Spectrum for a small inclusion of negative material

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Introduction: general setting

- Scattering by a negative material in electromagnetism in time-harmonic regime (at a given frequency):

Positive material
\[ \varepsilon > 0 \]
and \[ \mu > 0 \]

Negative material
\[ \varepsilon < 0 \]
and/or \[ \mu < 0 \]

Do such negative materials occur in practice?

- For metals at optical frequencies, \[ \varepsilon < 0 \] and \[ \mu > 0 \].
- Recently, artificial metamaterials have been realized which can be modelled (at some frequency of interest) by \[ \varepsilon < 0 \] and \[ \mu < 0 \].

Zoom on a metamaterial: practical realizations of metamaterials are achieved by a periodic assembly of small resonators.

Example of metamaterial (NASA).

Mathematical justification of the homogenized model (Bouchitté, Bourel, Felbacq 09).
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Introduction: in this talk

In this talk, we investigate a Dirichlet spectral problem for a small inclusion of negative material in a bounded domain.

Let $\Omega, \omega$ be smooth domains of $\mathbb{R}^3$ such that $O \in \omega$, $\omega \subset \Omega$. For $\delta \in (0; 1]$, we consider the problem

Find $(\lambda^\delta, u^\delta) \in \mathbb{C} \times (H^1_0(\Omega) \setminus \{0\})$ s.t.:

$$-\text{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{in } \Omega,$$

with

\begin{align*}
\Omega^\delta_1 &> \cdots > \Omega^\delta_2
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In this talk, we investigate a Dirichlet spectral problem for a small inclusion of negative material in a bounded domain.

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-\text{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{in } \Omega, \text{ with,}
\]

- $H^1_0(\Omega) := \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \partial \Omega \}$
- $\sigma^\delta =
\begin{align*}
\sigma_1 &> 0 \quad \text{in } \Omega_1^\delta := \Omega \setminus \overline{\delta \omega} \\
\sigma_2 &< 0 \quad \text{in } \Omega_2^\delta := \delta \omega.
\end{align*}$
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- $H^1_0(\Omega) := \{ u \in H^1(\Omega) \mid u = 0 \text{ on } \partial \Omega \}$
- $\sigma^\delta = \begin{cases} \sigma_1 > 0 & \text{in } \Omega^\delta_1 := \Omega \setminus \delta \omega \\ \sigma_2 < 0 & \text{in } \Omega^\delta_2 := \delta \omega. \end{cases}$

This problem is not classical because $\sigma^\delta$ changes sign.
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This problem is not classical because $\sigma^\delta$ changes sign.

- We define the operator $A^\delta : D(A^\delta) \to L^2(\Omega)$ such that

\[
\begin{align*}
D(A^\delta) &= \{u \in H^1_0(\Omega) | \text{div}(\sigma^\delta \nabla u) \in L^2(\Omega)\} \\
A^\delta u &= -\text{div}(\sigma^\delta \nabla u).
\end{align*}\]
Introduction: main question of the talk

Using boundary integral equations (see Costabel and Stephan 85, Dauge and Texier 97) or the T-coercivity approach (see Bonnet-Ben Dhia et al. 99,10,12,13), we can prove the:

**Proposition.** Assume that $\sigma_2/\sigma_1 \neq -1$. For $\delta > 0$, the operator $A^\delta$ is selfadjoint and has compact resolvent. Its spectrum $\mathcal{S}(A^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \leftarrow \ldots \lambda_{-n}^\delta \leq \ldots \leq \lambda_{-1}^\delta < 0 \leq \lambda_1^\delta \leq \lambda_2^\delta \leq \ldots \leq \lambda_n^\delta \ldots \rightarrow \infty.$$
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For all $\delta \in (0;1]$, $A^\delta$ has negative spectrum. At the limit $\delta = 0$, the inclusion of negative material vanishes and $\sigma^0$ is strictly positive.
\end{proof}
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- Using boundary integral equations (see Costabel and Stephan 85, Dauge and Texier 97) or the T-coercivity approach (see Bonnet-Ben Dhia et al. 99,10,12,13), we can prove the:

**PROPOSITION.** Assume that $\sigma_2/\sigma_1 \neq -1$. For $\delta > 0$, the operator $A^\delta$ is selfadjoint and has compact resolvent. Its spectrum $\mathcal{S}(A^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \xrightarrow{n \to +\infty} \ldots \lambda_{-n}^\delta \leq \ldots \leq \lambda_{-1}^\delta < 0 \leq \lambda_1^\delta \leq \lambda_2^\delta \leq \ldots \leq \lambda_n^\delta \ldots \xrightarrow{n \to +\infty} +\infty.$$

- For all $\delta \in (0; 1]$, $A^\delta$ has negative spectrum. At the limit $\delta = 0$, the inclusion of negative material vanishes and $\sigma^0$ is strictly positive.

? What happens to the negative spectrum when $\delta$ tends to zero?
Outline of the talk

1. **Limit operators**
   
   We introduce the two natural limit operators which appear when \( \delta \to 0 \).

2. **Results**
   
   We state the main results concerning the asymptotic behaviour of the eigenvalues when \( \delta \to 0 \).

3. **Numerical experiments**
   
   We illustrate the theoretical results with numerical experiments.
1 Limit operators

2 Results

3 Numerical experiments
Far field operator

- As $\delta \to 0$, the small inclusion of negative material disappears.
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- We introduce the far field operator $A^0$ such that

$$D(A^0) = \{ v \in H_0^1(\Omega) \mid \Delta v \in L^2(\Omega) \}$$

$$A^0 v = -\sigma_1 \Delta v.$$ 

Proposition. There holds $\mathcal{S}(A^0) = \{ \mu_n \}_{n \geq 1}$ with $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_n \cdots \xrightarrow{n \to +\infty} +\infty$. 

Near field operator

- Introduce the rapid coordinate $\xi := \delta^{-1} x$ and let $\delta \to 0$. 

\[ B_\infty w = -\text{div} (\sigma_\infty \nabla w) \]

Proposition. Assume that $\sigma_2 / \sigma_1 \neq -1$. The continuous spectrum of $B_\infty$ is equal to $[0; +\infty)$ while its discrete spectrum is a sequence of eigenvalues:

\[ \mathcal{S}(B_\infty) \setminus \mathbb{R}^+ = \{ \mu - n \} \quad n \geq 1 \]

with $0 > \mu - 1 \geq \cdots \geq \mu - n \ldots \to n \to +\infty - \infty$. 

Near field operator

- Introduce the rapid coordinate $\xi := \delta - 1/x$ and let $\delta \to 0$.

Define the near field operator $B_{\infty}$ such that $D(\sigma_{\infty} \nabla w) \in L^2(R^3)$.

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**Proposition.** Assume that $\sigma_2/\sigma_1 \neq -1$. The continuous spectrum of $B^\infty$ is equal to $[0; +\infty)$ while its discrete spectrum is a sequence of eigenvalues:

$$\mathcal{G}(B^\infty) \setminus \overline{\mathbb{R}^+} = \{ \mu_{-n} \}_{n \geq 1} \quad \text{with} \quad 0 > \mu_{-1} \geq \cdots \geq \mu_{-n} \ldots \quad n \to +\infty \to -\infty.
1 Limit operators

2 Results

3 Numerical experiments
Spectrum for a small inclusion: results

Assume that $\sigma_2/\sigma_1 \neq -1$ and that $B^\infty$ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda_{\pm n}^\delta$, $\mu_n^\delta$, $\mu_{-n}^\delta$ the eigenvalues of $A^\delta$, $A^0$, $B^\infty$ as in the previous slides.

**Theorem. (Positive spectrum)** For all $n \in \mathbb{N}^*$, $\varepsilon \in (0; 1)$, there exist constants $C, \delta_0 > 0$ depending on $n, \varepsilon$ but independent of $\delta$, such that

$$|\lambda_n^\delta - \mu_n| \leq C \delta^{3/2 - \varepsilon}, \quad \forall \delta \in (0; \delta_0].$$
Assume that \( \frac{\sigma_0}{\sigma_1} \neq 1 \) and that \( B^\infty \) is injective. For \( n \in \mathbb{N}^* \), we denote \( \lambda_{\delta n}, \mu_{\delta n}, \mu_{-\delta n} \) the eigenvalues of \( A_{\delta}, A_0, B^\infty \) as in the previous slides.

**Theorem. (Positive spectrum)** For all \( n \in \mathbb{N}^*, \varepsilon \in (0; 1) \), there exist constants \( C, \delta_0 > 0 \) depending on \( n, \varepsilon \) but independent of \( \delta \), such that
\[
|\lambda_{\delta n} - \mu_n| \leq C \delta^{3/2} - \varepsilon, \quad \forall \delta \in (0; \delta_0].
\]

**Idea of the proof:**

1. We prove the *a priori* estimate \( \|u^\delta\|_{H_0^1(\Omega)} \leq c \|A^\delta u^\delta\|_{\Omega} \) for \( \delta \) small enough (♠ hard part of the proof: weighted Sobolev spaces + overlapping cut-off functions + Nazarov’s technique).

2. If \((\mu_n, v_n)\) is an eigenpair of \( A^0 \), we construct \( u \) such that
\[
\|A^\delta u - \mu_n u\|_{\Omega} \leq c \delta^{\beta} \|u\|_{\Omega}, \quad \text{for some } \beta > 0.
\]

3. If \((\lambda_{\delta n}, u_{\delta n})\) is an eigenpair of \( A^\delta \), we construct \( v \) such that
\[
\|A^0 v - \lambda_{\delta n} v\|_{\Omega} \leq c \delta^{\beta} \|v\|_{\Omega}, \quad \text{for some } \beta > 0.
\]

4. We conclude with a classical lemma on quasi eigenvalues.
Spectrum for a small inclusion: results

Assume that \( \sigma_2 / \sigma_1 \neq -1 \) and that \( B^\infty \) is injective. For \( n \in \mathbb{N}^* \), we denote \( \lambda_\pm^n, \mu^n_\delta, \mu_{-n}^\delta \) the eigenvalues of \( A^\delta, A^0, B^\infty \) as in the previous slides.

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\[
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\]

**Theorem. (Negative spectrum)** For all \( n \in \mathbb{N}^* \), there exist constants \( C, \gamma, \delta_0 > 0 \), depending on \( n \) but independent of \( \delta \), such that

\[
|\lambda_\delta^n - \mu_{-n}^-\delta| \leq C \exp(-\gamma/\delta), \quad \forall \delta \in (0; \delta_0].
\]

**Proposition. (Localization effect)** For all \( n \in \mathbb{N}^* \), let \( u_\delta_{-n} \) be an eigenfunction corresponding to the negative eigenvalue \( \lambda_\delta_{-n} \). There exist constants \( C, \gamma, \delta_0 > 0 \), depending on \( n \) but independent of \( \delta \), such that

\[
\int_{\Omega} \left( |u_\delta_{-n}|^2 + |\nabla u_\delta_{-n}|^2 \right) e^{\gamma x/\delta} dx \leq C \|u_\delta_{-n}\|_\Omega, \quad \forall \delta \in (0; \delta_0].
\]
Assume that $\sigma_2/\sigma_1 \neq -1$ and that $B^\infty$ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, $\mu^\delta_n$, $\mu^\delta_{-n}$ the eigenvalues of $A^\delta$, $A^0$, $B^\infty$ as in the previous slides.

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\]

Why is it a \( \delta^{-2} \)?

- If \( (\lambda_\delta, u_\delta) \) is an eigenpair of \( A^\delta \), there holds

\[
\int_\Omega \sigma^\delta \nabla_x u^\delta \cdot \nabla_x v \, dx = \lambda^\delta \int_\Omega u^\delta v \, dx, \quad \forall v \in H^1_0(\Omega).
\]

- \( x = \delta \xi \Rightarrow \nabla_x = \delta^{-1} \nabla_\xi \). Denoting \( U^\delta(\xi) = u^\delta(\delta \xi) \), we deduce

\[
\int_{\delta^{-1}\Omega} \sigma^\infty \nabla_\xi U^\delta \cdot \nabla_\xi V \, d\xi = \delta^2 \lambda^\delta \int_{\delta^{-1}\Omega} U^\delta V \, d\xi, \quad \forall V \in H^1_0(\delta^{-1}\Omega).
\]

Why the convergence is exponential?

- If \( (\mu_\pm, v_\pm) \) is an eigenpair of \( B^\infty \), \( v_\pm \) is exponentially decaying at \( \infty \).
Assume that $\frac{\sigma_2}{\sigma_1} \neq -1$ and that $B^\infty$ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda_{\pm n}^\delta$, $\mu_n^\delta$, $\mu_{-n}^\delta$ the eigenvalues of $A^\delta$, $A^0$, $B^\infty$ as in the previous slides.

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$$|\lambda_{-n}^\delta - \delta^2 \mu_{-n}| \leq C \exp(-\gamma/\delta), \quad \forall \delta \in (0; \delta_0].$$
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Schematically, we have:

\[
\begin{array}{cccc}
\mathcal{S}(A^\delta) & \lambda_{-2}^\delta & \lambda_{-1}^\delta & \lambda_1^\delta & \lambda_2^\delta \\
\times & \times & \times & 0 & \to \\
\mathcal{S}(A^0) & \delta^{-2}\mu_2 & \delta^{-2}\mu_1 & \mu_1 & \mu_2 \\
\times & \times & \times & 0 & \to \\
\delta^{-2}\mathcal{S}(B^{\infty}) \cap (-\infty; 0) & \mathcal{S}(A^0)
\end{array}
\]
Assume that $\sigma_2/\sigma_1 \neq -1$ and that $B^\infty$ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda_{\pm n}^\delta$, $\mu_n^\delta$, $\mu_{-n}^\delta$ the eigenvalues of $A^\delta$, $A^0$, $B^\infty$ as in the previous slides.

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**Proposition. (Localization effect)** For all $n \in \mathbb{N}^*$, let $u_{-n}^\delta$ be an eigenfunction corresponding to the negative eigenvalue $\lambda_{-n}^\delta$. There exist constants $C, \gamma, \delta_0 > 0$, depending on $n$ but independent of $\delta$, such that

$$\int_{\Omega} (|u_{-n}^\delta|^2 + |\nabla u_{-n}^\delta|^2) e^{\gamma x/\delta} \, d\mathbf{x} \leq C \|u_{-n}^\delta\|_{\Omega}, \quad \forall \delta \in (0; \delta_0].$$
1 Limit operators

2 Results

3 Numerical experiments
Numerical experiments for the small inclusion

- Using FreeFem++, we approximate numerically the spectrum of $A^\delta$ using a usual P1 Finite Element Method. We solve the problem

\[
\text{Find } (\lambda_{h}^\delta, u_{h}^\delta) \in \mathbb{C} \times (V_{h} \setminus \{0\}) \text{ s.t.:
}\]

\[
\int_{\Omega} \sigma_{h}^\delta \nabla u_{h}^\delta \cdot \nabla v_{h} = \lambda_{h}^\delta \int_{\Omega} u_{h}^\delta v_{h}, \quad \forall v_{h} \in V_{h},
\]

where $V_{h}$ approximates $H_{0}^{1}(\Omega)$ as $h \to 0$ ($h$ is the mesh size).

- We consider the following 2D geometry:
Numerical experiments for the small inclusion

Using FreeFem++, we approximate numerically the spectrum of \( A^\delta \) using a usual P1 Finite Element Method. We solve the problem

\[
\begin{align*}
\text{Find } (\lambda_h^\delta, u_h^\delta) \in \mathbb{C} \times (V_h \setminus \{0\}) \text{ s.t.:} \\
\int_\Omega \sigma_h^\delta \nabla u_h^\delta \cdot \nabla v_h = \lambda_h^\delta \int_\Omega u_h^\delta v_h, \quad \forall v_h \in V_h,
\end{align*}
\]

where \( V_h \) approximates \( H_0^1(\Omega) \) as \( h \to 0 \) (\( h \) is the mesh size).

We consider the following 2D geometry:

We display the spectrum as \( \delta \to 0 \) (\( h \) is more or less fixed).
Numerical experiments for the small inclusion

Contrast $\kappa_\sigma = -2.5$

The positive part of $\mathcal{S}(A^\delta)$ converges to $\mathcal{S}(A^0)$ when $\delta \to 0$. 
The negative part of $\mathcal{G}(A^{\delta})$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathcal{G}(B^\infty)$ when $\delta \to 0$. 
Numerical experiments for the small inclusion

Contrast $\kappa_{\sigma} = -2.5$

The negative part of $\mathcal{G}(A^\delta)$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathcal{G}(B^\infty)$ when $\delta \to 0$. 
The eigenfunctions corresponding to the negative eigenvalues are localized around the small inclusion. Here, $\sigma_2/\sigma_1 = -2.5$. 
Thank you for your attention!!!

