Conference in honnor of the 60th birthday of Patrick Joly

Invisibility and complete reflectivity in waveguides with finite length branches

Lucas Chesnel[1]

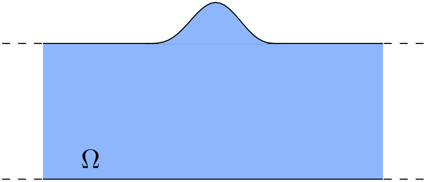
Coll. with S.A. Nazarov[2] and V. Pagneux[3].

[1]Defi team, CMAP, École Polytechnique, France
[2]FMM, St. Petersburg State University, Russia
[3]LAUM, Université du Maine, France

Gif-Sur-Yvette, 30/08/2017

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



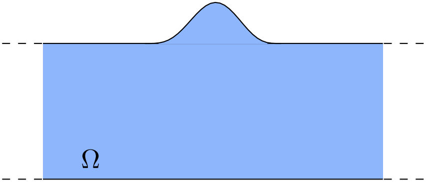
Find $v = v_{\mathrm{i}} + v_{\mathrm{s}}$ s. t.

$$\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \end{array}$$

v_{s} is outgoing.

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.

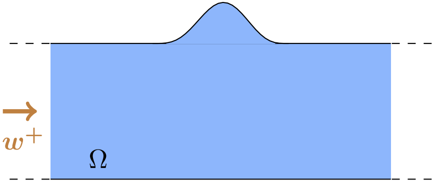


$$\left|\begin{array}{l} \text{Find } v = v_{\rm i} + v_{\rm s} \text{ s. t.} \\ \begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \end{array} \\ \qquad v_{\rm s} \text{ is outgoing.} \end{array}\right.$$

- For $k \in (0;\pi)$, $\boxed{\text{only 2 propagating modes } w^{\pm} = e^{\pm ikx}/\sqrt{2k}.}$

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.



Find $v = v_{\mathrm{i}} + v_{\mathrm{s}}$ s. t.

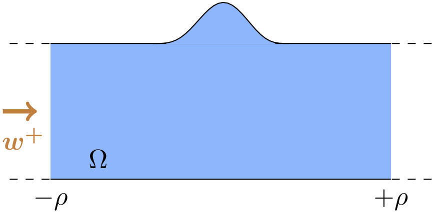
$$\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \end{array}$$

v_{s} is outgoing.

- For $k \in (0;\pi)$, $\boxed{\text{only 2 propagating modes } w^{\pm} = e^{\pm ikx}/\sqrt{2k}.}$ Set $v_{\mathrm{i}} = w^+$.

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



Find $v = v_i + v_s$ s. t.

$$\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega, \end{array}$$

v_s is outgoing.

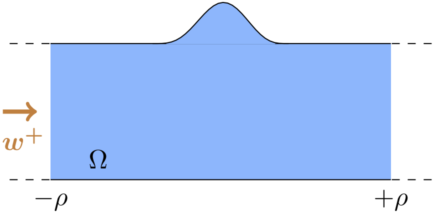
- For $k \in (0; \pi)$, $\boxed{\text{only 2 propagating modes } w^{\pm} = e^{\pm ikx}/\sqrt{2k}.}$ Set $v_i = w^+$.

- v_s is outgoing $\Leftrightarrow$ $\boxed{v_s = s^{\pm} w^{\pm} + \tilde{v}_s \quad \text{for } \pm x \geq \rho,}$

with $s^{\pm} \in \mathbb{C}$, $\tilde{v}_s$ exponentially decaying at $\pm\infty$.

Waveguide problem

- Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.



Find $v = v_{\mathrm{i}} + v_{\mathrm{s}}$ s. t.

$$\begin{array}{rcll} -\Delta v &=& k^2 v & \text{in } \Omega, \\ \partial_n v &=& 0 & \text{on } \partial\Omega, \\ && \multicolumn{2}{l}{v_{\mathrm{s}} \text{ is outgoing.}} \end{array}$$

- For $k \in (0;\pi)$, $\boxed{\text{only 2 propagating modes } w^{\pm} = e^{\pm ikx}/\sqrt{2k}.}$ Set $v_{\mathrm{i}} = w^+$.

- v_{s} is outgoing $\quad\Leftrightarrow\quad$ $\boxed{v_{\mathrm{s}} = s^{\pm} w^{\pm} + \tilde{v}_{\mathrm{s}} \quad \text{for } \pm x \geq \rho,}$

with $s^{\pm} \in \mathbb{C}$, $\tilde{v}_{\mathrm{s}}$ exponentially decaying at $\pm\infty$.

Definition:

$$\begin{array}{lcl} v_{\mathrm{i}} &=& \text{incident field} \\ v &=& \text{total field} \\ v_{\mathrm{s}} &=& \text{scattered field.} \end{array}$$

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

$$|R|^2 + |T|^2 = 1.$$

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

$$|R|^2 + |T|^2 = 1.$$

DEFINITION: Defect is said	non reflecting if $R = 0$ ($|T| = 1$) perfectly invisible if $T = 1$ ($R = 0$)

- For $T = 1$, defect cannot be detected from far field measurements.

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

$$|R|^2 + |T|^2 = 1.$$

DEFINITION: Defect is said
- non reflecting if $R = 0$ ($|T| = 1$)
- perfectly invisible if $T = 1$ ($R = 0$)
- completely reflecting if $T = 0$ ($|R| = 1$).

- For $T = 1$, defect cannot be detected from far field measurements.
- For $T = 0$, defect is like a mirror.

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

$$|R|^2 + |T|^2 = 1.$$

DEFINITION: Defect is said
- non reflecting if $R = 0$ ($|T| = 1$)
- perfectly invisible if $T = 1$ ($R = 0$)
- completely reflecting if $T = 0$ ($|R| = 1$).

- For $T = 1$, defect cannot be detected from far field measurements.
- For $T = 0$, defect is like a mirror.

GOAL

We explain how to construct waveguides such that

$$R = 0 \ (|T| = 1), \quad T = 1 \ (R = 0) \quad \text{or} \quad T = 0 \ (|R| = 1).$$

Invisibility and complete reflectivity

- At infinity, one measures the reflection coefficient $R = s^-$ and/or the transmission coefficient $T = 1 + s^+$ (other terms are too small).

- From conservation of energy, one has

$$|R|^2 + |T|^2 = 1.$$

DEFINITION: Defect is said
- non reflecting if $R = 0$ ($|T| = 1$)
- perfectly invisible if $T = 1$ ($R = 0$)
- completely reflecting if $T = 0$ ($|R| = 1$).

  - For $T = 1$, defect cannot be detected from far field measurements.
  - For $T = 0$, defect is like a mirror.

GOAL

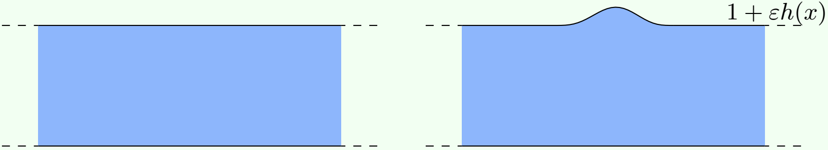
We explain how to construct waveguides such that

$$R = 0 \ (|T| = 1), \ T = 1 \ (R = 0) \ \text{ or } \ T = 0 \ (|R| = 1).$$

- We assume that k is given ($\neq$ A.-S. Bonnet-Ben Dhia's talk last Mond.).

First idea

- **Perturbative** technique: we can construct small non reflecting defects using the implicit functions theorem.



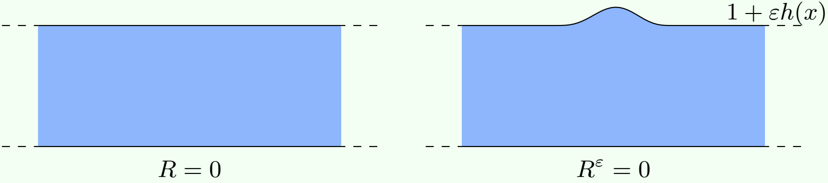
$R = 0$ $\qquad\qquad$ $R^{\varepsilon} = 0$

$\Rightarrow$ We obtain small defects such that $R = 0$ (harder to get $T = 1$).

Biblio.: Bonnet-Nazarov 13, Bonnet et al. 16.

First idea

► Perturbative technique: we can construct small non reflecting defects using the implicit functions theorem.



⇒ We obtain small defects such that $R=0$ (harder to get $T=1$).

Biblio.: Bonnet-Nazarov 13, Bonnet et al. 16.

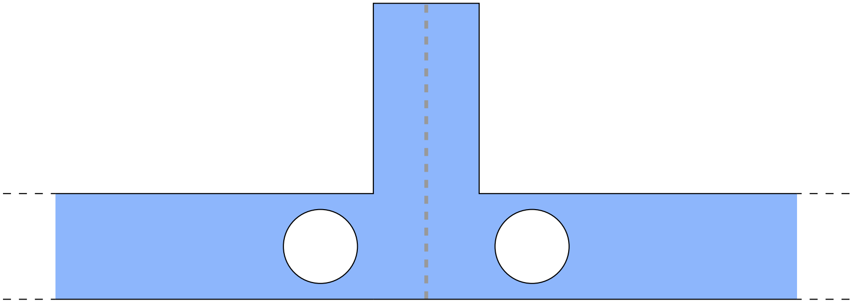
TALK

We propose another mechanism to get **large defects** s. t.

$R=0$ ($|T|=1$), $T=1$ ($R=0$) or $T=0$ ($|R|=1$).

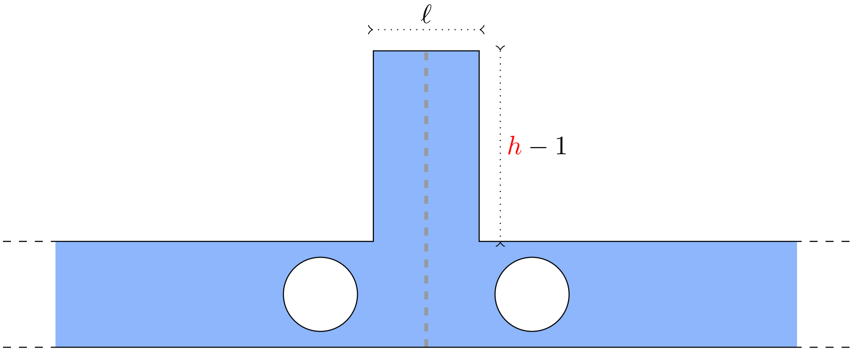
Geometrical setting

- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



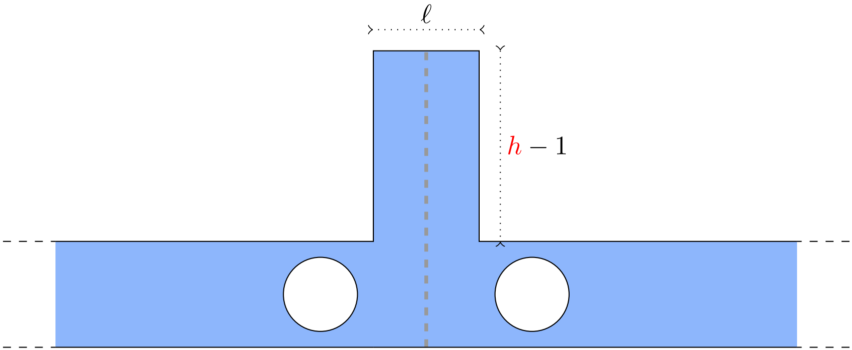
Geometrical setting

► We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



Geometrical setting

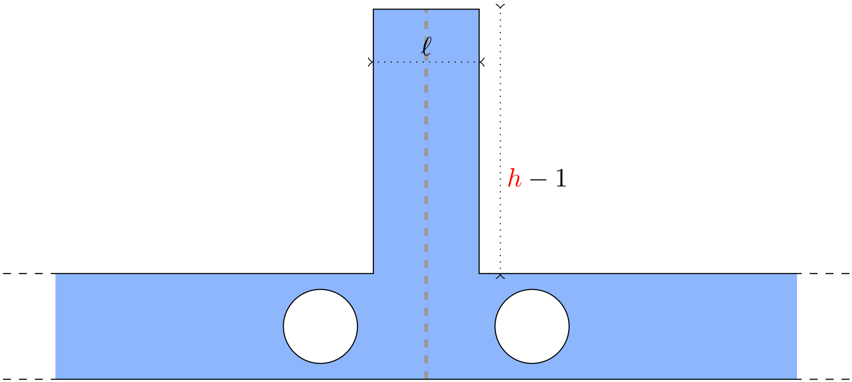
- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



$\rightarrow$ We will study the behaviour of the coefficients R, $T \in \mathbb{C}$ as $h \rightarrow +\infty$.

Geometrical setting

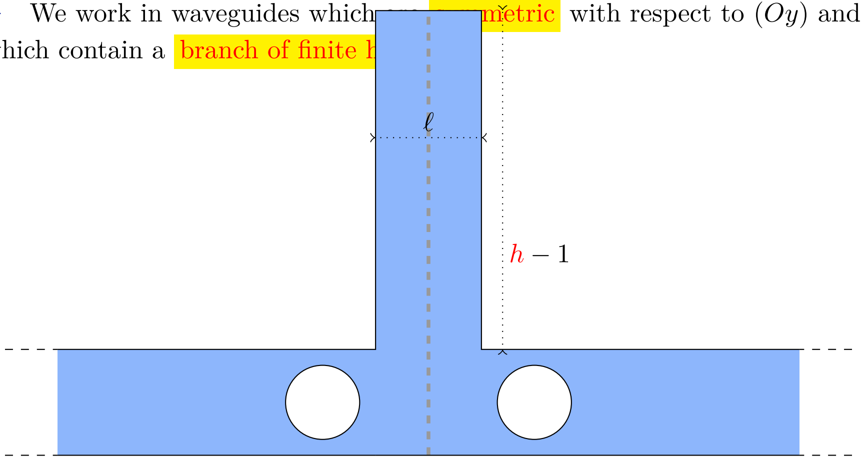
- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite height.



$\rightarrow$ We will study the behaviour of the coefficients R, $T \in \mathbb{C}$ as $h \to +\infty$.

Geometrical setting

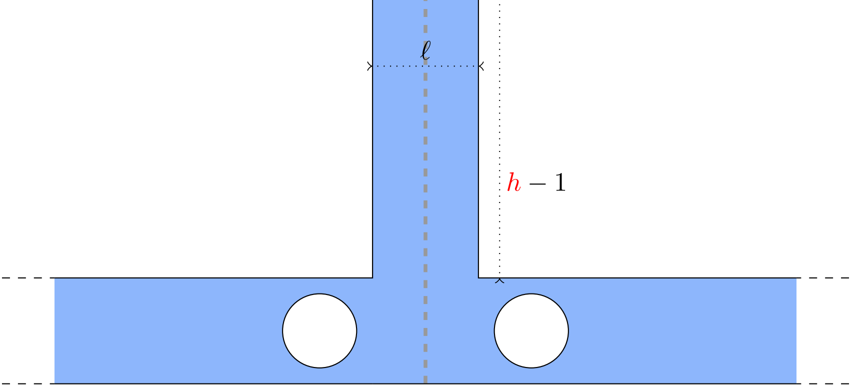
- We work in waveguides which are symmetric with respect to (Oy) and which contain a branch of finite h



$\rightarrow$ We will study the behaviour of the coefficients R, $T \in \mathbb{C}$ as $h \to +\infty$.

Geometrical setting

- We work in waveguides which metric with respect to (Oy) and which contain a branch of finite h



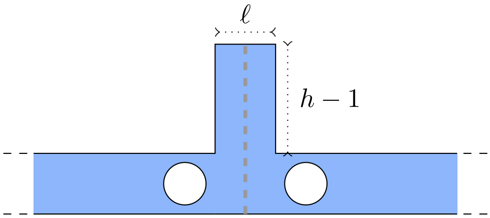
$\rightarrow$ We will study the behaviour of the coefficients R, $T \in \mathbb{C}$ as $h \rightarrow +\infty$.

Outline of the talk

1. Main analysis
2. Numerical results
3. Variants and extensions

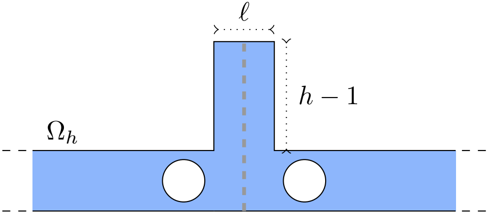
Half-waveguide problems

► Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



Half-waveguide problems

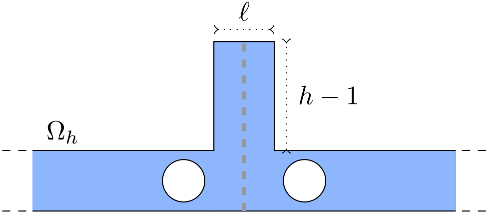
► Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



$$\left|\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega_h \\ \partial_n v & = & 0 & \text{on } \partial\Omega_h \end{array}\right.$$

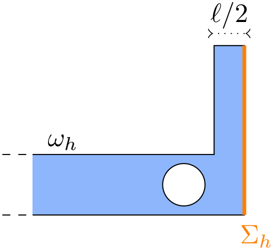
Half-waveguide problems

- Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



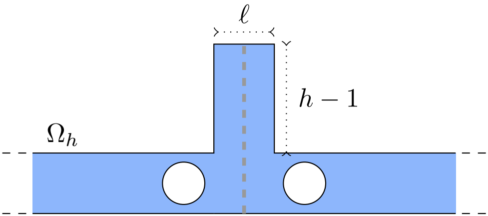
$$\left| \begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega_h \\ \partial_n v & = & 0 & \text{on } \partial\Omega_h \end{array} \right.$$

- Introduce the two half-waveguide problems



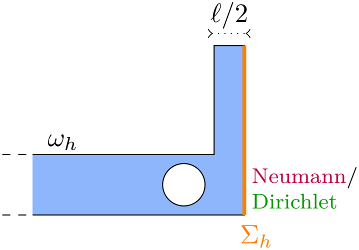
Half-waveguide problems

- Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



$$\left|\begin{array}{rcll} -\Delta v & = & k^2 v & \text{in } \Omega_h \\ \partial_n v & = & 0 & \text{on } \partial\Omega_h \end{array}\right.$$

- Introduce the two half-waveguide problems

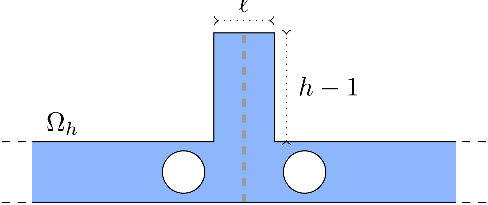


$$\left|\begin{array}{rcll} -\Delta u & = & k^2 u & \text{in } \omega_h \\ \partial_n u & = & 0 & \text{on } \partial\omega_h \end{array}\right.$$

$$\left|\begin{array}{rcll} -\Delta U & = & k^2 U & \text{in } \omega_h \\ \partial_n U & = & 0 & \text{on } \partial\omega_h \setminus \Sigma_h \\ U & = & 0 & \text{on } \Sigma_h. \end{array}\right.$$

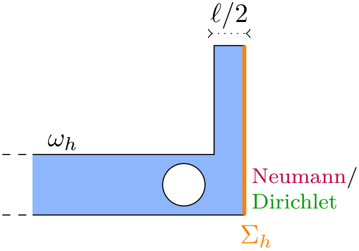
Half-waveguide problems

- Consider a waveguide which is symmetric with respect to (Oy) and which contains a branch of finite height.



$$\left|\begin{array}{rcll} -\Delta v &=& k^2 v & \text{in } \Omega_h \\ \partial_n v &=& 0 & \text{on } \partial\Omega_h \end{array}\right.$$

- Introduce the two half-waveguide problems



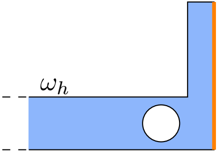
$$\left|\begin{array}{rcll} -\Delta u &=& k^2 u & \text{in } \omega_h \\ \partial_n u &=& 0 & \text{on } \partial\omega_h \end{array}\right.$$ **Neumann B.C.**

$$\left|\begin{array}{rcll} -\Delta U &=& k^2 U & \text{in } \omega_h \\ \partial_n U &=& 0 & \text{on } \partial\omega_h \setminus \Sigma_h \\ U &=& 0 & \text{on } \Sigma_h. \end{array}\right.$$ **Mixed B.C.**

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

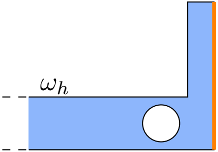
$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$



Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$
\begin{aligned}
u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h)\\
U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h).
\end{aligned}
$$



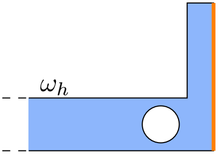
- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$

Relations for the scattering coefficients

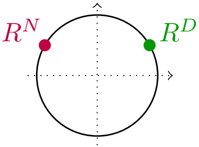
- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$



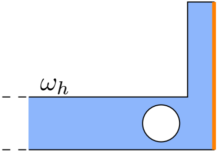
- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$

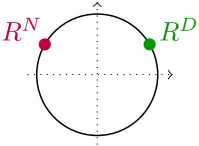


Relations for the scattering coefficients

- Half-waveguide problems admit the solutions



$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$



- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$

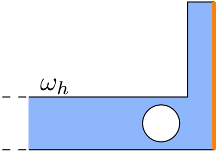
- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Relations for the scattering coefficients

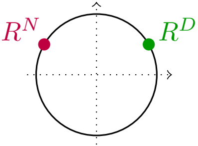
- Half-waveguide problems admit the solutions

$$
\begin{aligned}
u &= w^+ + R^N\, w^- + \tilde{u}, &\quad& \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\
U &= w^+ + R^D\, w^- + \tilde{U}, &\quad& \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h).
\end{aligned}
$$



- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



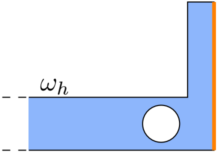
- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Non reflectivity
$\Leftrightarrow \; R^N = -R^D$

Relations for the scattering coefficients

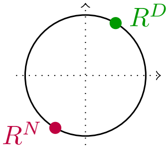
- Half-waveguide problems admit the solutions



$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, &\qquad \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, &\qquad \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$

- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

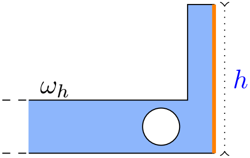
we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

Relations for the scattering coefficients

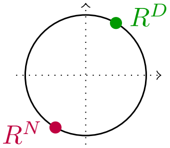
- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= w^+ + R^N\, w^- + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega_h) \\ U &= w^+ + R^D\, w^- + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega_h). \end{aligned}$$



- Due to conservation of energy, one has

$$|R^N| = |R^D| = 1.$$



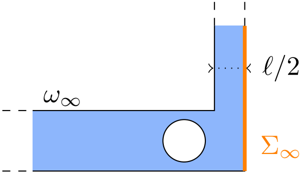
- Using that $v = \dfrac{u+U}{2}$ in ω_h, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega_h \setminus \overline{\omega_h}$,

we deduce that $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Non reflectivity
$\Leftrightarrow \quad R^N = -R^D$

$\rightarrow$ Now, we study the behaviour of $R^N = R^N(h)$, $R^D = R^D(h)$ as $h \to +\infty$.

Depend on the nb. of propagating modes in the vertical branch of ω_∞

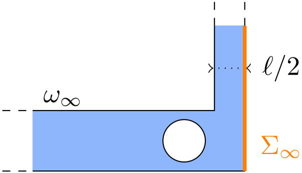


$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi &=& k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi &=& 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

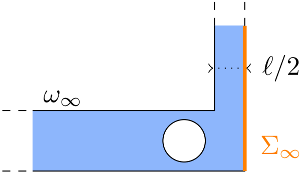
$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi &=& k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi &=& 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi &=& 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

- Analysis for R^D

- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

Asymptotics of R^N, R^D 1/2

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

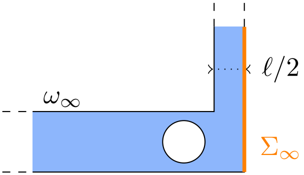
- Analysis for R^D

• For $\boldsymbol{\ell \in (0; \pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

• $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R_\infty^D\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R_\infty^D| = 1.$$

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

- Analysis for R^D

- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

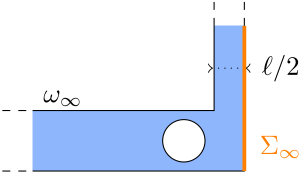
- $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R^D_\infty\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R^D_\infty| = 1.$$

($w_1^\pm = \chi_l\, w^\mp$ where χ_l is a cut-off function s.t. $\chi_l = 1$ for $x \leq -2\ell$, $\chi_l = 0$ for $x \geq -\ell$)

Asymptotics of R^N, R^D

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

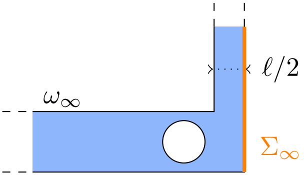
- Analysis for R^D

- For $\boldsymbol{\ell \in (0;\pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R^D_\infty\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R^D_\infty| = 1.$$

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\ \left|\ \begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\ \left|\ \begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

▶ Analysis for R^D

- For $\boldsymbol{\ell \in (0; \pi/k)}$, no prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

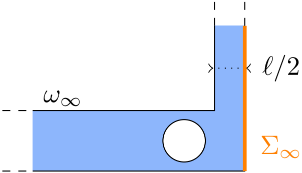
- $(\mathscr{P}^D)$ admits the solution

$$U_\infty = w_1^- + R^D_\infty\, w_1^+ + \tilde{U}_\infty, \qquad \text{with } \tilde{U}_\infty \in \mathrm{H}^1(\omega_\infty),\ |R^D_\infty| = 1.$$

- As $h \to +\infty$, we have $U = U_\infty + \dots$ which implies $|R^D - R^D_\infty| \le C\, e^{-\beta h}$.

Asymptotics of R^N, R^D 1/2

Depend on the nb. of propagating modes in the vertical branch of ω_∞



$$(\mathscr{P}^N)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \end{array}\right.$$

$$(\mathscr{P}^D)\left|\begin{array}{rcll} -\Delta\varphi & = & k^2\varphi & \text{in } \omega_\infty \\ \partial_n\varphi & = & 0 & \text{on } \partial\omega_\infty \setminus \Sigma_\infty \\ \varphi & = & 0 & \text{on } \Sigma_\infty. \end{array}\right.$$

- Analysis for R^D

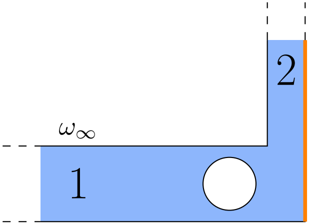
For $\boldsymbol{\ell \in (0; \pi/k)}$, $h \mapsto R^D(h)$ tends to a constant on $\mathscr{C} := \{z \in \mathbb{C},\ |z| = 1\}$.

Asymptotic of R^N, R^D 2/2

- Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$



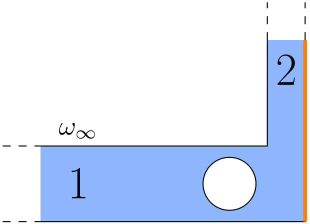
Asymptotic of R^N, R^D

▶ $\boxed{\text{Analysis for } R^N}$

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$

(χ_t is a cut-off function such that $\chi_t = 1$ for $y \geq 2$, $\chi_t = 0$ for $y \leq 1$)

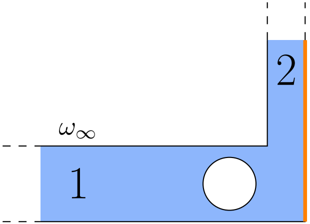


Asymptotic of R^N, R^D

- $\boxed{\text{Analysis for } R^N}$

• For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^\pm = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$



Asymptotic of R^N, R^D

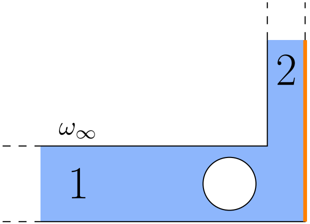
- Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^{\pm} = \chi_t\, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, \qquad \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, \qquad \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$



The scattering matrix

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$ is unitary.

- **Analysis for R^N**

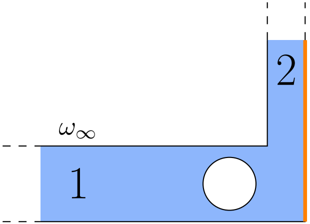
- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^\pm = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, && \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, && \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$

- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.



The scattering matrix

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$ is unitary.

Asymptotic of R^N, R^D

▶ Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

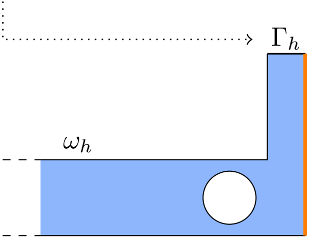
$$w_2^{\pm} = \chi_t\, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, \qquad &&\text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, \qquad &&\text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$

- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.

On Γ_h $\qquad 0 = \partial_n u = C\,(s_{12}e^{ikh} + a(h)\,(-e^{-ikh} + s_{22}e^{ikh})) + \dots$.



Asymptotic of R^N, R^D 2/2

- Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^\pm = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, & \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, & \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$

- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.

On Γ_h $\qquad 0 = \partial_n u = C\,(s_{12}e^{ikh} + a(h)\,(-e^{-ikh} + s_{22}e^{ikh})) + \dots$.

Asymptotic of R^N, R^D

▶ Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^\pm = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, &\quad& \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, &\quad& \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$

- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.

$$\text{On } \Gamma_h \qquad 0 = \partial_n u = C\,(s_{12}e^{ikh} + a(h)\,(-e^{-ikh} + s_{22}e^{ikh})) + \dots.$$

- This gives $a(h)$ and implies, as $h \to +\infty$,

$$\boxed{|R^N - R^N_{\text{asy}}(h)| \le C\, e^{-\beta h} \quad \text{with} \quad R^N_{\text{asy}}(h) = s_{11} + \frac{s_{12}\, s_{21}}{e^{-2ikh} - s_{22}}.}$$

- Analysis for R^N

- For $\boldsymbol{\ell \in (0; 2\pi/k)}$, 2 prop. modes in the vertical branch of ω_∞ for $(\mathscr{P}^N)$

$$w_2^\pm = \chi_t \, e^{\pm iky}/\sqrt{k\ell}$$

- $(\mathscr{P}^N)$ admits the solutions

$$\begin{aligned} u_\infty^1 &= w_1^- + s_{11}\, w_1^+ + s_{12}\, w_2^+ + \tilde{u}_\infty^1, \qquad && \text{with } \tilde{u}_\infty^1 \in \mathrm{H}^1(\omega_\infty) \\ u_\infty^2 &= w_2^- + s_{21}\, w_1^+ + s_{22}\, w_2^+ + \tilde{u}_\infty^2, \qquad && \text{with } \tilde{u}_\infty^2 \in \mathrm{H}^1(\omega_\infty). \end{aligned}$$

- If $s_{12} \neq 0$, we make the ansatz $u = u_\infty^1 + a(h)\, u_\infty^2 + \dots$.

On Γ_h $\qquad 0 = \partial_n u = C\,(s_{12}e^{ikh} + a(h)\,(-e^{-ikh} + s_{22}e^{ikh})) + \dots.$

- This gives $a(h)$ and implies, as $h \to +\infty$,

$$\boxed{|R^N - R^N_{\text{asy}}(h)| \le C\, e^{-\beta h} \quad \text{with} \quad R^N_{\text{asy}}(h) = s_{11} + \frac{s_{12}\, s_{21}}{e^{-2ikh} - s_{22}}.}$$

- Unitarity of $\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \Rightarrow h \mapsto R^N_{\text{asy}}(h)$ runs periodically on $\mathscr{C}$.

Asymptotic of R^N, R^D 2/2

- $\boxed{\text{Analysis for } R^N}$

For $\boldsymbol{\ell \in (0; 2\pi/k)}$, $h \mapsto R^N(h)$ runs continuously and almost period. on $\mathscr{C}$.

Conclusions for $\ell \in (0;\pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

Proposition: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

Conclusions for $\ell \in (0; \pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

PROPOSITION: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

PROPOSITION: There is an unbounded sequence (h_n) such that for $h = h_n$, $R^N = -R^D$ and so $R = 0$ (non reflectivity).

Conclusions for $\ell \in (0;\pi/k)$, $s_{12} \neq 0$

- Reminder: $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

PROPOSITION: Asympt. as $h \to +\infty$, R and T run on circles of radius $1/2$.

PROPOSITION: There is an unbounded sequence (h_n) such that for $h = h_n$, $R^N = -R^D$ and so $R = 0$ (non reflectivity).

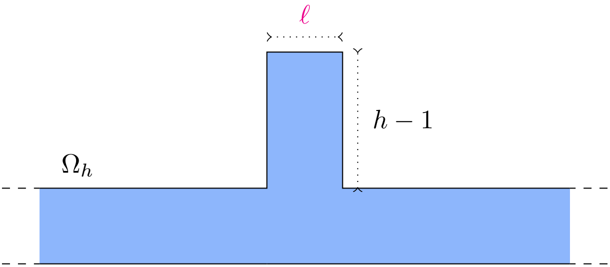
PROPOSITION: There is an unbounded sequence $(\mathcal{H}_n)$ such that for $h = \mathcal{H}_n$, $R^N = R^D$ and so $T = 0$ (complete reflectivity).

- Sequences (h_n) and $(\mathcal{H}_n)$ are almost periodic. As $n \to +\infty$, we have

$$h_{n+1} - h_n = \pi/k + \dots \qquad \text{and} \qquad \mathcal{H}_{n+1} - \mathcal{H}_n = \pi/k + \dots .$$

Setting

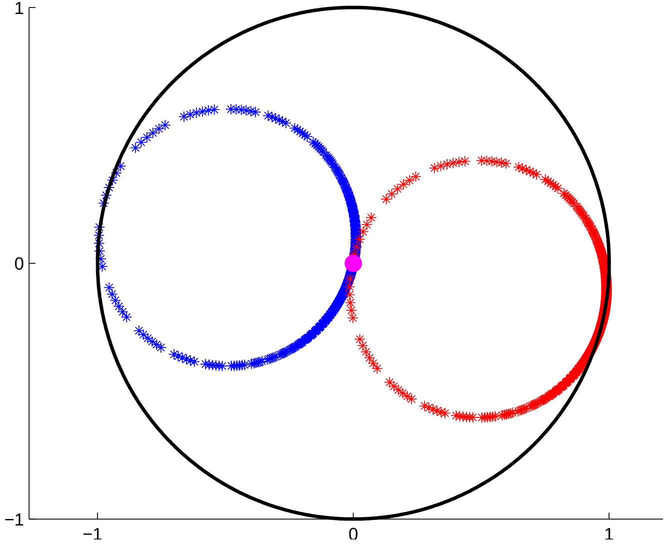
- We compute numerically R, T for $h \in (2; 10)$ in the geometry Ω_h



- We use a P2 finite element method with Dirichlet-to-Neumann maps.
- We set $k = 0.8\pi$ and $\ell = 1 \in (0; \pi/k)$.

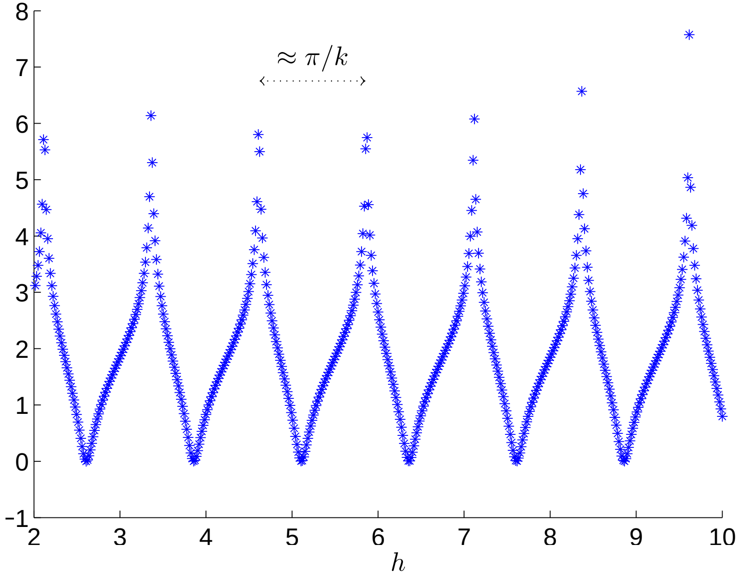
Numerical results

- Reflection coefficient R and transmission coefficient T for $h \in (2; 10)$.



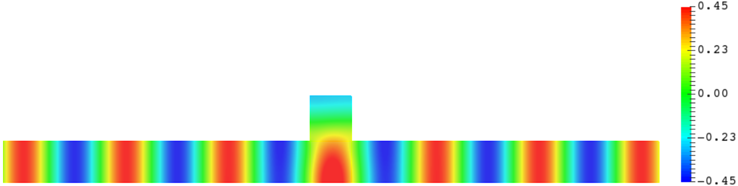
Non reflectivity

- Curve $h \mapsto -\ln|R|$. Peaks correspond to non reflectivity.

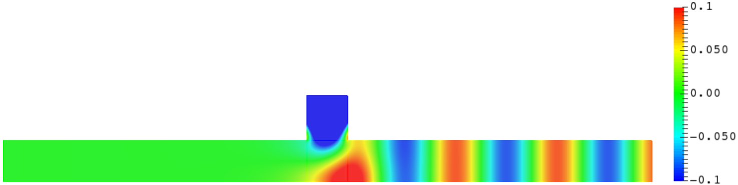


Non reflectivity

- Total field v for h such that $R = 0$.

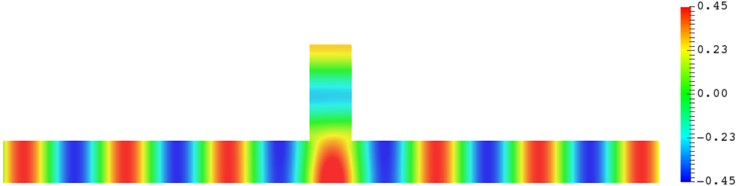


- Scattered field v_s.

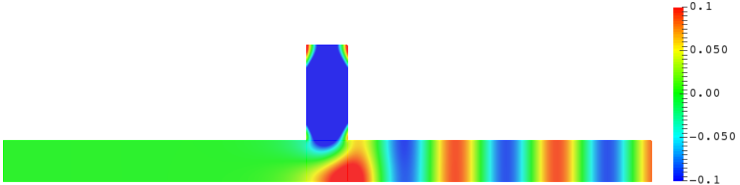


Non reflectivity

- Total field v for h such that $R = 0$.

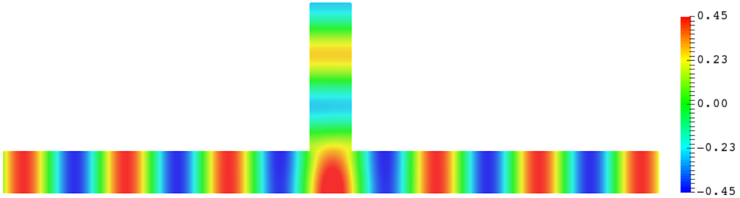


- Scattered field v_{s}.

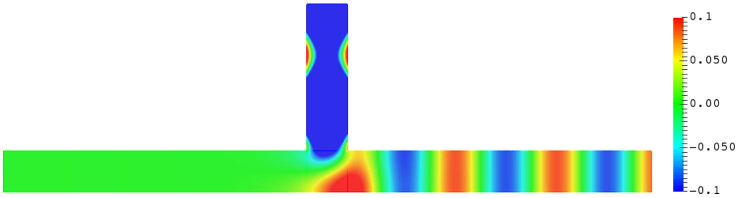


Non reflectivity

- Total field v for h such that $R = 0$.

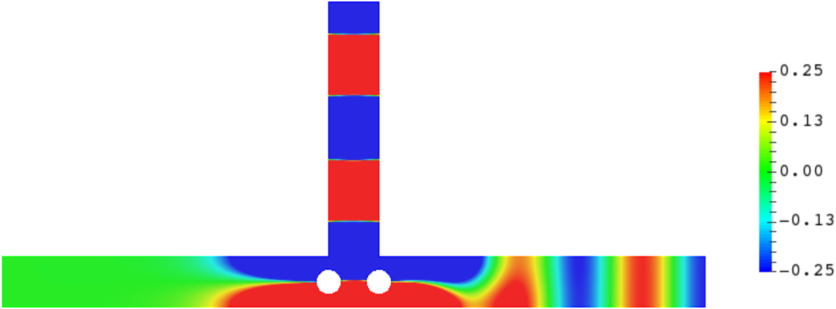


- Scattered field v_{s}.



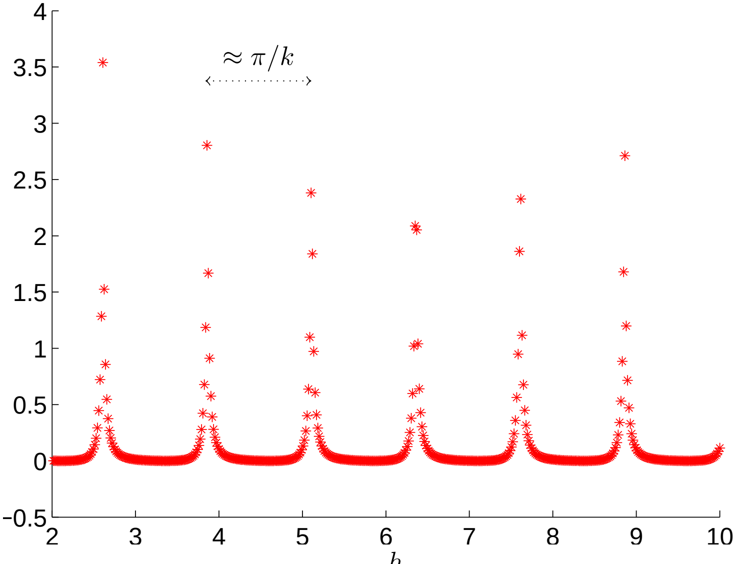
Other non reflecting geometry

- Scattered field v_s



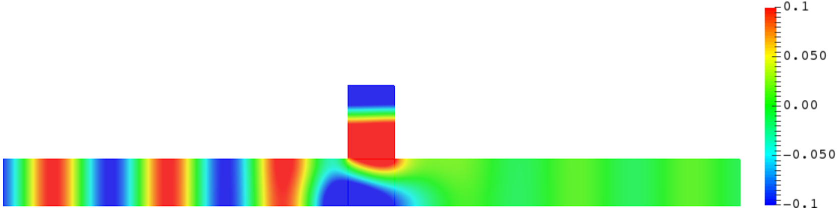
Complete reflectivity

- Curve $h \mapsto -\ln|T|$. Peaks correspond to complete reflectivity.



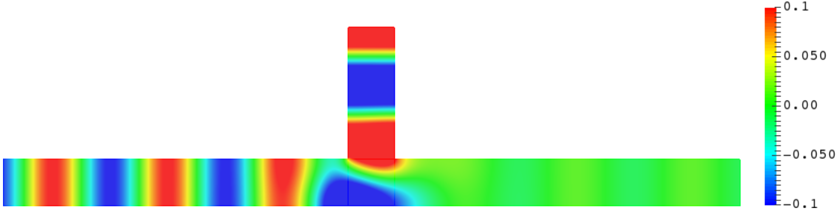
Complete reflectivity

- Total field v for h such that $T = 0$.



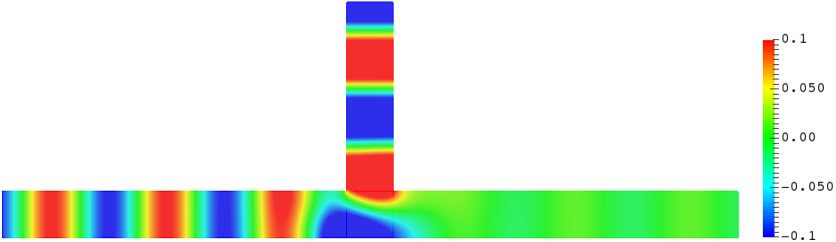
Complete reflectivity

- Total field v for h such that $\textcolor{red}{T} = 0$.

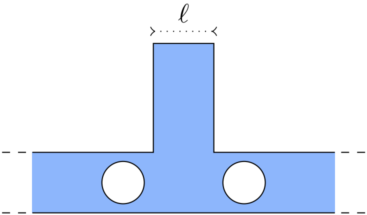


Complete reflectivity

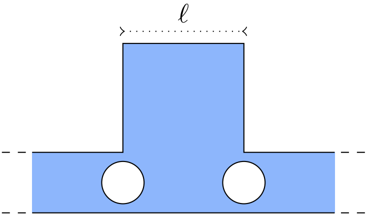
- Total field v for h such that $T = 0$.



Analysis for $\ell \in (\pi/k; 2\pi/k)$



We did $\ell \in (0; \pi/k)$



Now $\ell \in (\pi/k; 2\pi/k)$

Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

- Now 2 prop. modes exist in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

- Now 2 prop. modes exist in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- As before, we can show, with $\alpha = \sqrt{k^2 - (\pi/\ell)^2}$,

$$\boxed{|R^D - R^D_{\text{asy}}(h)| \leq C\, e^{-\beta h} \quad \text{with} \quad R^D_{\text{asy}}(h) = S_{11} + \frac{S_{12}\, S_{21}}{e^{-2i\alpha h} - S_{22}}.}$$

Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

- Now 2 prop. modes exist in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- As before, we can show, with $\alpha = \sqrt{k^2 - (\pi/\ell)^2}$,

$$|R^D - R^D_{\text{asy}}(h)| \le C\, e^{-\beta h} \quad \text{with} \quad R^D_{\text{asy}}(h) = S_{11} + \frac{S_{12}\, S_{21}}{e^{-2i\alpha h} - S_{22}}.$$

$h \mapsto R^N_{\text{asy}}(h)$, $h \mapsto R^D_{\text{asy}}(h)$ run period. on $\mathscr{C}$ with periods π/k, π/α.

Analysis for $\ell \in (\pi/k; 2\pi/k)$

- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

- Now 2 prop. modes exist in the vertical branch of ω_∞ for $(\mathscr{P}^D)$.

- As before, we can show, with $\alpha = \sqrt{k^2 - (\pi/\ell)^2}$,

$$|R^D - R^D_{\text{asy}}(h)| \le C\, e^{-\beta h} \quad \text{with} \quad R^D_{\text{asy}}(h) = S_{11} + \frac{S_{12}\, S_{21}}{e^{-2i\alpha h} - S_{22}}.$$

$h \mapsto R^N_{\text{asy}}(h)$, $h \mapsto R^D_{\text{asy}}(h)$ run period. on $\mathscr{C}$ with periods π/k, π/α.

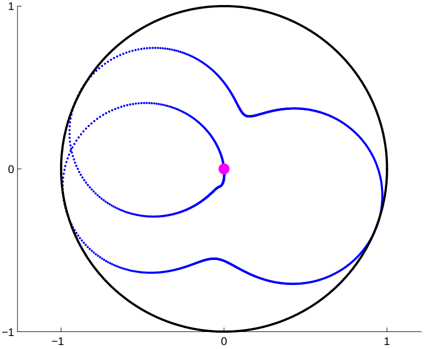
⋆ The curves $h \mapsto R(h)$, $T(h)$ still pass through zero an infinite nb. of times.

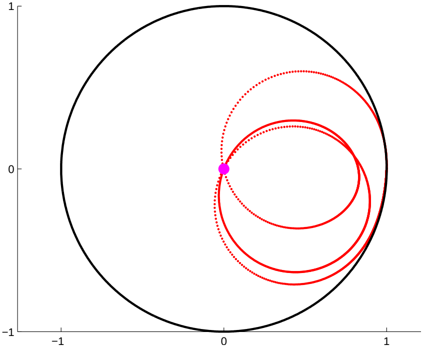
⋆ Behaviours of $h \mapsto R(h)$, $T(h)$ can be much more complex than before...

Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; +\infty)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} = 2.$$

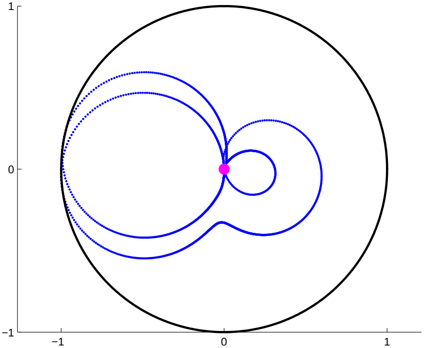


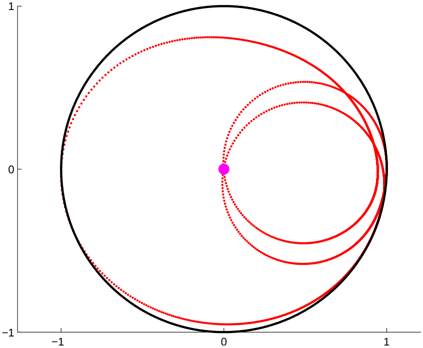


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; +\infty)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} = 3.$$

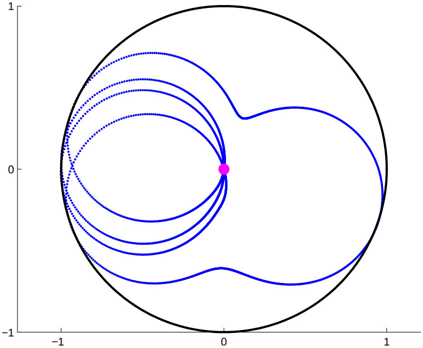


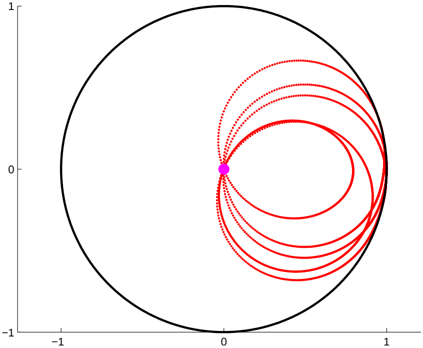


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; +\infty)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} = 4.$$

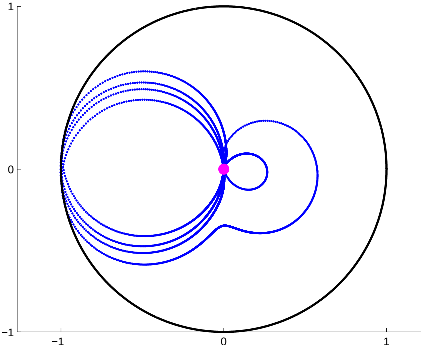


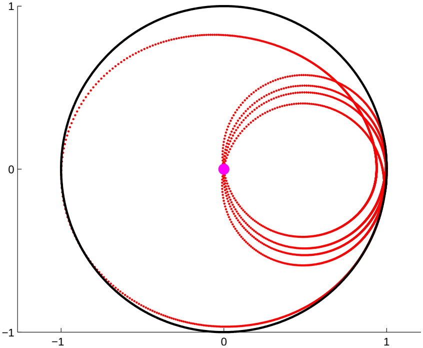


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; +\infty)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} = 5.$$

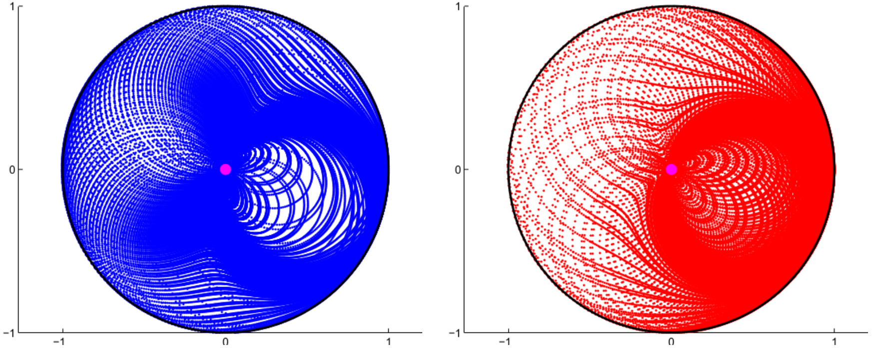




Numerical results for $\ell \in (\pi/k; 2\pi/k)$

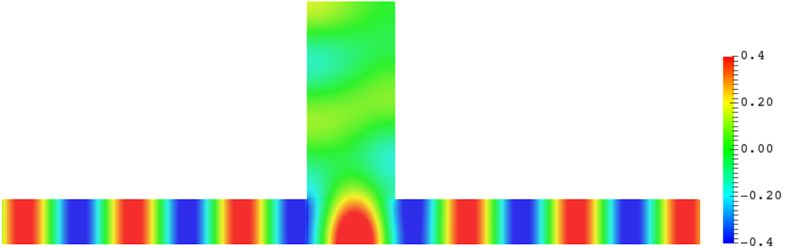
- Asympt. curves of $h \mapsto R(h)$, $T(h)$ for $h \in (0; 100)$ and ℓ such that

$$\frac{\pi/\alpha}{\pi/k} = \frac{k}{\sqrt{k^2 - (\pi/\ell)^2}} \notin \mathbb{Q}.$$

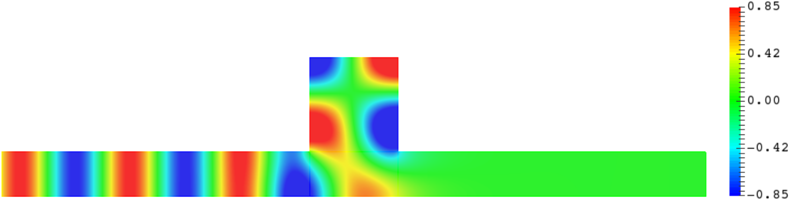


Numerical results for $\ell \in (\pi/k; 2\pi/k)$

- Non reflecting geometry ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).

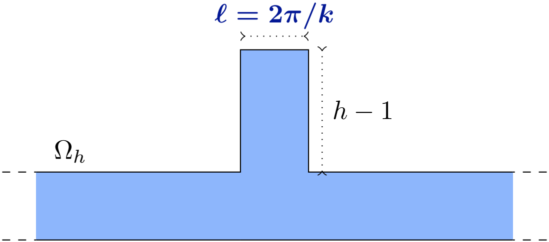


- Completely reflecting geometry ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



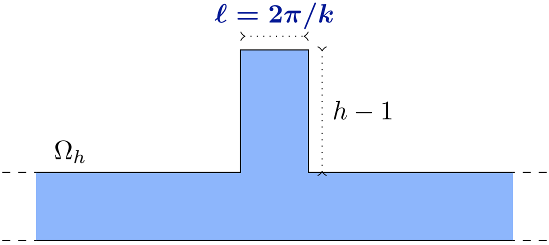
The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



The special case $\ell = 2\pi/k$

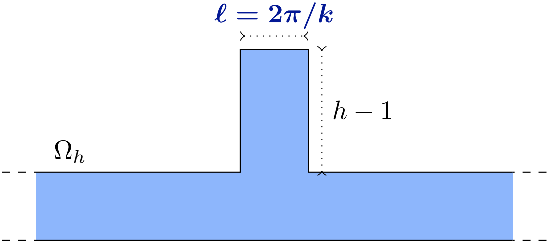
- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry

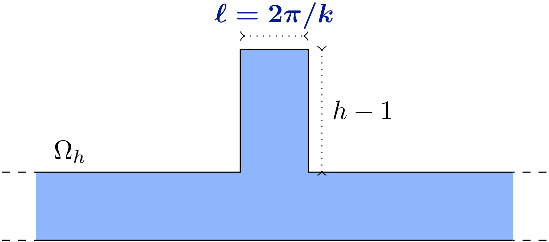


- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

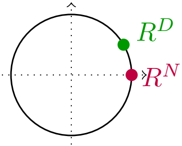
$\star$ $u = w^+ + w^- = C\,\cos(kx)$ solves the Neum. pb. in ω_h

The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



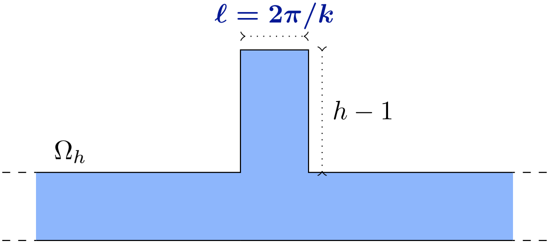
- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.



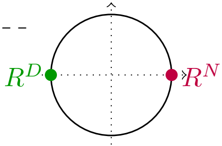
$\star$ $u = w^+ + w^- = C\cos(kx)$ solves the Neum. pb. in $\omega_h \Rightarrow R^N = 1, \forall h > 1.$

The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.

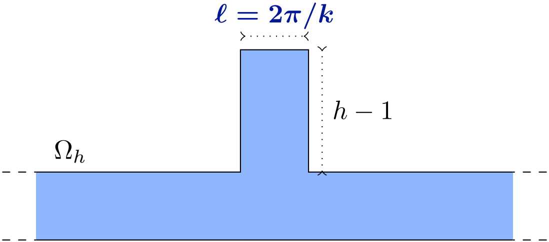


⋆ $u = w^+ + w^- = C\cos(kx)$ solves the Neum. pb. in $\omega_h \Rightarrow R^N = 1, \forall h > 1.$

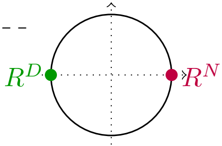
⋆ $h \mapsto R^D(h)$ still runs on the unit circle and goes through -1.

The special case $\ell = 2\pi/k$

- Now set $\boldsymbol{\ell = 2\pi/k}$ in the geometry



- We still have $\boxed{R = \dfrac{R^N + R^D}{2}}$ and $\boxed{T = \dfrac{R^N - R^D}{2}}$.



$\star$ $u = w^+ + w^- = C\cos(kx)$ solves the Neum. pb. in $\omega_h \Rightarrow R^N = 1, \forall h > 1.$

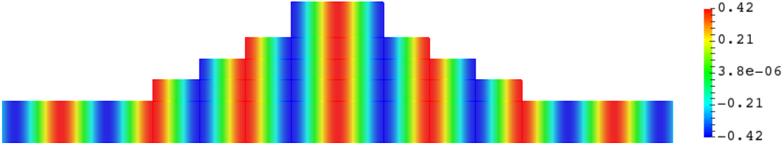
$\star$ $h \mapsto R^D(h)$ still runs on the unit circle and goes through -1.

There is a sequence (h_n) such that $T = 1$ (perfect invisibility)

The special case $\ell = 2\pi/k$ - perfect invisibility

- Works also in the geometry below (h is the height of the central branch).

- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).



- Reference waveguide ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$).

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^{\pm} = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^{\pm} = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$
$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying}.$$

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^\pm = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ and $\boxed{w_2^\pm = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$
$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^\pm = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^\pm = \dfrac{e^{-\gamma x} \mp ie^{\gamma x}}{\sqrt{2\gamma}}\cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$
$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and* $u_2 \in \mathrm{H}^1(\omega_h)$ *is a trapped mode.*

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^{\pm} = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^{\pm} = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$
$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and $u_2 \in \mathrm{H}^1(\omega_h)$ is a trapped mode.*

$\star$ $u = w_1^- + w_1^+$ solves the Neum. pb. in ω_h as in the previous slide

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^\pm = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^\pm = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$
$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and $u_2 \in \mathrm{H}^1(\omega_h)$ is a trapped mode.*

$\star$ $u = w_1^- + w_1^+$ solves the Neum. pb. in ω_h as in the previous slide

$$\Rightarrow \mathfrak{s}_{11} = 1 \qquad \Rightarrow |\mathfrak{s}_{22}| = 1, \qquad \forall h > 1.$$

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^{\pm} = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ $\quad$ and $\quad \boxed{w_2^{\pm} = \dfrac{e^{-\gamma x} \mp ie^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$
\begin{aligned}
u_1 &= w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, && \text{with } \tilde{u}_1 \text{ fastly expo. decaying} \\
u_2 &= w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, && \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}
\end{aligned}
$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and $u_2 \in \mathrm{H}^1(\omega_h)$ is a trapped mode.*

⋆ $u = w_1^- + w_1^+$ solves the Neum. pb. in ω_h as in the previous slide

$$\Rightarrow \mathfrak{s}_{11} = 1 \qquad \Rightarrow |\mathfrak{s}_{22}| = 1, \qquad \forall h > 1.$$

⋆ As previously, $h \mapsto \mathfrak{s}_{22}(h)$ runs on the unit circle and goes through -1.

The special case $\ell = 2\pi/k$ - trapped modes

- Set $\gamma = \sqrt{\pi^2 - k^2}$, $\quad w_1^{\pm} = \dfrac{e^{\mp ikx}}{\sqrt{2k}}$ and $\boxed{w_2^{\pm} = \dfrac{e^{-\gamma x} \mp i e^{\gamma x}}{\sqrt{2\gamma}} \cos(\pi y)}$.
- The Neumann problem in ω_h admits the solutions

$$u_1 = w_1^- + \mathfrak{s}_{11}\, w_1^+ + \mathfrak{s}_{12}\, w_2^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying}$$

$$u_2 = w_2^- + \mathfrak{s}_{21}\, w_1^+ + \mathfrak{s}_{22}\, w_2^+ + \tilde{u}_2, \qquad \text{with } \tilde{u}_2 \text{ fastly expo. decaying.}$$

- The augmented scattering matrix $\mathbb{S} = \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} \end{pmatrix}$ is **unitary**.

LEMMA: If $\mathfrak{s}_{22} = -1$, the Neumann problems in ω_h admits trapped modes.

Proof: $\mathfrak{s}_{22} = -1 \Rightarrow \mathfrak{s}_{21} = 0$ *($\mathbb{S}$ is unitary) and* $u_2 \in \mathrm{H}^1(\omega_h)$ *is a trapped mode.*

⋆ $u = w_1^- + w_1^+$ solves the Neum. pb. in ω_h as in the previous slide

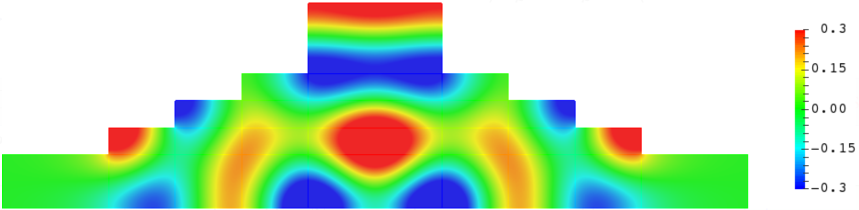
$$\Rightarrow \mathfrak{s}_{11} = 1 \qquad \Rightarrow |\mathfrak{s}_{22}| = 1, \qquad \forall h > 1.$$

⋆ As previously, $h \mapsto \mathfrak{s}_{22}(h)$ runs on the unit circle and goes through -1.

There is a sequence (h_n) such that trapped modes exist in ω_h.

The special case $\ell = 2\pi/k$ - trapped modes

- Symmetry argument w.r.t. (Oy) $\Rightarrow$ existence of trapped modes in Ω_h. It works also in the geometry below (h is the height of the central branch).



Non zero $v \in \mathrm{H}^1(\Omega_h)$ satisfying $\Delta v + k^2 v = 0$ in Ω_h, $\partial_n v = 0$ on $\partial\Omega_h$.

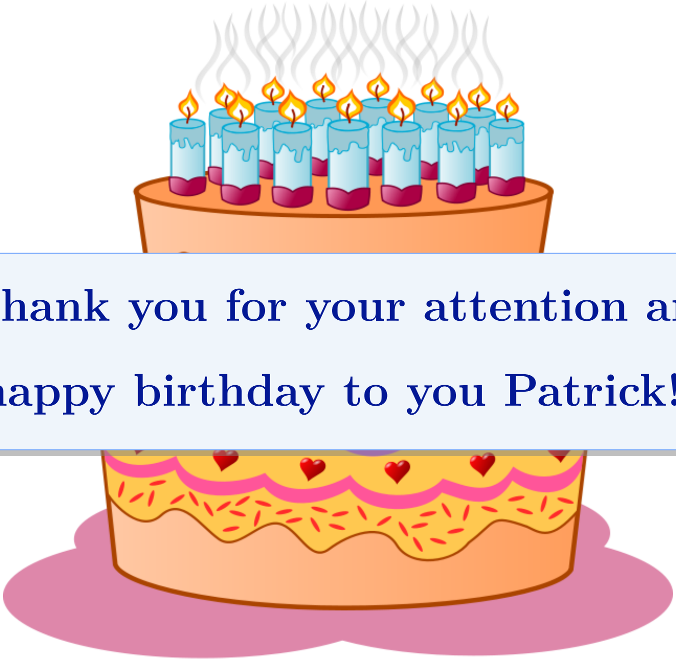
Conclusion

What we did

- ♠ We explained how to construct waveguides such that $R = 0$, $T = 0$ (the method works also for the Dirichlet problem) or $T = 1$.
- ♠ We showed how to construct waveguides supporting trapped modes.

Future work

1) When the symmetry is broken, we can still do things...
2) Can we work at higher frequencies (several propagating modes)?
3) Can we deal with multi-channel waveguides?
4) For a given perturbation, can we study the frequencies such that invisibility holds? $\Rightarrow$ A.-S. Bonnet-Ben Dhia's talk last Monday.



Thank you for your attention and happy birthday to you Patrick!!!