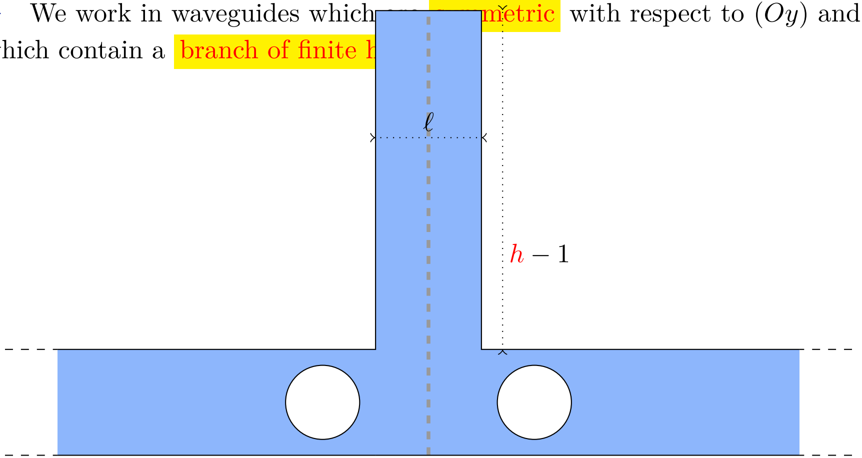

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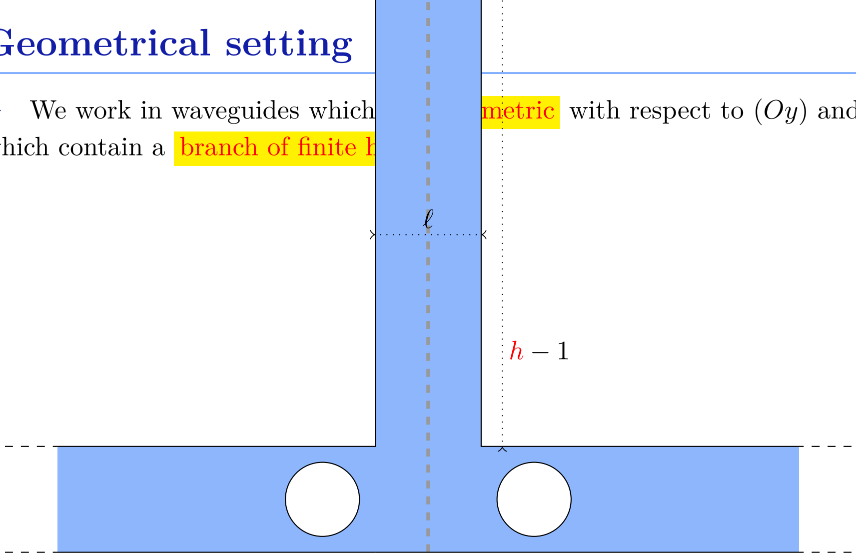

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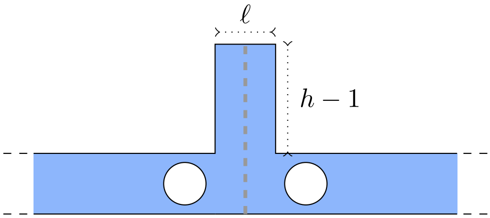
Half-waveguide problems

- Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



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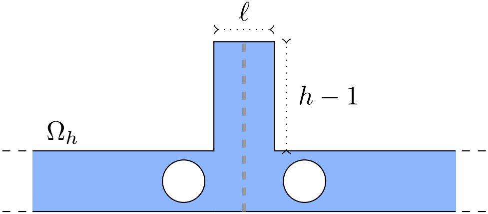

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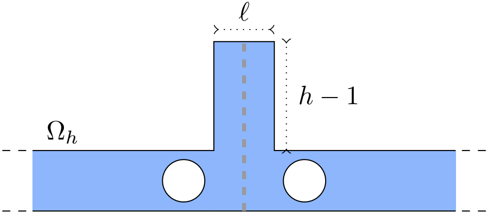

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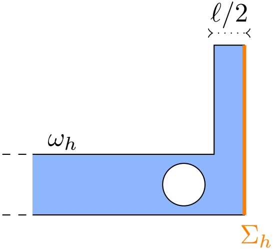
- Introduce the two half-waveguide problems



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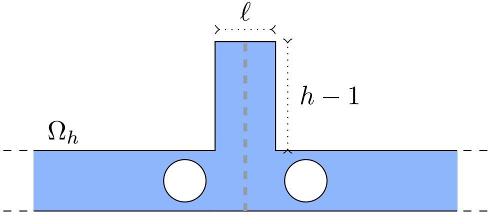

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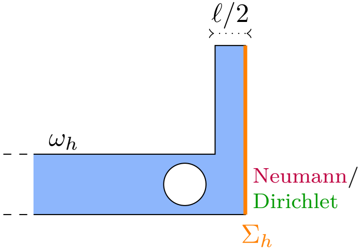

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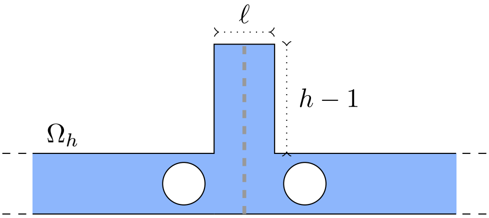

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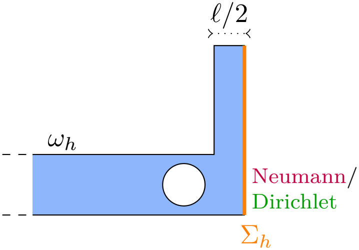

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Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$
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u &= w^+ + R^N w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h)\\
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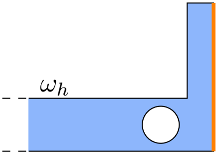

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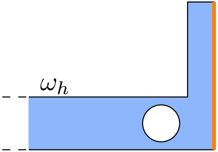

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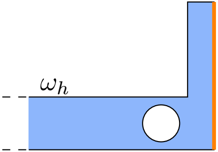

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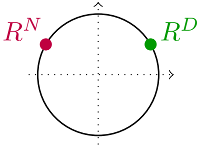

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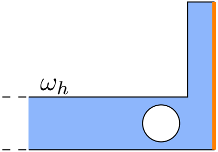

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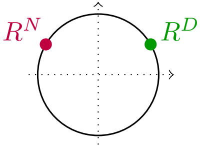

- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

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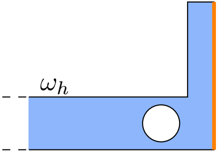

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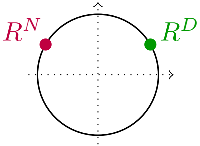

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- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

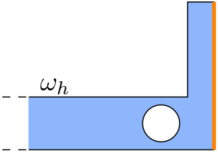
we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$



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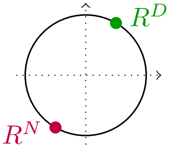

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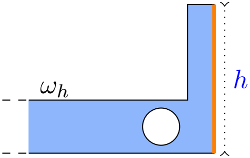

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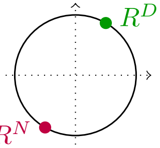

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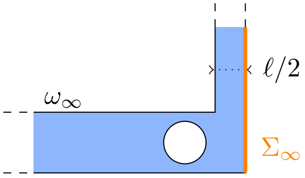
Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

$\rightarrow$ Now, we study the behaviour of $R^N = R^N(h)$, $R^D = R^D(h)$ as $h \rightarrow +\infty$.

Asymptotics of R^N, R^D

Depends on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

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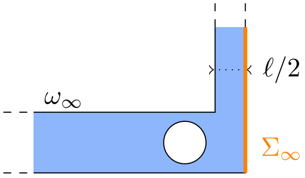

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- Analysis for R^D

For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$

$\Rightarrow h \mapsto R^D(h)$ tends to a constant on $\mathscr{C} := \{z \in \mathbb{C},\ |z| = 1\}$.

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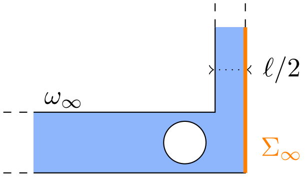

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- Analysis for R^N

For $\boldsymbol{\ell \in (0;2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$\Rightarrow h \mapsto R^N(h)$ runs continuously and almost periodically on $\mathscr{C}$.

Conclusions for $\ell \in (0; \pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

PROPOSITION: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

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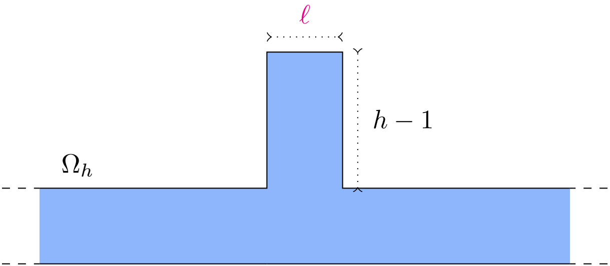
PROPOSITION: There is an unbounded sequence $(\mathcal{H}_n)$ such that for $h = \mathcal{H}_n$, $R^N = R^D$ and so $T = 0$ (complete reflectivity).

- Sequences (h_n) and $(\mathcal{H}_n)$ are almost periodic. As $n \to +\infty$, we have

$$h_{n+1} - h_n = \pi/k + \dots \qquad \text{and} \qquad \mathcal{H}_{n+1} - \mathcal{H}_n = \pi/k + \dots.$$

Setting

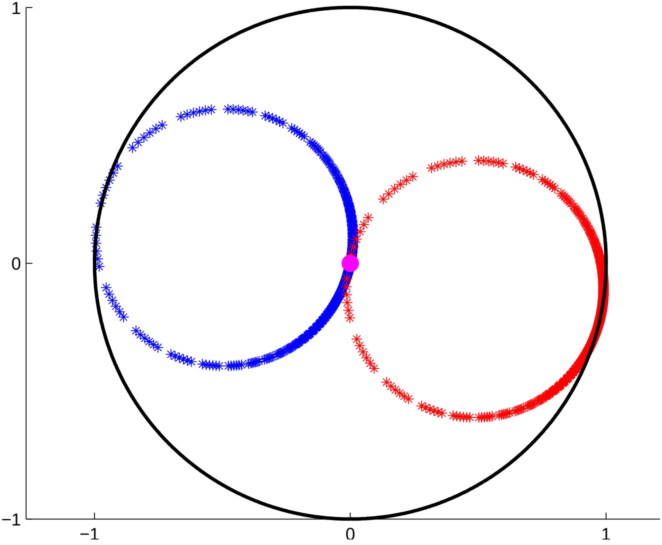
- We compute numerically R, T for $h \in (2;10)$ in the geometry Ω_h



- We use a P2 finite element method with Dirichlet-to-Neumann maps.
- We set $k = 0.8\pi$ and $\ell = 1 \in (0;\pi/k)$.

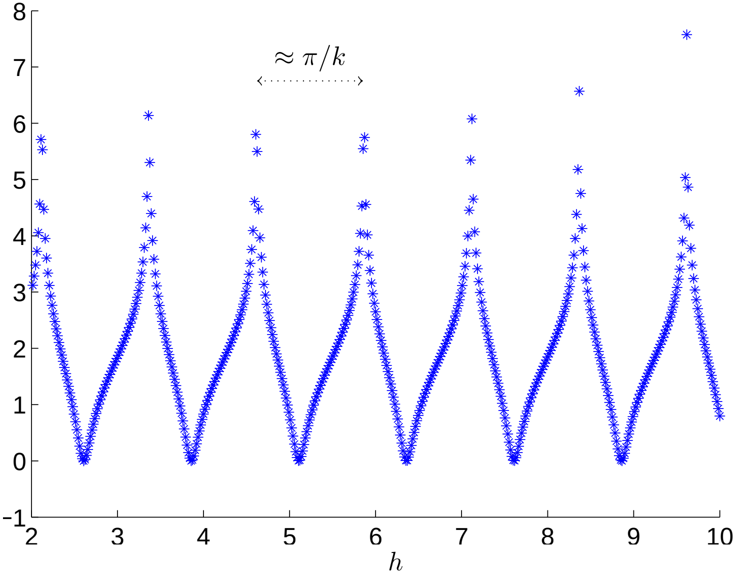
Numerical results

- Reflection coefficient R and transmission coefficient T for $h \in (2; 10)$.



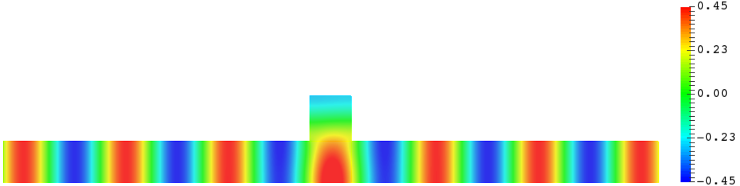
Non reflectivity

- Curve $h \mapsto -\ln|R|$. Peaks correspond to non reflectivity.

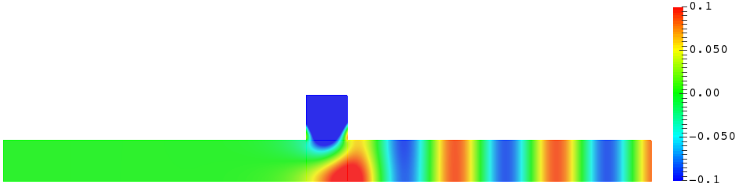


Non reflectivity

- Total field v for h such that $R = 0$.



- Scattered field v_s.



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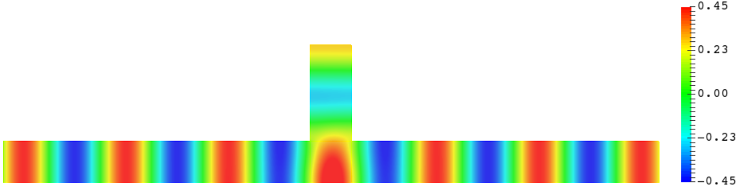

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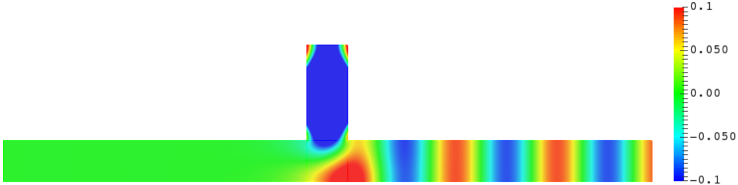

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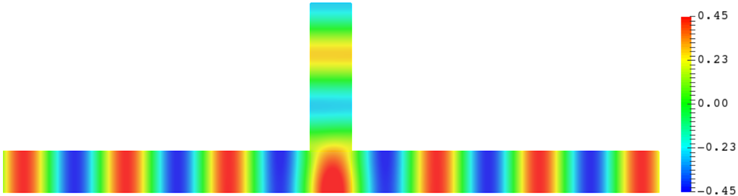

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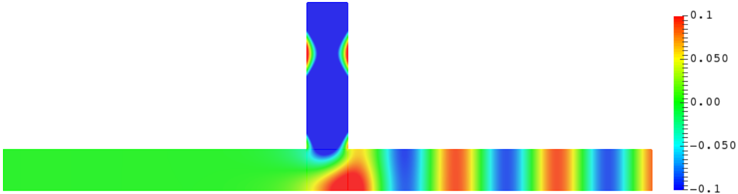
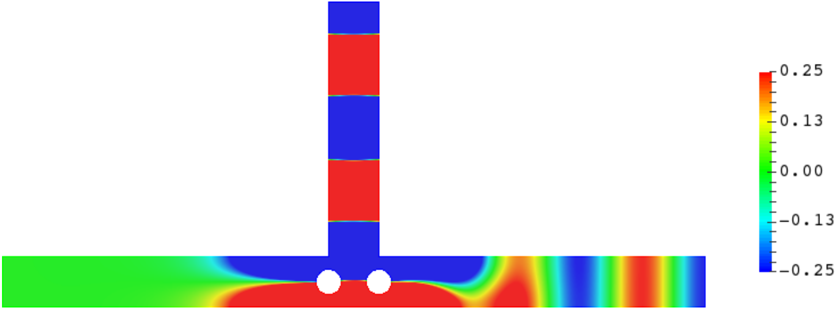


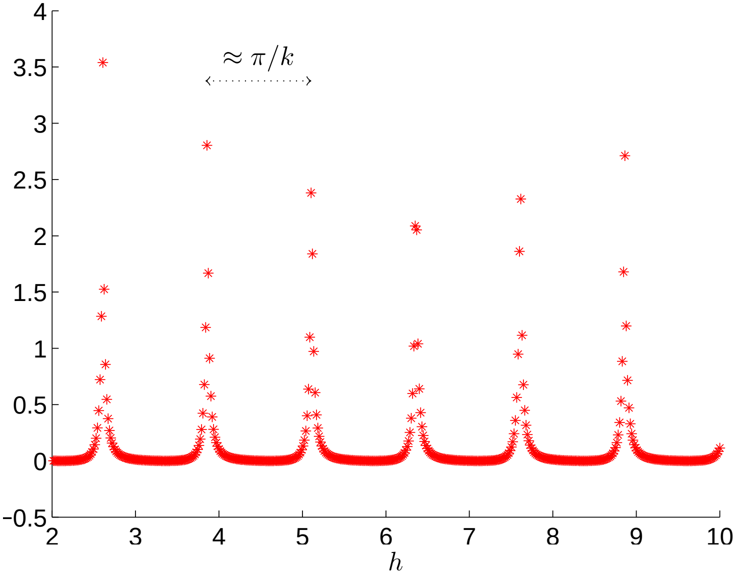
Other non reflecting geometry

- Scattered field v_s.



Complete reflectivity

- Curve $h \mapsto -\ln|T|$. Peaks correspond to complete reflectivity.



Complete reflectivity

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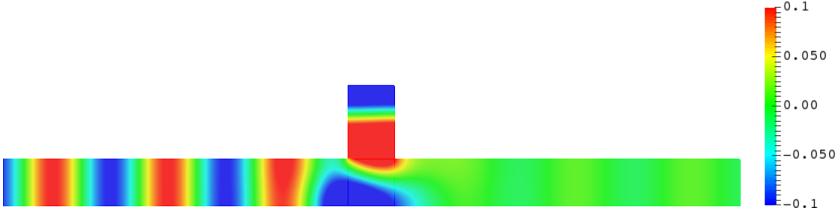

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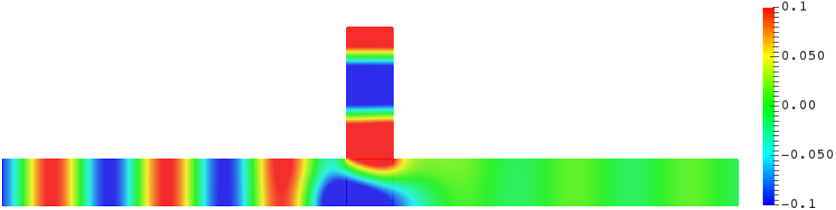

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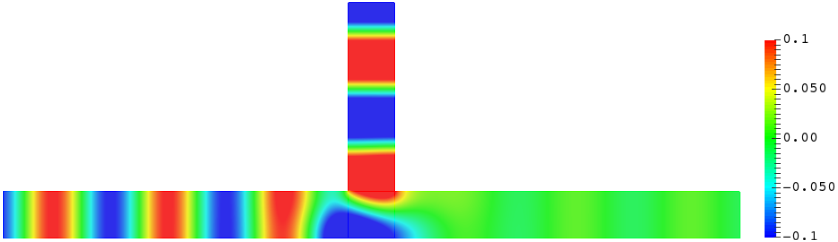
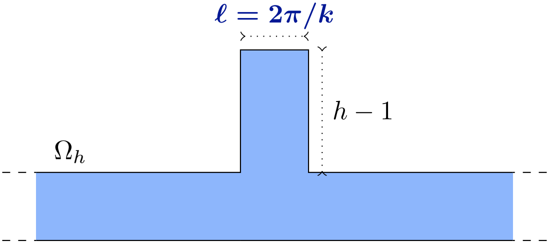


The special case $\ell = 2\pi/k$ - perfect invisibility

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



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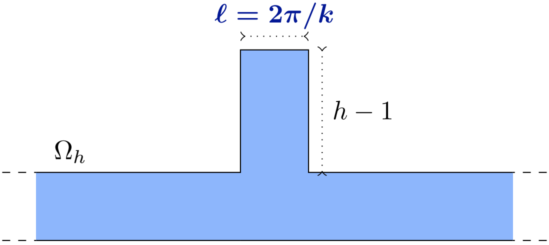

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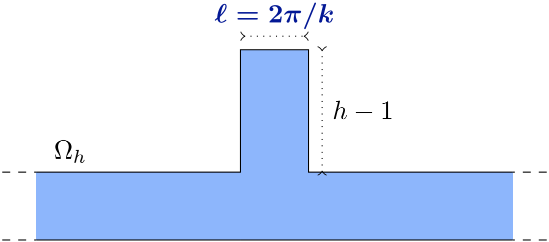

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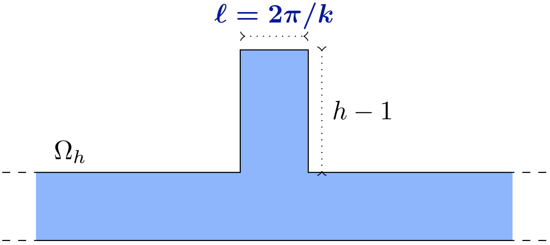

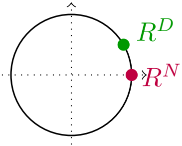

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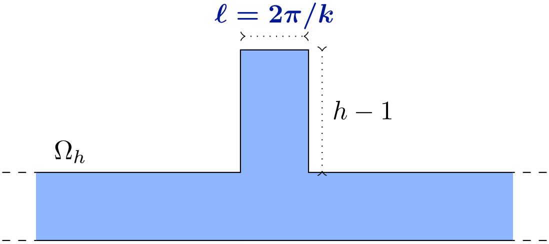

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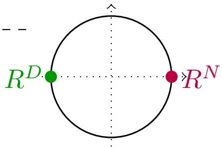

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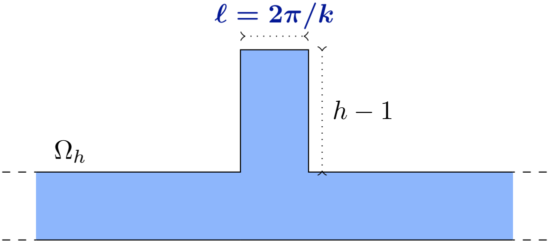

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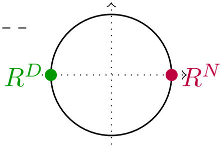

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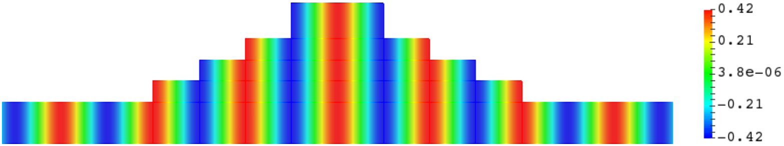
There is a sequence (h_n) such that $T = 1$ (perfect invisibility)

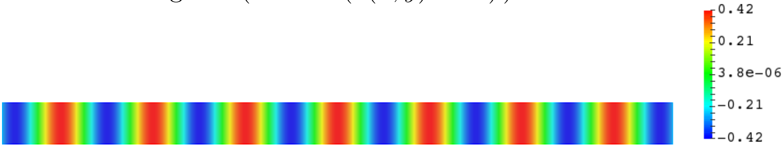
The special case $\ell = 2\pi/k$ - perfect invisibility

- Works also in the geometry below (h is the height of the central branch).

- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



- Reference waveguide ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



Outline of the talk

1. First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

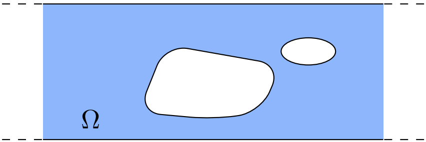
k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3. A spectral approach to determine non reflecting wavenumbers

For a given geometry, we explain how to find non reflecting k solving a spectral problem.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} \Delta v + k^2 v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \\ & & v_s \text{ is outgoing.} & \end{array}$$

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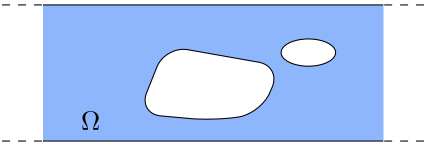

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- For this problem, the modes are

$$\begin{array}{l|l} \text{Propagating} & w_n^\pm(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!] \\ \text{Evanescent} & w_n^\pm(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N. \end{array}$$

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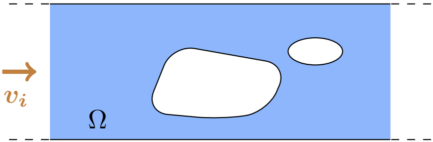

Find $v = v_i + v_s$ s. t.

$$\left|\begin{array}{rcll} \Delta v + k^2 v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \\ & & \multicolumn{2}{l}{v_s \text{ is outgoing.}} \end{array}\right.$$

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- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

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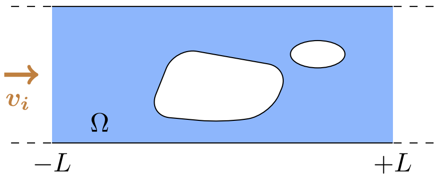

Find $v = v_i + v_s$ s. t.

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Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y)$, $\beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

- Set $\boldsymbol{v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+}$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

- v_s is outgoing $\Leftrightarrow$ $\boxed{v_s = \sum_{n=0}^{+\infty} \gamma_n^{\pm} w_n^{\pm} \quad \text{for } \pm x \geq L, \text{ with } (\gamma_n^{\pm}) \in \mathbb{C}^{\mathbb{N}}.}$

Goal of the section

DEFINITION: v is a non reflecting mode if v_s is expo. decaying for $x \leq -L$
$\Leftrightarrow \quad \gamma_n^- = 0,\ n \in [\![0, N-1]\!] \quad \Leftrightarrow$ energy is completely transmitted.

GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

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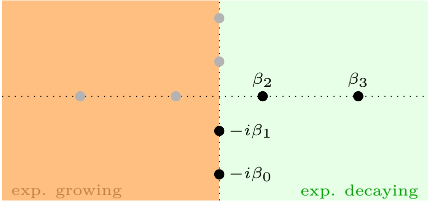
GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

$\rightarrow$ Note that non reflection occurs for **particular** $\boldsymbol{v_i}$ to be computed.

Classical complex scaling to compute v_s 1/2

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm}\, e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm}\, e^{\mp\beta_n x} \cos(n\pi y), \ \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

Classical complex scaling to compute v_s 1/2

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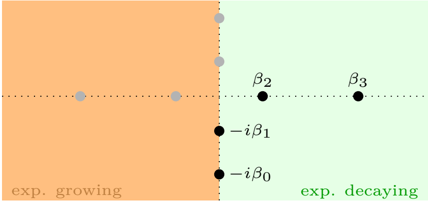

`Modal exponents for` v_s `(`$x \leq -L$`)`

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{I}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\, e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\, e^{i\theta} & \text{for } x \geq L. \end{array} \right.$$

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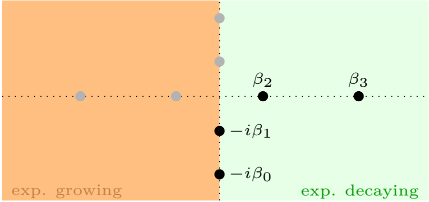

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- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y)$.

1) $v_\theta = v_s$ for $|x| < L$.

2) v_θ is exp. decaying at infinity.

Classical complex scaling to compute v_s

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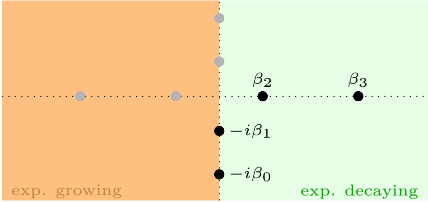

Modal exponents for v_s ($x \leq -L$)

$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^{\pm}\, e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^{\pm}\, e^{\mp \tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y)$.

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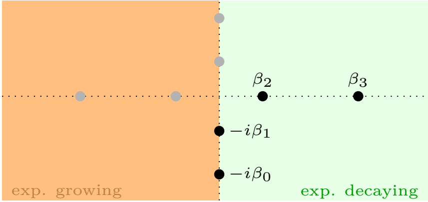
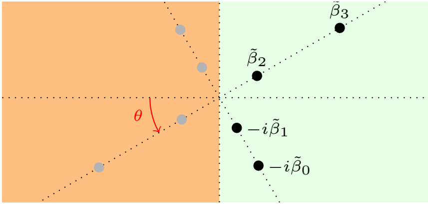


Modal exponents for v_s $(x \leq -L)$



Modal exponents for v_θ $(x \leq -L)$

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1) $v_\theta = v_s$ for $|x| < L$.
2) v_θ is exp. decaying at infinity.

Classical complex scaling to compute v_s 2/2

- v_θ solves $\boxed{(*) \left| \begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array} \right.}$

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$\alpha_\theta(x) = 1$ for $|x| < L$ $\qquad \alpha_\theta(x) = e^{-i\theta}$ for $|x| \geq L$

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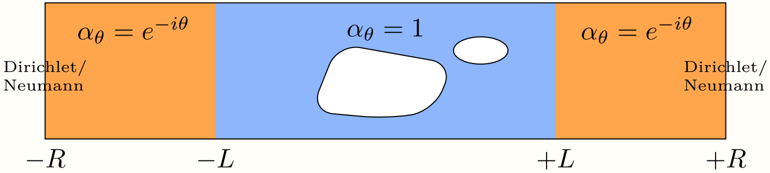
- Numerically we solve $(*)$ in the truncated domain



$\Rightarrow$ We obtain a good approximation of v_s for $|x| < L$.

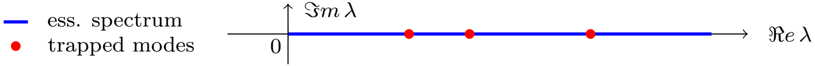
- This is the method of Perfectly Matched Layers (PMLs).

Spectral analysis

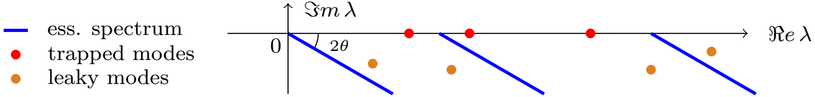
- Define the operators A, A_θ of $\mathrm{L}^2(\Omega)$ such that

$$Av = -\Delta v, \qquad A_\theta v = -\left(\alpha_\theta \frac{\partial}{\partial x}\left(\alpha_\theta \frac{\partial v}{\partial x}\right) + \frac{\partial^2 v}{\partial y^2}\right) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\mathrm{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.

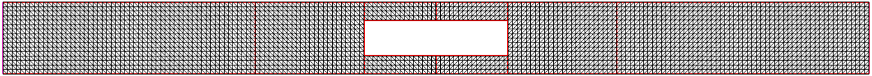


- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
- $\sigma_{\mathrm{ess}}(A_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\}$.
- real eigenvalues of A_θ = real eigenvalues of A.



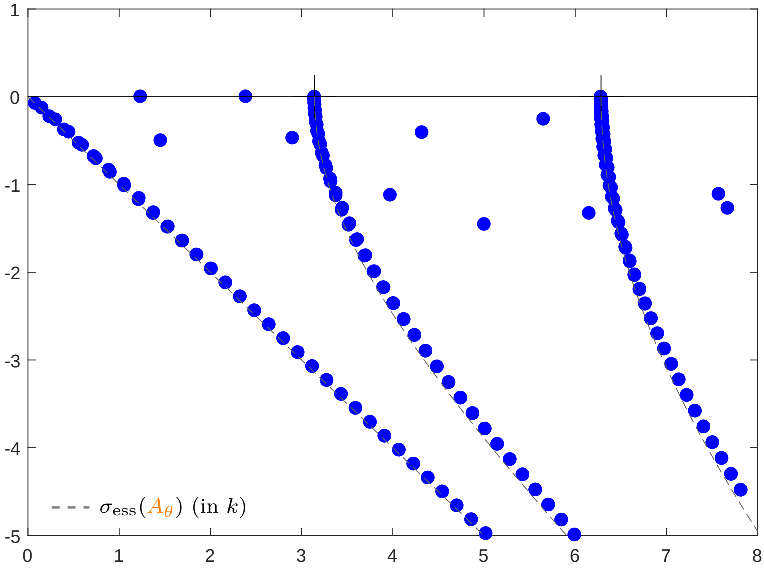
Numerical results

- We work in the geometry



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



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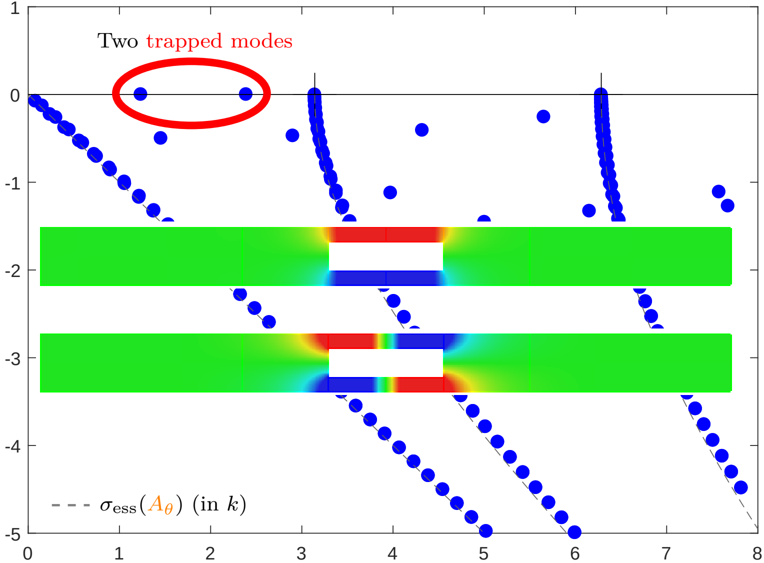

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: **general** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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Important remark: **non reflecting** v decompose as

$$v = v_i + \cancel{\sum_{n=0}^{N-1} \gamma_n^- \, w_n^-} + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \le -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \ge L.$$

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IMPORTANT REMARK: **non reflecting** v decompose as

$$v = \sum_{\boldsymbol{n=0}}^{\boldsymbol{N-1}} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

- In other words, **non reflecting** v are

ingoing at $-\infty$ and outgoing at $+\infty$.

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- In other words, **non reflecting** v are

ingoing at $-\infty$ and outgoing at $+\infty$.

Let us **change the sign** of the complex scaling at $-\infty$!

A new complex spectrum for non reflecting v

- For $\theta \in (0;\pi/2)$, consider the complex change of variables

$$\mathcal{J}_\theta(x) = \left| \begin{array}{lll} -L + (x+L)\ e^{-i\theta} & & \text{for } x \leq -L \\ \qquad x & & \text{for } |x| < L \\ +L + (x-L)\ e^{+i\theta} & & \text{for } x \geq L. \end{array} \right.$$

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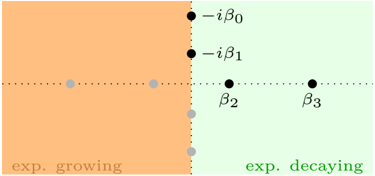
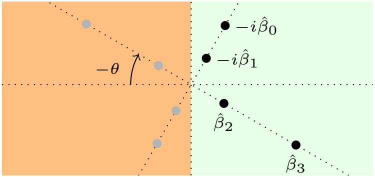
- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

1) $u_\theta = v$ for $|x| < L$.

2) u_θ is exp. decaying at infinity.



Modal exponents for v $(x \le -L)$



Modal exponents for u_θ $(x \le -L)$

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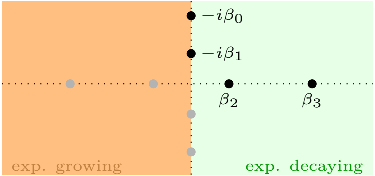

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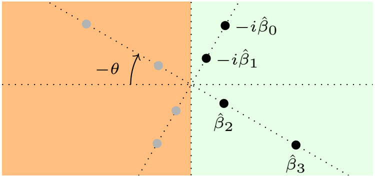

Modal exponents for u_θ $(x \le -L)$

- u_θ solves

$$(*) \left| \begin{array}{rcl} \beta_\theta \dfrac{\partial}{\partial x}\left(\beta_\theta \dfrac{\partial u_\theta}{\partial x}\right) + \dfrac{\partial^2 u_\theta}{\partial y^2} + k^2 u_\theta & = 0 & \text{in } \Omega \\ \partial_n u_\theta & = 0 & \text{on } \partial\Omega. \end{array} \right.$$

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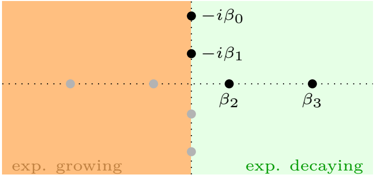
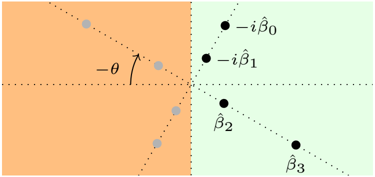


`Modal exponents for` v $(x \le -L)$



`Modal exponents for` u_θ $(x \le -L)$

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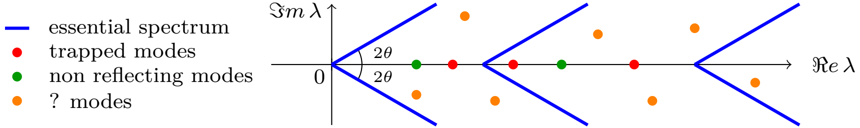
$\beta_\theta(x) = 1$ for $|x| < L$, $\beta_\theta(x) = e^{i\theta}$ for $x \le -L$, $\beta_\theta(x) = e^{-i\theta}$ for $x \ge L$

Spectral analysis

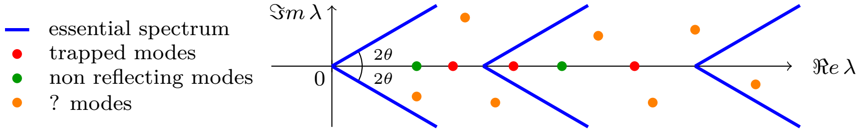
- Define the operator B_θ of $\mathrm{L}^2(\Omega)$ such that

$$B_\theta v = -\Big(\beta_\theta \frac{\partial}{\partial x}\Big(\beta_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- B_θ is not selfadjoint. $\sigma(B_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0, \gamma \in [-2\theta; 2\theta]\}$.
- $\sigma_{\mathrm{ess}}(B_\theta) = \cup_{n\in\mathbb{N}}\{n^2\pi^2 + t\, e^{-2i\theta},\ t \geq 0\} \cup \{n^2\pi^2 + t\, e^{2i\theta},\ t \geq 0\}$.
- real eigenvalues of B_θ = real eigenvalues of A+non reflecting k^2.



Remarks

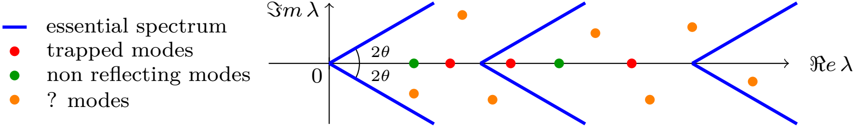


1) ● **? modes** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

Different from **leaky modes** which are exp. growing both at $\pm\infty$ ...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

Remarks



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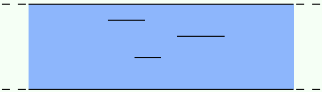
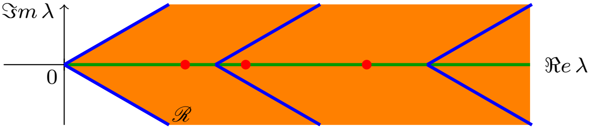


$\rightarrow$ **Not true in general!**

$\boxed{e^{ikx} \circ \mathcal{J}_\theta \text{ is an eigenfunction for all } k \in \mathscr{R}.}$

Remarks

- essential spectrum
- trapped modes
- non reflecting modes
- ? modes



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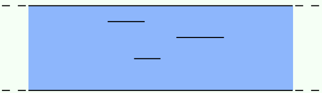

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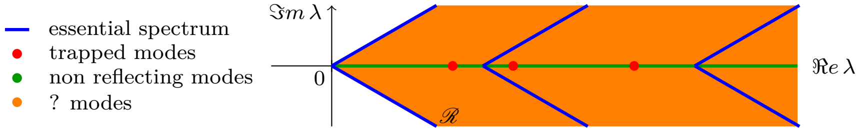

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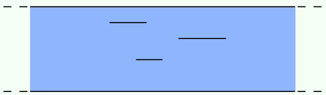
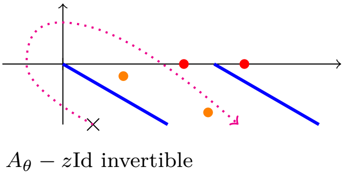


$\rightarrow$ **Not true in general!**

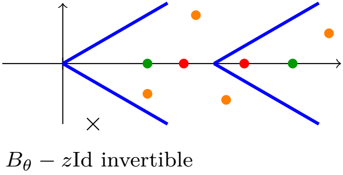
$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

$\rightarrow \mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

Remarks



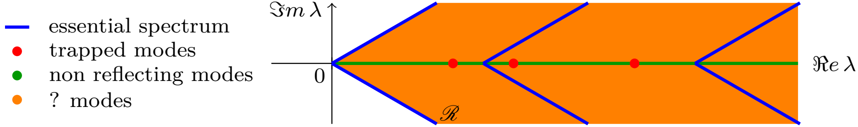
Usual PMLs



Conjugated PMLs

$\rightarrow \mathbb{C} \setminus \sigma_{\mathrm{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

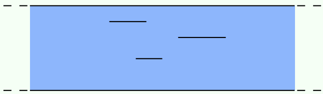
Remarks



1) ● **? modes** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

Different from **leaky modes** which are exp. growing both at $\pm\infty$ ...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.



$\rightarrow$ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

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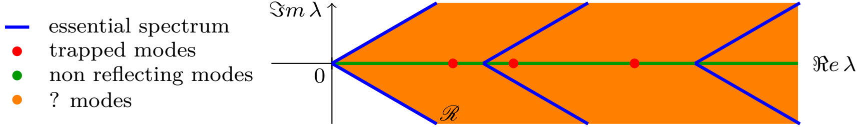

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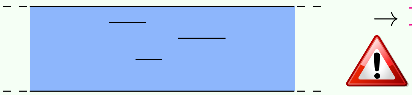


$\rightarrow$ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

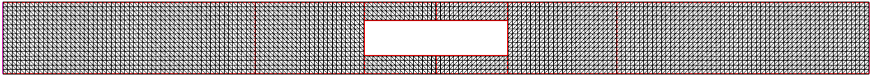
$\rightarrow$ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

$\rightarrow$ A compact perturbation can change drastically the spectrum (B_θ is not selfadjoint).

Numerical consequences?

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

PROP.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y)\in\Omega\}$, B_θ is $\mathcal{PT}$ symmetric:

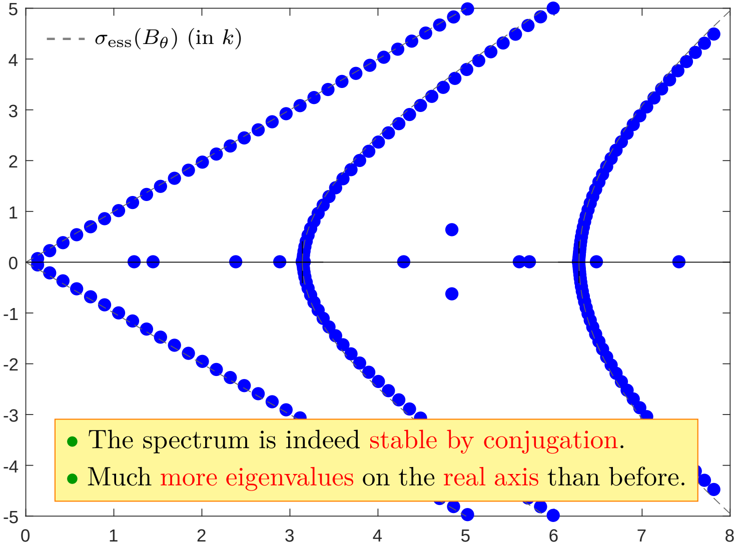
$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If λ is an "isolated" eigenvalue located close to the real axis, then $\lambda\in\mathbb{R}$!

Numerical results

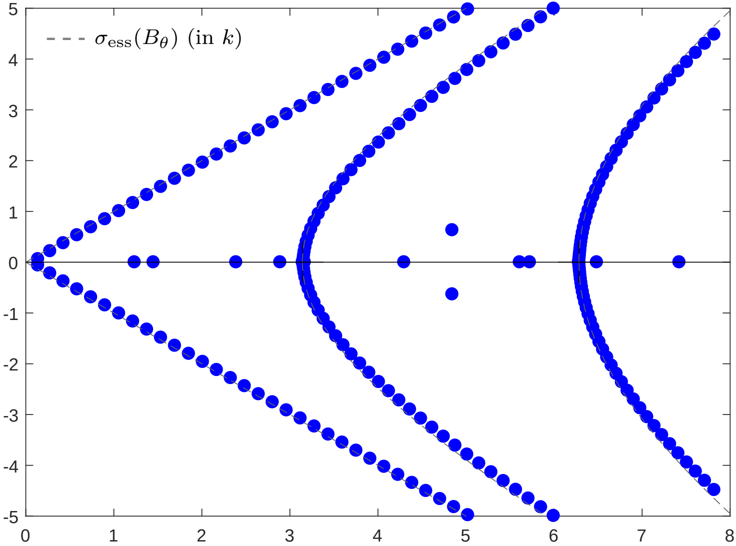
- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



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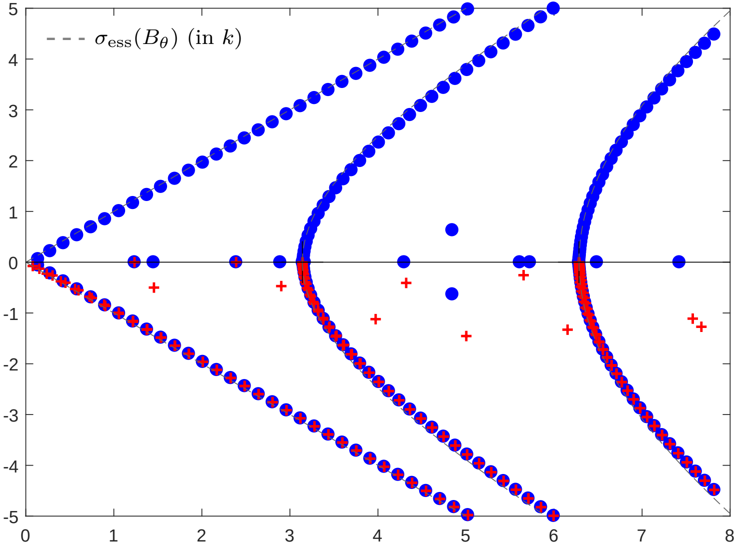
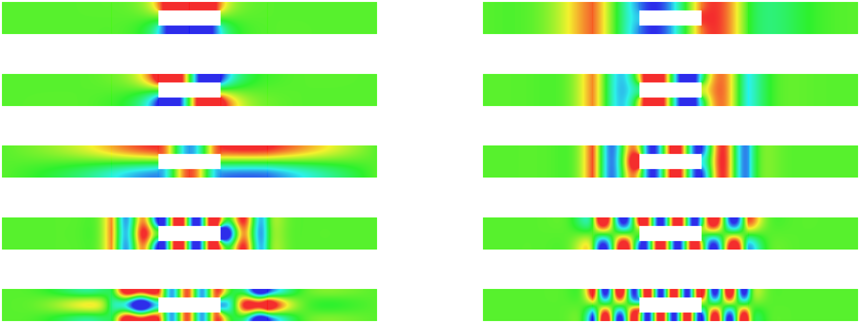


● PMLs with different signs ✚ Classical PMLs

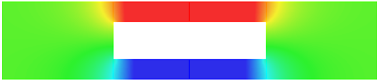
Numerical results

- We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



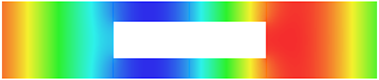
Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

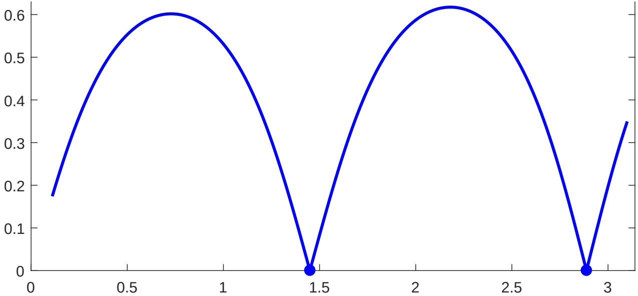


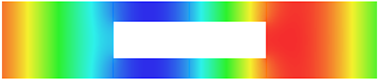
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





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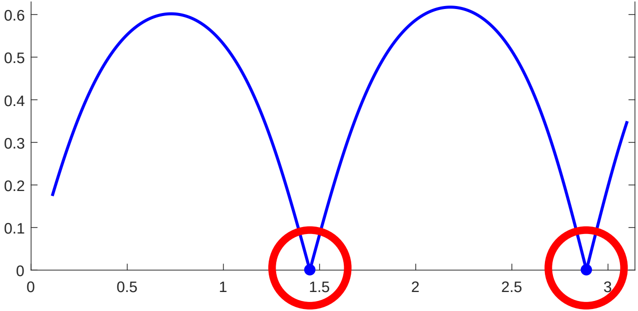
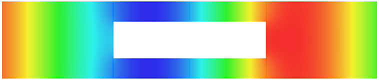




First non reflecting mode

$k = 1.4513...$

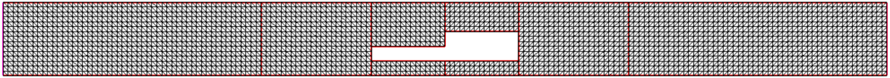
Second non reflecting mode

$k = 2.8896...$

There is perfect agreement!

Numerical results

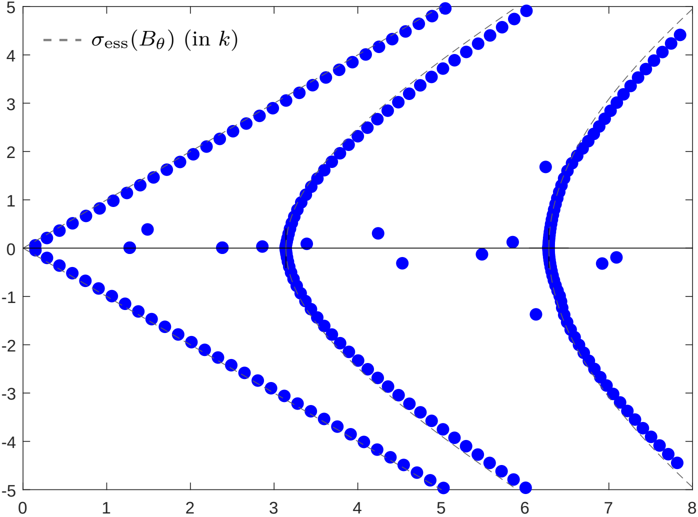
- Now the geometry is not symmetric in x nor in y:



- The operator B_θ is no longer $\mathcal{PT}$-symmetric and we expect:
  - No trapped modes
  - No invariance of the spectrum by complex conjugation.

Numerical results

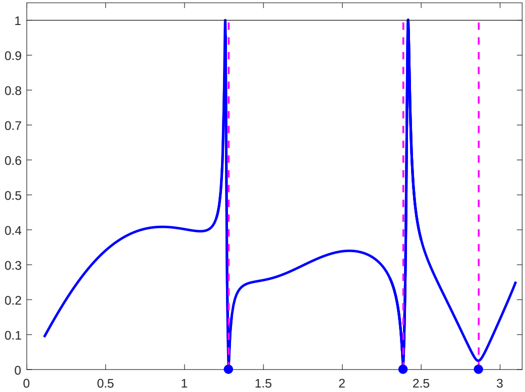
- Discretized spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is not symmetric w.r.t. the real axis.

Numerical results

- We compute $\boxed{k \mapsto |R(k)|}$ for $0 < k < \pi$.



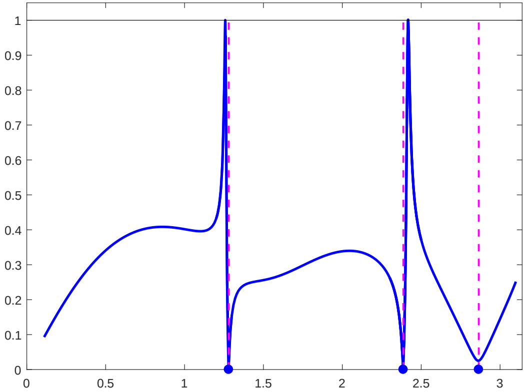
$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

Numerical results

- We compute $\boxed{k \mapsto |R(k)|}$ for $0 < k < \pi$.



$k = 1.28 + 0.0003i$

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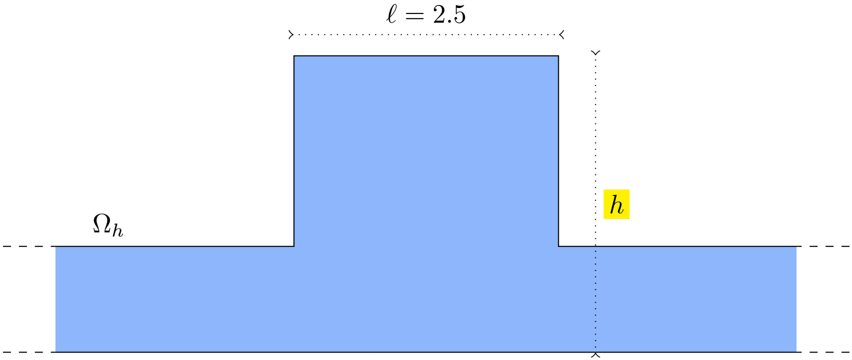
Complex eigenvalues also contain information on almost no reflection.

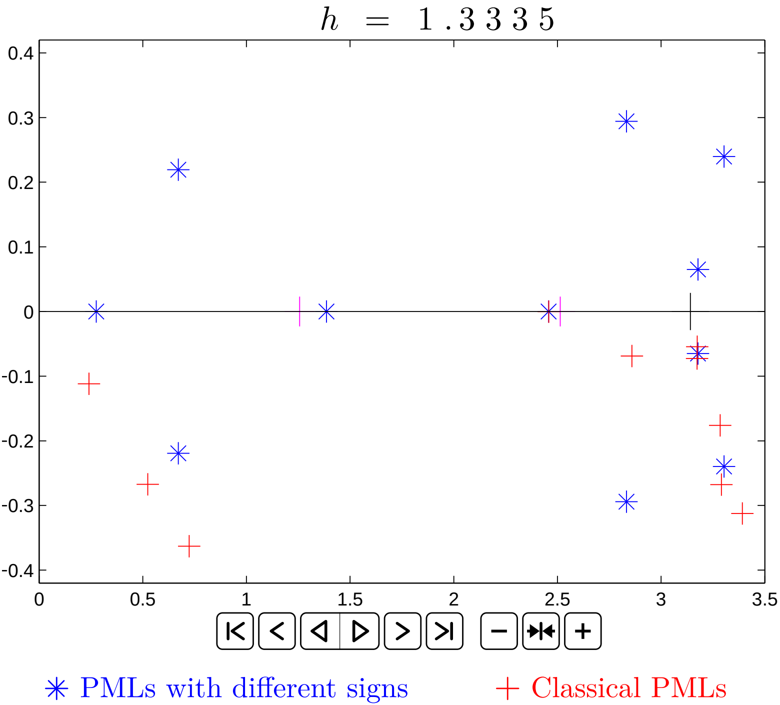
Spectra for a changing geometry

- Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.



- The magenta marks on the real axis correspond to $k = \pi/\ell$ & $k = 2\pi/\ell$. For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain h.

- We zoom at the region $0 < \Re e\, k < \pi$.



$h = 1.3335$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
PMLs with different signs
Classical PMLs

Outline of the talk

1. First constructive method

k is given, we use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

2. Second constructive method

k is given, we use an approach based on symmetries to construct geometries such that $R = 0$, $T = 1$ or $T = 0$ and even a bit more...

3. A spectral approach to determine non reflecting wavenumbers

For a given geometry, we explain how to find non reflecting k solving a spectral problem.

Conclusion

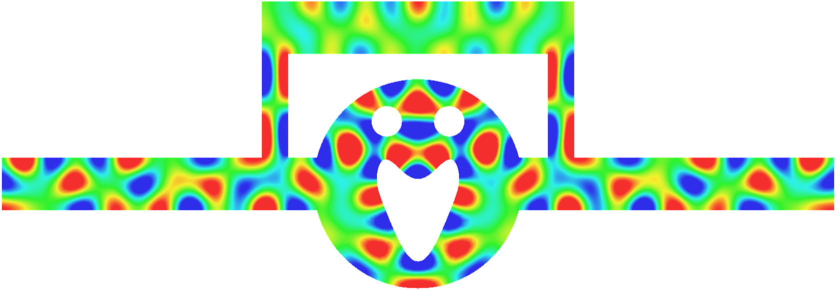
What we did

- ♠ We presented two methods to construct geometries such that $R = 0$, $T = 0$, $T = 1$ at a given frequency $k \in (0; \pi)$.
- ♠ We proposed a spectral approach to compute non reflecting k ($R = 0$) for a given geometry.

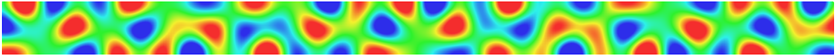
Future work

1) How to construct invisible or completely reflecting defects for a given $k > \pi$ (several propagating modes)?
2) Can we find a spectral approach to compute completely reflecting or completely invisible k for a given geometry?
3) Can we prove existence of non reflecting k for the $\mathcal{PT}$-symmetric pb?
4) Can we work in free space with a finite number of directions? on other equations (electromagnetism, elasticity,...)?

v



v_i



Thank you for your attention!