Non-scattering wavenumbers and far field invisibility for a finite set of incident/scattering directions

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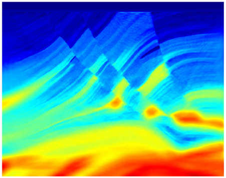
General setting

- We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

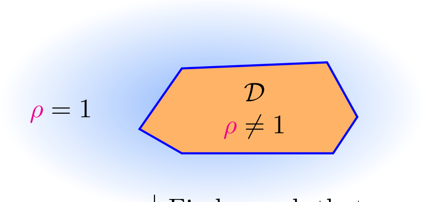
- General principle of the methods:
  - i) send waves in the medium;
  - ii) measure the scattered field;
  - iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Model problem

► Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



$$\left|\begin{array}{l}\text{Find } u \text{ such that} \\ \begin{array}{rcll} -\Delta u &=& k^2 \rho\, u & \text{in } \mathbb{R}^2, \\ u &=& u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \end{array} \\ \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0. \end{array}\right. \qquad (1)$$

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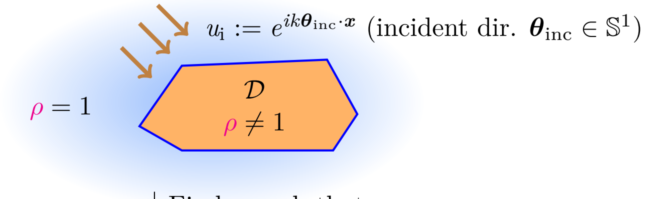

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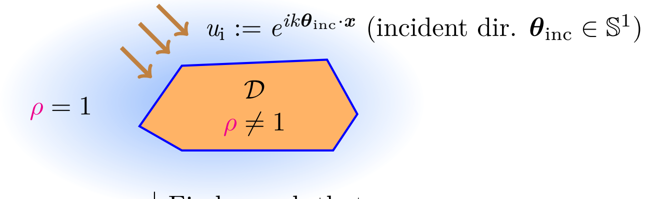


Find u such that

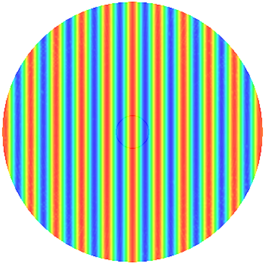
$$\begin{aligned} -\Delta u &= k^2 \rho\, u && \text{in } \mathbb{R}^2, \\ u &= u_{\mathrm{i}} + u_{\mathrm{s}} && \text{in } \mathbb{R}^2, \\ \lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) &= 0. \end{aligned} \tag{1}$$

Definition:
- u_{i} = incident field (data)
- u = total field (uniquely defined by (1))
- u_{s} = scattered field (uniquely defined by (1)).

Illustration of the scattering of a plane wave

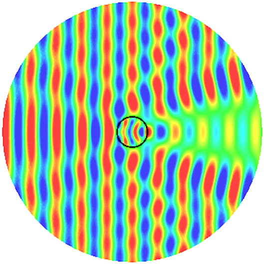
- Below, the movies represent a numerical approximation of the solution of the previous problem.

Incident field



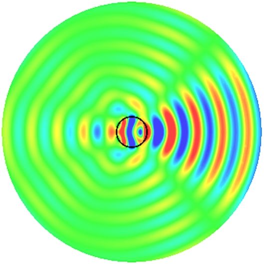
$t \mapsto \Re e\left(e^{-i\omega t} u_{\mathrm{i}}(\boldsymbol{x})\right)$

Total field



$t \mapsto \Re e\left(e^{-i\omega t} u(\boldsymbol{x})\right)$

Scattered field



$t \mapsto \Re e\left(e^{-i\omega t} u_{\mathrm{s}}(\boldsymbol{x})\right)$

- The pulsation ω is defined by $\omega = k/c$ where $c = 1$ is the celerity of the waves in the homogeneous medium.

Far field pattern

► The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\text{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\text{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\text{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \left(u_{\text{s}}^{\infty}(\boldsymbol{\theta}_{\text{sca}}, \boldsymbol{\theta}_{\text{inc}}) + O(1/r) \right)$$

as $r = |\boldsymbol{x}| \to +\infty$, uniformly in $\boldsymbol{\theta}_{\text{sca}} \in \mathbb{S}^1$.

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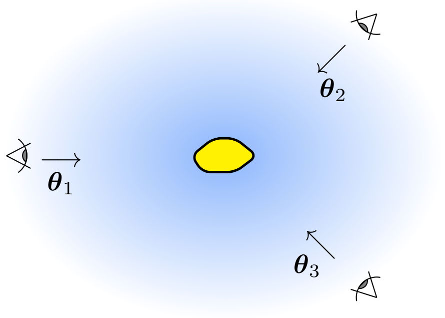
- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_{\mathrm{s}}^{\infty}(\cdot,\cdot)$ on a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.
  - In literature, most of the techniques require a continuum of data.
  - In practice, one has a finite number of emitters and receivers.

Setting

- Let $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ be given directions of the unit circle $\mathbb{S}^1$.



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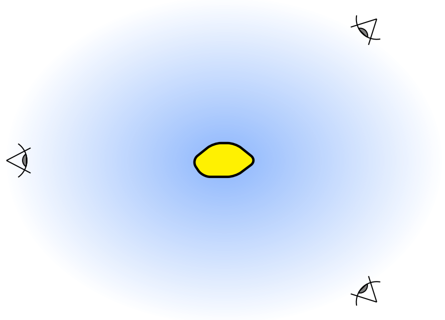

- We assume that emitters and receivers coincide:
  - We send the plane wave $e^{ik\boldsymbol{\theta}_1 \cdot \boldsymbol{x}}$ (direction $\boldsymbol{\theta}_1$) and measure the resulted scattered fields in the directions $-\boldsymbol{\theta}_1, \ldots, -\boldsymbol{\theta}_N$.

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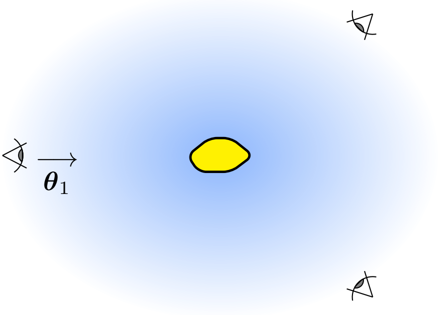

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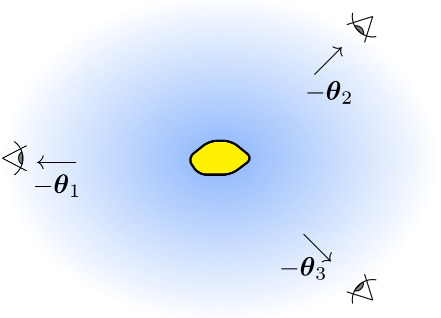

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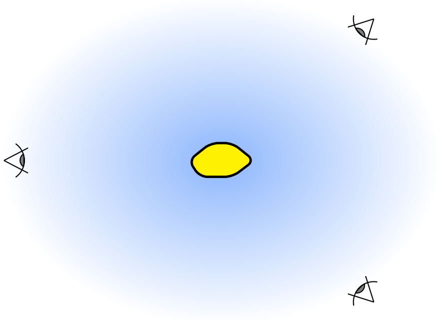

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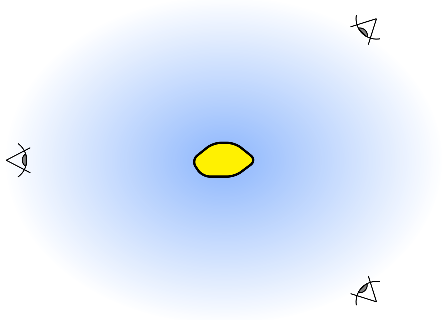

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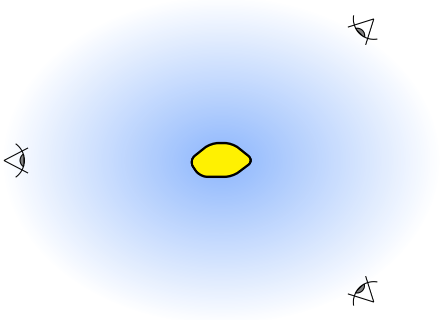

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$N \times N$ multistatic backscattering measurements

Relative scattering matrix

- For $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ given directions of $\mathbb{S}^1$, we introduce the relative scattering matrix

$$\boxed{\mathscr{S}(k) := \begin{pmatrix} u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_1, \boldsymbol{\theta}_1) & \cdots & u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_1, \boldsymbol{\theta}_N) \\ \vdots & \ddots & \vdots \\ u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_N, \boldsymbol{\theta}_1) & \cdots & u_{\mathrm{s}}^{\infty}(-\boldsymbol{\theta}_N, \boldsymbol{\theta}_N) \end{pmatrix} \in \mathbb{C}^{N \times N}.}$$

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We are interested in defects that cannot be detected and in invisibility.

1) Is there an incident wave which does not scatter at infinity?
$\rightarrow$ $\ker \mathscr{S}(k) \neq \{0\}$?

2) Can it be that all incident waves do not scatter at infinity?
$\rightarrow$ $\mathscr{S}(k) = 0$?

Outline of the talk

1. Introduction

2. Non-scattering wavenumbers

   Is there an incident wave which does not scatter at infinity?

3. Invisible inclusions

   Can it be that all incident waves do not scatter at infinity?

4. Conclusion

Non-scattering wavenumbers

Definition. Values of $k > 0$ for which $\mathscr{S}(k)$ has a non trivial kernel are called non-scattering wavenumbers.

- For k non-scat. wavenumber, there is some $(\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N \setminus \{0\}$ s.t.

$$u_{\mathrm{i}} = \sum_{n=1}^{N} \alpha_n e^{ik\boldsymbol{\theta}_n \cdot \boldsymbol{x}}$$

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We want to prove that non-scattering wavenumbers form a discrete set because we want to avoid them to implement reconstruction techniques.

Discreteness of non-scattering wavenumbers

Idea of the approach:

1. We show that $k \mapsto \mathscr{S}(k)$ can be meromorphically extended to $\mathbb{C} \setminus \{0\}$.

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$$\boxed{c\,\overline{\alpha}^{\top}\mathscr{S}(k)\,\alpha = \int_{\mathbb{R}^2} |\nabla u_{\mathrm{s}}|^2 + |k|^2\rho\,|u_{\mathrm{s}}|^2 + |k|^2 \int_{\mathcal{D}} (1-\rho)|u_{\mathrm{i}}|^2.}$$

where $u_{\mathrm{i}} = \sum_{n=1}^{N} \alpha_n e^{ik\boldsymbol{\theta}_n\cdot\boldsymbol{x}}$, $\alpha = (\alpha_1, \ldots, \alpha_N)^{\top}$ and $c \neq 0$ is a constant.

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4. Using the principle of isolated zeros, we obtain the following result:

PROPOSITION. Suppose that $\rho < 1$. Then the set of non-scattering wavenumbers is discrete and countable.

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Invisible inclusions: setting

- In the previous section, for a given obstacle, we have studied the k such that $\ker \mathscr{S}(k) \neq \{0\}$ ($\mathscr{S}(k)$ is the relative scattering matrix).

- Now, we assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

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- These inclusions cannot be detected from far field measurements.

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Formulation of the problem:

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution of the problem

$$\left|\begin{array}{l} \text{Find } u = u_{\mathrm{s}} + e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho\, u \qquad \text{in } \mathbb{R}^2, \\ \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

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Origin of the method:

- The idea we will use has been introduced in `Nazarov 11` to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

- It has been adapted in `Bonnet-Ben Dhia & Nazarov 13` to build invisible perturbations of waveguides (see also `Bonnet-Ben Dhia, Nazarov & Taskinen 14` for an application to a water-wave problem).

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$
(N complex measurements $\Rightarrow$ $2N$ real measurements)

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
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- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

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- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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- Taylor: $F(\varepsilon\mu) = F(0) + \varepsilon dF(0)(\mu) + \varepsilon^2 \tilde{F}^\varepsilon(\mu)$

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If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$. Set $\sigma^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (existence of an invisible inclusion).

Calculus of $dF(0)$

- For our problem, we have ($\sigma = \rho - 1$)

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$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot\boldsymbol{x}}\, d\boldsymbol{x} + O(\varepsilon^2).$$

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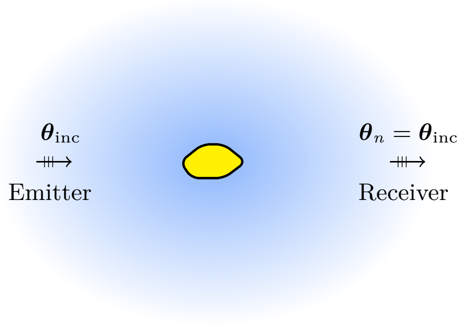
$$\begin{aligned} dF(0)(\mu) = \Big(&\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \\ &\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}\Big) \end{aligned}$$

Calculus of $dF(0)$



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Construction of the shape functions

1. If $\boldsymbol{\theta}_{\text{inc}} \neq \boldsymbol{\theta}_n$ for $n = 1, \dots, N$,

$$\mathscr{M} := \{\cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x}), \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n) \cdot \boldsymbol{x})\}_{n=1}^{N},$$

is a family of linearly independent functions. Using the Gram matrix, we can build $\mu_{1,1}, \dots, \mu_{1,N}, \mu_{2,1}, \dots, \mu_{2,N} \in \operatorname{span}(\mathscr{M})$ such that

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2. We need to construct some $\mu_0 \in \ker dF(0)$, *i.e.* some μ_0 satisfying

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2 We take

$$\boxed{\mu_0 = \mu_0^{\#} - \sum_{m=1}^{N} \left(\int_{\mathcal{D}} \mu_{1,m}\, \mu_0^{\#}\, d\boldsymbol{x} \right) \mu_{1,m} - \sum_{m=1}^{N} \left(\int_{\mathcal{D}} \mu_{2,m}\, \mu_0^{\#}\, d\boldsymbol{x} \right) \mu_{2,m}}$$

where $\mu_0^{\#} \notin \mathrm{span}\{\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N}\}$.

Main result

PROPOSITION: Assume that $\boldsymbol{\theta}_{\text{inc}} \neq \boldsymbol{\theta}_n$ for $n = 1, \dots, N$. For ε small enough, define $\rho^{\,\text{sol}} = 1 + \varepsilon\mu^{\,\text{sol}}$ with

$$\mu^{\,\text{sol}} = \mu_0 + \sum_{m=1}^{N} \tau^{\text{sol}}_{1,m}\,\mu_{1,m} + \sum_{m=1}^{N} \tau^{\text{sol}}_{2,m}\,\mu_{2,m}.$$

Then the solution of the scattering problem

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u^\varepsilon_{\text{s}} + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2\rho^{\,\text{sol}}\,u \quad \text{ in } \mathbb{R}^2, \\ \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\text{s}}}{\partial r} - iku_{\text{s}}\right) = 0 \end{array}\right.$$

verifies $u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

COMMENTS:

$\to$ Proving that G^ε is a contraction is not a big deal.

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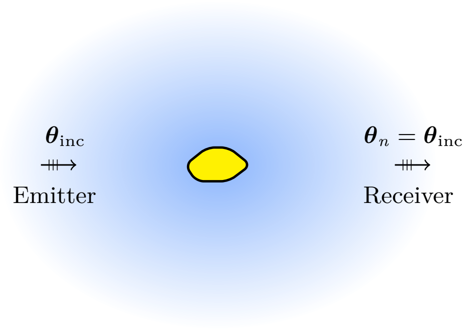

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- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- `Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014`
- `Corners and edges always scatter, J. Elschner, G. Hu, 2015`

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

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Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \ldots, 0)^\top$ then define

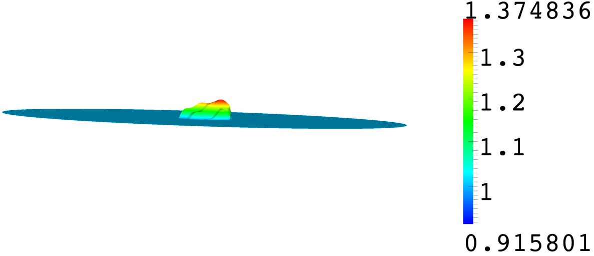
$$\boxed{\vec{\tau}^{\,n+1} = G^\varepsilon(\vec{\tau}^{\,n}).}$$

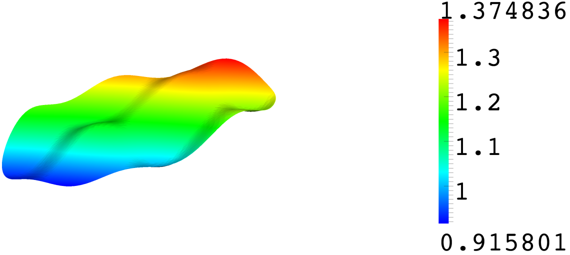
- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{ll} \boldsymbol{\theta}_{\mathrm{inc}} = (\cos(\psi_{\mathrm{inc}}), \sin(\psi_{\mathrm{inc}})), & \psi_{\mathrm{inc}} = 0^\circ \\ \boldsymbol{\theta}_1 = (\cos(\psi_1), \sin(\psi_1)), & \psi_1 = 90^\circ \\ \boldsymbol{\theta}_2 = (\cos(\psi_2), \sin(\psi_2)), & \psi_2 = 180^\circ \\ \boldsymbol{\theta}_3 = (\cos(\psi_3), \sin(\psi_3)), & \psi_3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

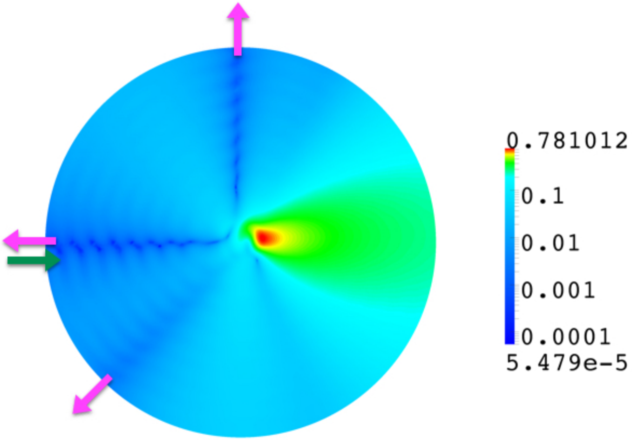


Figure: $|u_s|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles 90°, 180° and 225°. The domain is equal to B_8.

Results: far field pattern

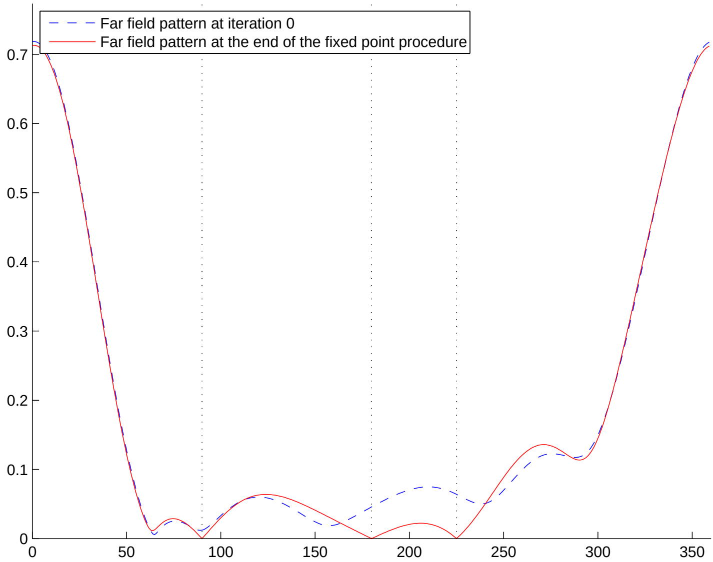


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

Conclusion

Discreteness of non-scattering eigenvalues

For a given obstacle, is there an incident field that does not scatter?

- ♠ How to proceed to prove discreteness of non-scattering wavenumbers for situations other than multistatic backscattering measurements?
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- ♠ Can we prove existence of non-scattering wavenumbers in this setting?

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Invisibility

For a given frequency, how to build an invisible obstacle?

- An important issue: can we reiterate the process to construct larger defects in the reference medium?
- Can we hide small Dirichlet obstacles (flies)? *Work in progress...*

Thank you for your attention!!!