Non-scattering wavenumbers and far field invisibility for a finite set of incident/scattering directions

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General setting

- We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

- General principle of the methods:
  i) send waves in the medium;
  ii) measure the scattered field;
  iii) deduce information on the structure.

- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...
Model problem

Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients $\rho$) in $\mathbb{R}^2$.

\[ \rho = 1 \quad \text{in} \quad \mathcal{D} \]

\[ \rho \neq 1 \]

Find $u$ such that

\[ -\Delta u = k^2 \rho u \quad \text{in} \quad \mathbb{R}^2, \]

\[ u = u_i + u_s \quad \text{in} \quad \mathbb{R}^2, \]

\[ \lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - ik u_s \right) = 0. \]
Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients $\rho$) in $\mathbb{R}^2$.

Define:
- $u_i := e^{ik\theta_{\text{inc}} \cdot x}$ (incident dir. $\theta_{\text{inc}} \in S^1$)
- $\rho = 1$
- $\rho \neq 1$

Find $u$ such that

\begin{align}
-\Delta u &= k^2 \rho u \quad \text{in } \mathbb{R}^2, \\
u &= u_i + u_s \quad \text{in } \mathbb{R}^2, \\
\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - ik u_s \right) &= 0.
\end{align} (1)
Model problem

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$$u_i := e^{ik\theta_{\text{inc}} \cdot x} \text{ (incident dir. } \theta_{\text{inc}} \in \mathbb{S}^1)$$

$$\rho = 1$$

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Find $u$ such that

$$-\Delta u = \kappa^2 \rho \ u \quad \text{in } \mathbb{R}^2,$$

$$u = u_i + u_s \quad \text{in } \mathbb{R}^2,$$

$$\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - iku_s \right) = 0.$$ (1)

**Definition:**

$u_i =$ incident field (data)

$u =$ total field (uniquely defined by (1))

$u_s =$ scattered field (uniquely defined by (1)).
Illustration of the scattering of a plane wave

Below, the movies represent a numerical approximation of the solution of the previous problem.

- The pulsation $\omega$ is defined by $\omega = k/c$ where $c = 1$ is the celerity of the waves in the homogeneous medium.
The scattered field of an incident plane wave of direction $\theta_{\text{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_s(x, \theta_{\text{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left( u_s^\infty(\theta_{\text{sca}}, \theta_{\text{inc}}) + O(1/r) \right)$$

as $r = |x| \to +\infty$, uniformly in $\theta_{\text{sca}} \in S^1$. 

Definition: The map $u_s^\infty(\cdot, \cdot) : S^1 \times S^1 \to \mathbb{C}$ is called the far field pattern. The far field pattern is the quantity one can measure at infinity (the other terms are too small).

- In literature, most of the techniques require a continuum of data.
- In practice, one has a finite number of emitters and receivers.
Far field pattern

- The scattered field of an incident plane wave of direction $\theta_{\text{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_s(x, \theta_{\text{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left( u_{\text{sc}}^{\infty}(\theta_{\text{sca}}, \theta_{\text{inc}}) + O(1/r) \right)$$

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Far field pattern

- The scattered field of an incident plane wave of direction $\theta_{inc}$ behaves in each direction like a cylindrical wave at infinity:

$$u_s(x, \theta_{inc}) = \frac{e^{ikr}}{\sqrt{r}} \left( u_{\infty}^{s}(\theta_{sca}, \theta_{inc}) + O(1/r) \right)$$

as $r = |x| \to +\infty$, uniformly in $\theta_{sca} \in S^1$.

**Definition:** The map $u_{\infty}^{s}(\cdot, \cdot) : S^1 \times S^1 \to \mathbb{C}$ is called the far field pattern.

The far field pattern is the quantity one can measure at infinity (the other terms are too small).

- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_{\infty}^{s}(\cdot, \cdot)$ on a subset of $S^1 \times S^1$.

  - In literature, most of the techniques require a continuum of data.
  - In practice, one has a finite number of emitters and receivers.
Let $\theta_1, \ldots, \theta_N$ be given directions of the unit circle $\mathbb{S}^1$. 

- We send the plane wave $e^{i k \theta_1 \cdot x}$ (direction $\theta_1$) and measure the resulted scattered fields in the directions $-\theta_1, \ldots, -\theta_N$.
- We repeat the experiment sending successively plane waves in the directions $\theta_2, \ldots, \theta_N$. 

$N \times N$ multistatic backscattering measurements.
Setting

Let $\theta_1, \ldots, \theta_N$ be given directions of the unit circle $S^1$.

We assume that emitters and receivers coincide:

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$N \times N$ multistatic backscattering measurements
Relative scattering matrix

For $\theta_1, \ldots, \theta_N$ given directions of $\mathbb{S}^1$, we introduce the relative scattering matrix

$\mathcal{S}(k) := \begin{pmatrix}
    u_s^\infty(-\theta_1, \theta_1) & \cdots & u_s^\infty(-\theta_1, \theta_N) \\
    \vdots & \ddots & \vdots \\
    u_s^\infty(-\theta_N, \theta_1) & \cdots & u_s^\infty(-\theta_N, \theta_N)
\end{pmatrix} \in \mathbb{C}^{N \times N}.$
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- Note that $\mathcal{S}(k) = 0$ when there is no obstacle ($\Rightarrow$ “relative”).

- We are interested in defects that cannot be detected and in invisibility.

1) Is there an incident wave which does not scatter at infinity?

2) Can it be that all incident waves do not scatter at infinity?
Relative scattering matrix

For $\theta_1, \ldots, \theta_N$ given directions of $\mathbb{S}^1$, we introduce the relative scattering matrix

$$S(k) := \begin{pmatrix} u_s(\theta_1, \theta_1) & \cdots & u_s(\theta_1, \theta_N) \\ \vdots & \ddots & \vdots \\ u_s(\theta_N, \theta_1) & \cdots & u_s(\theta_N, \theta_N) \end{pmatrix} \in \mathbb{C}^{N \times N}.$$ 

Note that $S(k) = 0$ when there is no obstacle (⇒ “relative”).

We are interested in defects that cannot be detected and in invisibility.

1) Is there an incident wave which does not scatter at infinity?
   \[ \rightarrow \ker S(k) \neq \{0\}? \]

2) Can it be that all incident waves do not scatter at infinity?
   \[ \rightarrow S(k) = 0? \]
Outline of the talk

1. Introduction

2. Non-scattering wavenumbers
   Is there an incident wave which does not scatter at infinity?

3. Invisible inclusions
   Can it be that all incident waves do not scatter at infinity?

4. Conclusion
Introduction

Non-scattering wavenumbers

Invisible inclusions

Conclusion
Non-scattering wavenumbers

**DEFINITION.** Values of $k > 0$ for which $\mathcal{I}(k)$ has a non trivial kernel are called non-scattering wavenumbers.

For $k$ non-scat. wavenumber, there is some $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N \setminus \{0\}$ s.t.

$$u_i = \sum_{n=1}^{N} \alpha_n e^{ik \theta_n \cdot x}$$

does not scatter at infinity in the directions $-\theta_1, \ldots, -\theta_N$. 
Non-scattering wavenumbers

**Definition.** Values of $k > 0$ for which $\mathcal{S}(k)$ has a non trivial kernel are called non-scattering wavenumbers.

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We want to prove that non-scattering wavenumbers form a discrete set because we want to avoid them to implement reconstruction techniques.
Discreteness of non-scattering wavenumbers

Idea of the approach:

1. We show that $k \mapsto \mathcal{J}(k)$ can be meromorphically extended to $\mathbb{C} \setminus \{0\}$.

2. For $k \in \mathbb{R} \setminus \{0\}$, using integration by parts, we prove the energy identity

$$c^T \mathcal{J}(k) \alpha = \int_{\mathbb{R}^2} |\nabla u_s|^2 + |k|_2 \rho |u_s|^2 + |k|_2 \int_D (1 - \rho) |u_i|^2.$$ 

where $u_i = \sum_{n=1}^N \alpha_n e^{ik \theta_n \cdot x}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^T$ and $c \neq 0$ is a constant.

3. For $k \in \mathbb{R} \setminus \{0\}$, $\rho < 1$, we deduce that $\mathcal{J}(k)$ is invertible.

4. Using the principle of isolated zeros, we obtain the following result:

Proposition. Suppose that $\rho < 1$. Then the set of non-scattering wavenumbers is discrete and countable.
Discreteness of non-scattering wavenumbers

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$$c \overline{\alpha}^\top \mathcal{I}(k) \alpha = \int_{\mathbb{R}^2} |\nabla u_s|^2 + |k|^2 \rho |u_s|^2 + |k|^2 \int_D (1 - \rho)|u_i|^2.$$

where $u_i = \sum_{n=1}^N \alpha_n e^{ik\theta_n \cdot x}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^\top$ and $c \neq 0$ is a constant.
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$$c \bar{\alpha}^\top \mathcal{S}(k) \alpha = \int_{\mathbb{R}^2} |\nabla u_s|^2 + |k|^2 \rho |u_s|^2 + |k|^2 \int_D (1 - \rho) |u_i|^2.$$

where $u_i = \sum_{n=1}^N \alpha_n e^{ik\theta_n \cdot x}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^\top$ and $c \neq 0$ is a constant.

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\[
c \, \overline{\alpha}^\top \mathcal{I}(k) \alpha = - \int_{\mathbb{R}^2} |\nabla u_s|^2 + |k|^2 |u_s|^2 - |k|^2 \int_D (\rho - 1) |u|^2.
\]

where \( u_i = \sum_{n=1}^{N} \alpha_n e^{ik\theta_n \cdot x} \), \( \alpha = (\alpha_1, \ldots, \alpha_N)^\top \) and \( c \neq 0 \) is a constant.

3. For \( k \in \mathbb{R}i \setminus \{0\}, \rho > 1 \), we deduce that \( \mathcal{I}(k) \) is invertible.

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**Proposition.** Suppose that \( \rho > 1 \). Then the set of non-scattering wavenumbers is discrete and countable.
1 Introduction

2 Non-scattering wavenumbers

3 Invisible inclusions

4 Conclusion
Invisible inclusions: setting

- In the previous section, for a given obstacle, we have studied the $k$ such that $\ker \mathcal{I}(k) \neq \{0\}$ ($\mathcal{I}(k)$ is the relative scattering matrix).

- Now, we assume that $k$ and the support of the inclusion $\overline{D}$ are given.

We explain how to construct non trivial inclusions such that $\mathcal{I}(k) = 0$. 
**Invisible inclusions: setting**

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  >> We explain how to construct non trivial inclusions such that $\mathcal{I}(k) = 0$.

- These inclusions cannot be detected from far field measurements.
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- Now, we assume that $k$ and the support of the inclusion $\overline{D}$ are given.

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- To simplify the presentation, assume that there is only one incident direction $\theta_{\text{inc}}$. Let $\theta_1, \ldots, \theta_N$ be given scattering directions.
Invisible inclusions: setting

In the previous section, for a given obstacle, we have studied the $k$ such that $\ker S(k) \neq \{0\}$ ($S(k)$ is the relative scattering matrix).

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To simplify the presentation, assume that there is only one incident direction $\theta_{\text{inc}}$. Let $\theta_1, \ldots, \theta_N$ be given scattering directions.

**Formulation of the problem:**

Find a real valued function $\rho \neq 1$, with $\rho - 1$ supported in $\bar{D}$, such that the solution of the problem

\[
\begin{align*}
-\Delta u &= k^2 \rho u \quad \text{in} \ \mathbb{R}^2, \\
\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - ik u_s \right) &= 0
\end{align*}
\]

verifies $u_s^\infty(\theta_1) = \cdots = u_s^\infty(\theta_N) = 0$.

Origin of the method:

- The idea we will use has been introduced in Nazarov 11 to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.
- It has been adapted in Bonnet-Ben Dhia & Nazarov 13 to build invisible perturbations of waveguides (see also Bonnet-Ben Dhia, Nazarov & Taskinen 14 for an application to a water-wave problem).
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Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
  \[ F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}. \]

  ($N$ complex measurements $\Rightarrow 2N$ real measurements)
Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
  
  $$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$  

- No obstacle leads to null measurements $\Rightarrow F(0) = 0$. 
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Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
\[
F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
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Our goal: to find \( \sigma \in L^\infty(\mathcal{D}) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).
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Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma)) \top \in \mathbb{R}^{2N}.$$ 

Our goal: to find $\sigma \in L^\infty(D)$ such that $F(\sigma) = 0$ (with $\sigma \neq 0$).

We look for small perturbations of the reference medium: $\sigma = \varepsilon \mu$ where $\varepsilon > 0$ is a small parameter and where $\mu$ has be to determined.
Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
  
  $$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$ 

  Our goal: to find $\sigma \in L^\infty(D)$ such that $F(\sigma) = 0$ (with $\sigma \neq 0$).

- Taylor: $F(\varepsilon \mu) = F(0) + \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$
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Assume that $dF(0) : L^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ is onto.
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\[
\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(\mathcal{D}) \text{ s.t. } \begin{cases}
  dF(0)(\mu_0) = 0 \\
  [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = Id_{2N}.
\end{cases}
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\end{array} \right\}$$

Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the $\tau_n$ are real parameters to set:
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- Define $\sigma = \rho - 1$ and gather the measurements in the vector
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- Taylor: \[ F(\varepsilon\mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu). \]

Assume that $dF(0) : L^\infty(\mathcal{D}) \rightarrow \mathbb{R}^{2N}$ is onto.

\[ \exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(\mathcal{D}) \text{ s.t.} \quad \begin{bmatrix} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \text{Id}_{2N}. \end{bmatrix} \]

- Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the $\tau_n$ are real parameters to set:
  \[ 0 = F(\varepsilon\mu) \quad \Leftrightarrow \quad 0 = \varepsilon \sum_{n=1}^{2N} \tau_n dF(0)(\mu_n) + \varepsilon^2 \tilde{F}^\varepsilon(\mu) \]
Sketch of the method

Define $\sigma = \rho - 1$ and gather the measurements in the vector

$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$ 

Our goal: to find $\sigma \in L^\infty(D)$ such that $F(\sigma) = 0$ (with $\sigma \neq 0$).

Taylor: $F(\varepsilon \mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$.

Assume that $dF(0): L^\infty(D) \to \mathbb{R}^{2N}$ is onto.

\[\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(D) \text{ s.t. }\begin{cases} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = Id_{2N}. \end{cases}\]

Take $\mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n$ where the $\tau_n$ are real parameters to set:

\[0 = F(\varepsilon \mu) \iff 0 = \varepsilon \tau + \varepsilon^2 \tilde{F}^\varepsilon(\mu)\]
Sketch of the method

Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
\[
F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
\]

Our goal: to find \( \sigma \in L^\infty(\mathcal{D}) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).

Taylor: \( F(\varepsilon \mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu) \).

Assume that \( dF(0) : L^\infty(\mathcal{D}) \to \mathbb{R}^{2N} \) is onto.

\[
\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(\mathcal{D}) \text{ s.t. } dF(0)(\mu_0) = 0 \Rightarrow [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = Id_{2N}.
\]

Take \( \mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n \) where the \( \tau_n \) are real parameters to set:
\[
0 = F(\varepsilon \mu) \iff 0 = \varepsilon \tilde{\tau} + \varepsilon^2 \tilde{F}^\varepsilon(\mu)
\]
where \( \tilde{\tau} = (\tau_1, \ldots, \tau_{2N})^\top \).
Sketch of the method

Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
\[
F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
\]

Our goal: to find \( \sigma \in L^\infty(\mathcal{D}) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).

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\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(\mathcal{D}) \text{ s.t. } \begin{cases}
  dF(0)(\mu_0) = 0 \\
  [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = Id_{2N}.
\end{cases}
\]

Take \( \mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n \) where the \( \tau_n \) are real parameters to set:
\[
0 = F(\varepsilon \mu) \iff \vec{\tau} = G^\varepsilon(\vec{\tau})
\]

where \( \vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top \)
Sketch of the method

Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
\[
F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
\]

Our goal: to find \( \sigma \in L^\infty(\mathcal{D}) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).

Taylor: \( F(\varepsilon \mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu) \).

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Take \( \mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n \) where the \( \tau_n \) are real parameters to set:
\[
0 = F(\varepsilon \mu) \quad \Leftrightarrow \quad \vec{\tau} = G^\varepsilon(\vec{\tau})
\]
where \( \vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top \) and \( G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu) \).
Sketch of the method

Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
\[
F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
\]

Our goal: to find \( \sigma \in L^\infty(D) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).

Taylor: \( F(\varepsilon \mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu) \).

Assume that \( dF(0) : L^\infty(D) \to \mathbb{R}^{2N} \) is onto.

\[
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dF(0)(\mu_0) = 0 \\
[dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \text{Id}_{2N}.
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\]

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\[
0 = F(\varepsilon \mu) \quad \Leftrightarrow \quad \vec{\tau} = G^\varepsilon(\vec{\tau})
\]

where \( \vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top \) and \( G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu) \).

If \( G^\varepsilon \) is a contraction, the fixed-point equation has a unique solution \( \vec{\tau}^{\text{sol}} \).
Sketch of the method

- Define \( \sigma = \rho - 1 \) and gather the measurements in the vector
  \[
  F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.
  \]

Our goal: to find \( \sigma \in L^\infty(D) \) such that \( F(\sigma) = 0 \) (with \( \sigma \neq 0 \)).

- Taylor: \( F(\varepsilon \mu) = \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu) \).

Assume that \( dF(0) : L^\infty(D) \to \mathbb{R}^{2N} \) is onto.

\[
\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in L^\infty(D) \text{ s.t. } \begin{cases}
  dF(0)(\mu_0) = 0 \\
  [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \text{Id}_{2N}.
\end{cases}
\]

- Take \( \mu = \mu_0 + \sum_{n=1}^{2N} \tau_n \mu_n \) where the \( \tau_n \) are real parameters to set:

\[
0 = F(\varepsilon \mu) \iff \vec{\tau} = G^\varepsilon(\vec{\tau})
\]

where \( \vec{\tau} = (\tau_1, \ldots, \tau_{2N})^\top \) and \( G^\varepsilon(\vec{\tau}) = -\varepsilon \tilde{F}^\varepsilon(\mu) \).

If \( G^\varepsilon \) is a contraction, the fixed-point equation has a unique solution \( \vec{\tau}^\text{sol} \).

Set \( \sigma^\text{sol} := \varepsilon \mu^\text{sol} \). We have \( F(\sigma^\text{sol}) = 0 \) (existence of an invisible inclusion).
Calculus of $dF(0)$

For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re u_\infty(\theta_1), \ldots, \Re u_\infty(\theta_N), \Im u_\infty(\theta_1), \ldots, \Im u_\infty(\theta_N)).$$
Calculus of $dF(0)$

For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re u^\infty_s(\theta_1), \ldots, \Re u^\infty_s(\theta_N), \Im u^\infty_s(\theta_1), \ldots, \Im u^\infty_s(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$. 

"Emitter" "Receiver"
Calculus of $dF(0)$

For our problem, we have $(\sigma = \rho - 1)$

\[ F(\sigma) = (\Re u_s^\infty(\theta_1), \ldots, \Re u_s^\infty(\theta_N), \Im m u_s^\infty(\theta_1), \ldots, \Im m u_s^\infty(\theta_N)). \]

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

We denote $u^\varepsilon, u_s^\varepsilon$ the functions satisfying

\[
\begin{align*}
\text{Find } u^\varepsilon &= u_s^\varepsilon + e^{ik\theta_{\text{inc}} \cdot x}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\
-\Delta u^\varepsilon &= k^2 \rho^\varepsilon u^\varepsilon \quad \text{in } \mathbb{R}^2.
\end{align*}
\]
Calculus of $dF(0)$

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re u_s^\infty(\theta_1), \ldots, \Re u_s^\infty(\theta_N), \Im m u_s^\infty(\theta_1), \ldots, \Im m u_s^\infty(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

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-\Delta u^\varepsilon &= k^2 \rho^\varepsilon u^\varepsilon \text{ in } \mathbb{R}^2.
\end{align*}$$

- $u_s^\varepsilon(\theta_n) = c k^2 \int_D (\rho^\varepsilon - 1) (u_i + u_s^\varepsilon) e^{-ik\theta_n \cdot x} \, d\mathbf{x}$ \quad (c = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}).$
Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re u_s^\infty(\theta_1), \ldots, \Re u_s^\infty(\theta_N), \Im m u_s^\infty(\theta_1), \ldots, \Im m u_s^\infty(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

- We denote $u^\varepsilon$, $u^\varepsilon_s$ the functions satisfying

Find $u^\varepsilon = u^\varepsilon_s + e^{ik\theta_{\text{inc}} \cdot x}$, with $u^\varepsilon_s$ outgoing, such that

$$-\Delta u^\varepsilon = k^2 \rho^\varepsilon u^\varepsilon \text{ in } \mathbb{R}^2.$$ 

- $u^\varepsilon_s(\theta_n) = c k^2 \int_{\mathcal{D}} (\rho^\varepsilon - 1) (u_i + u^\varepsilon_s) e^{-ik\theta_n \cdot x} d\mathbf{x}.$
Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re u_s^\infty(\theta_1), \ldots, \Re u_s^\infty(\theta_N), \Im m u_s^\infty(\theta_1), \ldots, \Im m u_s^\infty(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

- We denote $u^\varepsilon$, $u^\varepsilon_s$ the functions satisfying

$$\begin{align*}
\text{Find } u^\varepsilon & = u^\varepsilon_s + e^{ik\theta_{\text{inc}} \cdot x}, \text{with } u^\varepsilon_s \text{ outgoing, such that} \\
-\Delta u^\varepsilon & = k^2 \rho^\varepsilon u^\varepsilon \text{ in } \mathbb{R}^2.
\end{align*}$$

- $u^\varepsilon_s(\theta_n) = \varepsilon c k^2 \int_{\mathcal{D}} \mu (u_i + u^\varepsilon_s) e^{-ik\theta_{\text{inc}} \cdot x} \, d\mathbf{x}$. 


Calculus of $dF(0)$

▶ For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re u_\infty^s(\theta_1), \ldots, \Re u_\infty^s(\theta_N), \Im u_\infty^s(\theta_1), \ldots, \Im u_\infty^s(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

▶ We denote $u^\varepsilon$, $u_\varepsilon^s$ the functions satisfying

\[
\begin{align*}
\text{Find } u^\varepsilon &= u_\varepsilon^s + e^{ik\theta_{\text{inc}} \cdot x}, \text{with } u_\varepsilon^s \text{ outgoing, such that} \\
-\Delta u^\varepsilon &= k^2 \rho^\varepsilon \ u^\varepsilon \quad \text{in } \mathbb{R}^2.
\end{align*}
\]

- $u_\varepsilon^s \infty(\theta_n) = \varepsilon c k^2 \int_D \mu(u_i + u_\varepsilon^s) e^{-ik\theta_n \cdot x} d\mathbf{x}.$

- We can prove that $u_\varepsilon^s = O(\varepsilon)$. 
Calculus of $dF(0)$

- For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re u_\infty(\theta_1), \ldots, \Re u_\infty(\theta_N), \Im m u_\infty(\theta_1), \ldots, \Im m u_\infty(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

- We denote $u^\varepsilon, u_s^\varepsilon$ the functions satisfying

$$\begin{align*}
\text{Find } u^\varepsilon &= u_s^\varepsilon + e^{i k \theta_{\text{inc}} \cdot x}, \text{ with } u_s^\varepsilon \text{ outgoing, such that} \\
-\Delta u^\varepsilon &= k^2 \rho^\varepsilon u^\varepsilon \text{ in } \mathbb{R}^2.
\end{align*}$$

- $u_s^\varepsilon(\theta_n) = \varepsilon c k^2 \int_D \mu u_i e^{-i k \theta_n \cdot x} \, d\mathbf{x} + O(\varepsilon^2)$.

- We can prove that $u_s^\varepsilon = O(\varepsilon)$. 
Calculus of $dF(0)$

- For our problem, we have $(\sigma = \rho - 1)$

\[ F(\sigma) = (\Re u^\infty_s(\theta_1), \ldots, \Re u^\infty_s(\theta_N), \Im m u^\infty_s(\theta_1), \ldots, \Im m u^\infty_s(\theta_N)) \]

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

- We denote $u^\varepsilon, u^\varepsilon_s$ the functions satisfying

\[
\begin{align*}
\text{Find } & u^\varepsilon = u^\varepsilon_s + e^{ik\theta_{\text{inc}} \cdot x}, \text{with } u^\varepsilon_s \text{ outgoing, such that} \\
-\Delta u^\varepsilon = k^2 \rho^\varepsilon u^\varepsilon \quad & \text{in } \mathbb{R}^2.
\end{align*}
\]

- We obtain the expansion (Born approx.), for small $\varepsilon$

\[ u^\varepsilon_s(\theta_n) = 0 + \varepsilon \ c \ k^2 \int_D \mu e^{ik(\theta_{\text{inc}} - \theta_n) \cdot x} \, d\mathbf{x} + O(\varepsilon^2). \]
Calculus of $dF(0)$

For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re \frac{u_s^\infty(\theta_1)}{ck^2}, \ldots, \Re \frac{u_s^\infty(\theta_N)}{ck^2}, \Im \frac{u_s^\infty(\theta_1)}{ck^2}, \ldots, \Im \frac{u_s^\infty(\theta_N)}{ck^2}).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

We denote $u^\varepsilon, u_s^\varepsilon$ the functions satisfying

$$\begin{align*}
\text{Find } u^\varepsilon &= u_s^\varepsilon + e^{ik\theta_{\text{inc}} \cdot x}, \text{with } u_s^\varepsilon \text{ outgoing, such that} \\
-\Delta u^\varepsilon &= k^2 \rho^\varepsilon u^\varepsilon \text{ in } \mathbb{R}^2.
\end{align*}$$

We obtain the expansion (Born approx.), for small $\varepsilon$

$$u_s^\varepsilon^\infty(\theta_n) = 0 + \varepsilon c k^2 \int_D \mu e^{ik(\theta_{\text{inc}} - \theta_n) \cdot x} d\mathbf{x} + O(\varepsilon^2).$$
Calculus of $dF(0)$

For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re \frac{u_S^\infty(\theta_1)}{ck^2}, \ldots, \Re \frac{u_S^\infty(\theta_N)}{ck^2}, \Im \frac{u_S^\infty(\theta_1)}{ck^2}, \ldots, \Im \frac{u_S^\infty(\theta_N)}{ck^2}).$$

To compute $dF(0)(\mu)$, we take $\rho^\varepsilon = 1 + \varepsilon \mu$ with $\mu$ supported in $\overline{D}$.

We denote $u^\varepsilon, u^\varepsilon_s$ the functions satisfying

$$\begin{aligned}
\text{Find } u^\varepsilon &= u^\varepsilon_s + e^{ik_{\text{inc}} \cdot x}, \text{with } u^\varepsilon_s \text{ outgoing, such that} \\
-\Delta u^\varepsilon &= k^2 \rho^\varepsilon u^\varepsilon \quad \text{in } \mathbb{R}^2.
\end{aligned}$$

We obtain the expansion (Born approx.), for small $\varepsilon$

$$u^\varepsilon_s^\infty(\theta_n) = 0 + \varepsilon c k^2 \int_D \mu e^{ik(\theta_{\text{inc}} - \theta_n) \cdot x} \, d\mu + O(\varepsilon^2).$$

$$dF(0)(\mu) = \left( \int_D \mu \cos(k(\theta_{\text{inc}} - \theta_1) \cdot x) \, d\mu, \ldots, \right)
\int_D \mu \cos(k(\theta_{\text{inc}} - \theta_N) \cdot x) \, d\mu,
\int_D \mu \sin(k(\theta_{\text{inc}} - \theta_1) \cdot x) \, d\mu, \ldots, \int_D \mu \sin(k(\theta_{\text{inc}} - \theta_N) \cdot x) \, d\mu \right).$$
Calculus of $dF(0)$

For our problem, we have $(\sigma = \rho - 1)$

$$F(\sigma) = (\Re u_\infty s(\theta_1), \ldots, \Re u_\infty s(\theta_N), \Im u_\infty s(\theta_1), \ldots, \Im u_\infty s(\theta_N)).$$

To compute $dF(0)(\mu)$, we take $\rho = 1 + \epsilon \mu$ with $\mu$ supported in $D$.

We denote $u_\epsilon, u_\epsilon s$ the functions satisfying

Find $u_\epsilon = u_\epsilon s + e^{ik \theta \text{inc}} \cdot x$, with $u_\epsilon s$ outgoing, such that

$$-\Delta u_\epsilon = k^2 \rho_\epsilon u_\epsilon \text{in } \mathbb{R}^2.$$

- $u_\epsilon \infty s(\theta_n) = \epsilon c_2 \int_D \mu e^{-ik \theta \text{inc} \cdot x} dx + O(\epsilon^2)$.

We can prove that $u_\epsilon s = O(\epsilon)$.

We obtain the expansion (Born approx.), for small $\epsilon$

$$u_\epsilon \infty s(\theta_n) = 0 + \epsilon c_2 \int_D \mu e^{ik(\theta \text{inc} - \theta_n) \cdot x} dx + O(\epsilon^2).$$

$$dF(0)(\mu) = \left( \int_D \mu \cos(k(\theta \text{inc} - \theta_1) \cdot x) \, dx, \ldots, \int_D \mu \cos(k(\theta \text{inc} - \theta_N) \cdot x) \, dx, \right)$$

$$\left( \int_D \mu \sin(k(\theta \text{inc} - \theta_1) \cdot x) \, dx, \ldots, \int_D \mu \sin(k(\theta \text{inc} - \theta_N) \cdot x) \, dx \right).$$
Construction of the shape functions

1 If $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$,

$$
\mathcal{M} := \{\cos(k(\theta_{\text{inc}} - \theta_n) \cdot x), \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x)\}_{n=1}^N
$$

is a family of linearly independent functions. Using the Gram matrix, we can build $\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N} \in \text{span}(\mathcal{M})$ such that

$$
\int_D \mu_{1,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = \delta^{mn}, \quad \int_D \mu_{1,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = 0
$$

$$
\int_D \mu_{2,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = 0, \quad \int_D \mu_{2,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = \delta^{mn}
$$
Construction of the shape functions

1 If $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$,

$$\mathcal{M} := \{\cos(k(\theta_{\text{inc}} - \theta_n) \cdot x), \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x)\}_{n=1}^N,$$

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\[
\begin{align*}
\int_{\mathcal{D}} \mu_{1,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= \delta^{mn}, \\
\int_{\mathcal{D}} \mu_{1,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= 0, \\
\int_{\mathcal{D}} \mu_{2,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= 0, \\
\int_{\mathcal{D}} \mu_{2,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= \delta^{mn}
\end{align*}
\]

2 We need to construct some $\mu_0 \in \ker dF(0)$, i.e. some $\mu_0$ satisfying

\[
\begin{align*}
\int_{\mathcal{D}} \mu_0 \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= 0, \\
\int_{\mathcal{D}} \mu_0 \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx &= 0.
\end{align*}
\]
Construction of the shape functions

1. If $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$,

$$M := \{\cos(k(\theta_{\text{inc}} - \theta_n) \cdot x), \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x)\}_{n=1}^N,$$

is a family of linearly independent functions. Using the Gram matrix, we can build $\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N} \in \text{span}(M)$ such that

$$\int_D \mu_{1,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = \delta^{mn}, \quad \int_D \mu_{1,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = 0,$$

$$\int_D \mu_{2,m} \cos(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = 0, \quad \int_D \mu_{2,m} \sin(k(\theta_{\text{inc}} - \theta_n) \cdot x) \, dx = \delta^{mn}.$$ 

2. We take

$$\mu_0 = \mu_0^\# - \sum_{m=1}^N \left( \int_D \mu_{1,m} \mu_0^\# \, dx \right) \mu_{1,m} - \sum_{m=1}^N \left( \int_D \mu_{2,m} \mu_0^\# \, dx \right) \mu_{2,m}$$

where $\mu_0^\# \notin \text{span}\{\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N}\}$. 
Main result

**Proposition:** Assume that $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$. For $\varepsilon$ small enough, define $\rho^{\text{sol}} = 1 + \varepsilon \mu^{\text{sol}}$ with

$$\mu^{\text{sol}} = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}^{\text{sol}} \mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}^{\text{sol}} \mu_{2,m}.$$ 

Then the solution of the scattering problem

$$\text{Find } u^\varepsilon = u^\varepsilon_s + e^{ik \theta_{\text{inc}} \cdot x} \text{ such that }$$

$$-\Delta u = k^2 \rho^{\text{sol}} u \text{ in } \mathbb{R}^2,$$

$$\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - iku_s \right) = 0$$

verifies $u_s^\infty(\theta_1) = \cdots = u_s^\infty(\theta_N) = 0$.

**Comments:**

→ Proving that $G^\varepsilon$ is a contraction is not a big deal.

→ We have $\mu^{\text{sol}} \neq 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$. 
**Main result**

**PROPOSITION:** Assume that $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$. For $\varepsilon$ small enough, define $\rho^{\text{sol}} = 1 + \varepsilon \mu^{\text{sol}}$ with

$$
\mu^{\text{sol}} = \mu_0 + \sum_{m=1}^{N} \tau_{1,m}^{\text{sol}} \mu_{1,m} + \sum_{m=1}^{N} \tau_{2,m}^{\text{sol}} \mu_{2,m}.
$$

Then the solution of the scattering problem

Find $u^\varepsilon = u^\varepsilon_s + e^{ik\theta_{\text{inc}} \cdot x}$ such that

$$
-\Delta u = k^2 \rho^{\text{sol}} u \text{ in } \mathbb{R}^2,
$$

$$
\lim_{r \to +\infty} \sqrt{r} \left( \frac{\partial u_s}{\partial r} - iku_s \right) = 0
$$

verifies $u^\infty_s(\theta_1) = \cdots = u^\infty_s(\theta_N) = 0$.

**COMMENTS:**

→ Proving that $G^\varepsilon$ is a contraction is not a big deal.

→ We have $\mu^{\text{sol}} \neq 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

→ This proof of existence of invisible inclusions may appear not so surprising since $\mathcal{I}(k) \in \mathbb{R}^{2N}$, $\rho \in L^\infty(D)$. 
**Main result**

**Proposition:** Assume that $\theta_{\text{inc}} \neq \theta_n$ for $n = 1, \ldots, N$. For $\varepsilon$ small enough, define $\rho^{\text{sol}} = 1 + \varepsilon \mu^{\text{sol}}$ with

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Then the solution of the scattering problem

$$
\text{Find } u^{\varepsilon} = u^{\varepsilon}_s + e^{i k \theta_{\text{inc}} \cdot x} \text{ such that }

\begin{align*}
-\Delta u &= k^2 \rho^{\text{sol}} u \quad \text{in } \mathbb{R}^2, \\
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**Comments:**

→ Proving that $G^{\varepsilon}$ is a contraction is not a big deal.

→ We have $\mu^{\text{sol}} \neq 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\text{sol}})$.

→ This proof of existence of invisible inclusions may appear not so surprising since $\mathcal{J}(k) \in \mathbb{R}^{2N}$, $\rho \in L^\infty(D)$. The case $\theta_{\text{inc}} = \theta_n$ shows that nothing is obvious...
The case $\theta_{\text{inc}} = \theta_n$

- In the previous approach, we needed to assume $\theta_{\text{inc}} \neq \theta_n$, $n = 1, \ldots, N$.

What if $\theta_{\text{inc}} = \theta_n$?

![Diagram](image)

- No solution if $D$ has corners and under certain assumptions on $\rho$.
  - Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014
- And if $D$ is smooth? $\Rightarrow$ The problem seems open.
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- This allows to prove the formula (use Colton, Kress 98)

$$\Im m (c^{-1} u_s^\infty(\theta_{\text{inc}})) = k \int_{S^1} |u_s^\infty(\theta)|^2 \, d\theta.$$
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- This allows to prove the formula (use Colton, Kress 98)

$$\mathcal{S} m \left( c^{-1} u_s^\infty(\theta_{inc}) \right) = k \int_{\mathbb{S}^1} |u_s^\infty(\theta)|^2 \, d\theta.$$  

Imposing invisibility in the direction $\theta_{inc}$ requires to impose invisibility in all directions $\theta \in \mathbb{S}^1$!
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Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^0 = (0, \ldots, 0)^\top$ then define

$$
\vec{\tau}^{n+1} = G^\varepsilon(\vec{\tau}^n).
$$

- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball $B_8$. On $\partial B_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = B_1$, $M = 3$ (3 directions of observation) and

  $$
  \begin{align*}
  \theta_{\text{inc}} &= (\cos(\psi_{\text{inc}}), \sin(\psi_{\text{inc}})), \quad \psi_{\text{inc}} = 0^\circ \\
  \theta_1 &= (\cos(\psi_1), \sin(\psi_1)), \quad \psi_1 = 90^\circ \\
  \theta_2 &= (\cos(\psi_2), \sin(\psi_2)), \quad \psi_2 = 180^\circ \\
  \theta_3 &= (\cos(\psi_3), \sin(\psi_3)), \quad \psi_3 = 225^\circ
  \end{align*}
  $$
Results: coefficient $\rho$ at the end of the process
Results: scattered field

Figure: $|u_s|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $D$ in the directions corresponding to the angles $90^\circ$, $180^\circ$ and $225^\circ$. The domain is equal to $B_8$. 
Results: far field pattern

Figure: The dotted lines show the directions where we want \( u_s^\infty \) to vanish.
1. Introduction

2. Non-scattering wavenumbers

3. Invisible inclusions

4. Conclusion
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Discreteness of non-scattering eigenvalues

For a given obstacle, is there an incident field that does not scatter?

♠ How to proceed to prove discreteness of non-scattering wavenumbers for situations other than multistatic backscattering measurements?

♠ Can we relax assumptions on $\rho$?

♠ Can we prove existence of non-scattering wavenumbers in this setting?

Invisibility

For a given frequency, how to build an invisible obstacle?

♠ An important issue: can we reiterate the process to construct larger defects in the reference medium?

♠ Can we hide small Dirichlet obstacles (flies)?

Work in progress...
Conclusion

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Thank you for your attention!!!