Transmission eigenvalue problems with sign-changing coefficients

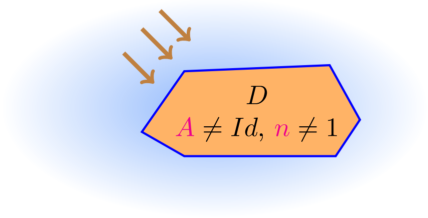
PICOF 2012

A.-S. Bonnet-Ben Dhia[†], L. Chesnel[†], P. Ciarlet[†], H. Haddar[‡]
[†]POems team, Ensta, Paris, France
[‡]DeFI team, École Polytechnique, Palaiseau, France

École Polytechnique, Palaiseau, France, April 2–4, 2012

Presentation of the ITEP

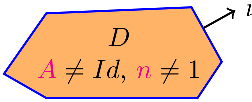
- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.



Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):



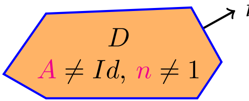
Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):

  ☛ u is the total field in D

$$\left| \begin{array}{rcll} \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \end{array} \right.$$



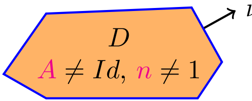
Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):

  ☛ u is the total field in D ☛ w is the incident field in D

$$
\left|\begin{array}{lcll}
\operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\
\Delta w + k^2 w & = & 0 & \text{in } D
\end{array}\right.
$$

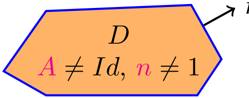


Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):
  - ☛ u is the total field in D
  - ☛ w is the incident field in D

$$\left| \begin{array}{lcll} \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array} \right.$$



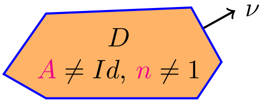
TRANSMISSION CONDITIONS ON ∂D

Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):
  - ☛ u is the total field in D
  - ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \text{Find } (u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \text{ such that:} & & & \\ \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array}\right.$$



TRANSMISSION CONDITIONS ON ∂D

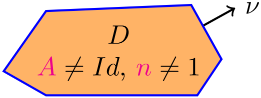
Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):

  ☛ u is the total field in D ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \text{Find } (u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) & \text{such that:} & & \\ \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu \cdot A\nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial D. \end{array}\right.$$



TRANSMISSION CONDITIONS ON ∂D

DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u,w) are called transmission eigenvalues.

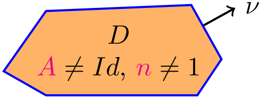
Presentation of the ITEP

- Scattering in time-harmonic regime by an inclusion D (coefficients A and n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.

- This leads to study the Interior Transmission Eigenvalue Problem (cf. F. Cakoni's talk):

  ☛ u is the total field in D ☛ w is the incident field in D

$$\left|\begin{array}{lcll} \text{Find } (u,w) \in \mathrm{H}^1(D)\times\mathrm{H}^1(D) \text{ such that:} & & & \\ \operatorname{div}(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu\cdot A\nabla u - \nu\cdot\nabla w & = & 0 & \text{on } \partial D. \end{array}\right.$$



TRANSMISSION CONDITIONS ON ∂D

DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u,w) are called transmission eigenvalues.

- The goal in this talk is to prove that the set of transmission eigenvalues is at most discrete.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X} \setminus \{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_D A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \;=\; k^2 \int_D (n u \overline{u'} - w \overline{w'}),$$

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u,w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u',w') \in \mathrm{X}$,

$$\int_D A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \;=\; k^2 \int_D (nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X} \backslash \{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_D A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \quad = \quad k^2 \int_D (n u \overline{u'} - w \overline{w'}),$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X} \backslash \{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\boxed{\int_D A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}} = k^2 \int_D (n u \overline{u'} - w \overline{w'}),$$

not coercive on X

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X} \backslash \{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\underbrace{\int_D A \nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'}}_{\text{not coercive on X}} = k^2 \underbrace{\int_D (n u \overline{u'} - w \overline{w'})}_{\text{not an inner product on X}},$$

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_D A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \quad = \quad k^2 \int_D (n u \overline{u'} - w \overline{w'}),$$

not coercive on X — not an inner product on X

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

- This problem has been widely studied since 1986-1988 (Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Sylvester, Päivärinta, Rynne, Sleeman...)

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u, w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u', w') \in \mathrm{X}$,

$$\int_D A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} = k^2 \int_D (nu\overline{u'} - w\overline{w'}),$$

not coercive on X — not an inner product on X

with $\mathrm{X} = \{(u, w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

- This problem has been widely studied since 1986-1988 (Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Sylvester, Päivärinta, Rynne, Sleeman...)

- In this talk, we want to highlight an

Variational formulation for the ITEP

- k is a transmission eigenvalue if and only if there exists $(u,w) \in \mathrm{X}\backslash\{0\}$ such that, for all $(u',w') \in \mathrm{X}$,

$$\int_D A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} \;=\; k^2 \int_D (n u\overline{u'} - w\overline{w'}),$$

not coercive on X — not an inner product on X

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$.

- This is a non standard eigenvalue problem.

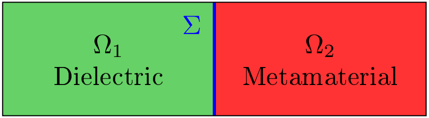
- This problem has been widely studied since 1986-1988 (Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Sylvester, Päivärinta, Rynne, Sleeman...)

- In this talk, we want to highlight an

Idea 1: Analogy with another non standard transmission problem ...

Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

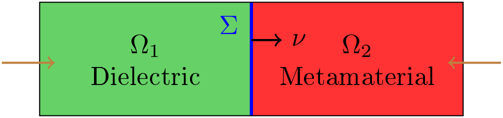
- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:



Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:

$$\varepsilon_1 := \varepsilon|_{\Omega_1} > 0 \qquad \mu_1 := \mu|_{\Omega_1} > 0$$

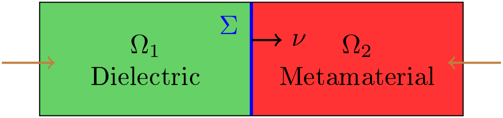


$$\varepsilon_2 := \varepsilon|_{\Omega_2} < 0 \qquad \mu_2 := \mu|_{\Omega_2} < 0$$

Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:

$$\left.\begin{array}{l}\varepsilon_1 := \varepsilon|_{\Omega_1} > 0 \\ \mu_1 := \mu|_{\Omega_1} > 0\end{array}\right|$$



$$\left|\begin{array}{l}\varepsilon_2 := \varepsilon|_{\Omega_2} < 0 \\ \mu_2 := \mu|_{\Omega_2} < 0\end{array}\right.$$

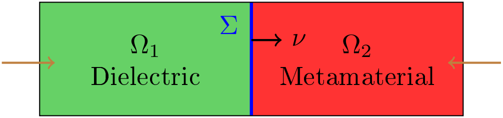
- Eigenvalue problem for E_z in 2D:

$$\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \operatorname{div}(\mu^{-1}\nabla v) + k^2 \varepsilon v = 0 \quad \text{in } \Omega.\end{array}\right.$$

Dielectric/Metamaterial Transmission Eigenvalue Problem (DMTEP)

- Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain Ω of $\mathbb{R}^2$:

$\varepsilon_1 := \varepsilon|_{\Omega_1} > 0$, $\mu_1 := \mu|_{\Omega_1} > 0$



$\varepsilon_2 := \varepsilon|_{\Omega_2} < 0$, $\mu_2 := \mu|_{\Omega_2} < 0$

- Eigenvalue problem for E_z in 2D:

$$\left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \operatorname{div}(\mu^{-1}\nabla v) + k^2 \varepsilon v = 0 \quad \text{in } \Omega. \end{array} \right.$$

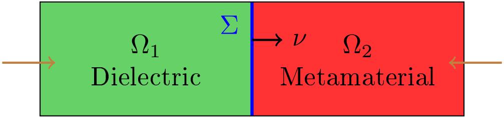
- k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}^1_0(\Omega)\backslash\{0\}$ such that, for all $v' \in \mathrm{H}^1_0(\Omega)$,

$$\int_{\Omega_1} \mu_1^{-1} \nabla v \cdot \overline{\nabla v'} - \int_{\Omega_2} |\mu_2|^{-1} \nabla v \cdot \overline{\nabla v'} = k^2 \left(\int_{\Omega_1} \varepsilon_1 v \overline{v'} - \int_{\Omega_2} |\varepsilon_2| v \overline{v'} \right).$$

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:

$$\varepsilon_1 = n, \quad \mu_1 = A$$



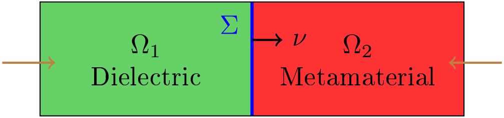
$$\varepsilon_2 = -1, \quad \mu_2 = -1$$

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:

$$\left.\begin{aligned}\varepsilon_1 &= n\\ \mu_1 &= A\end{aligned}\right|$$

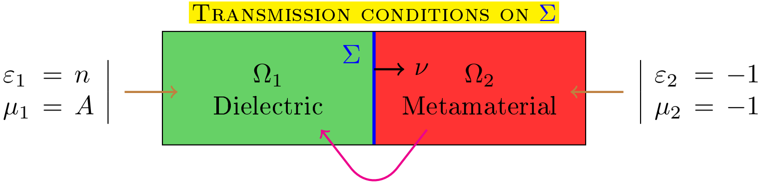
TRANSMISSION CONDITIONS ON Σ



$$\left|\begin{aligned}\varepsilon_2 &= -1\\ \mu_2 &= -1\end{aligned}\right.$$

Equivalence DMTEP/ITEP

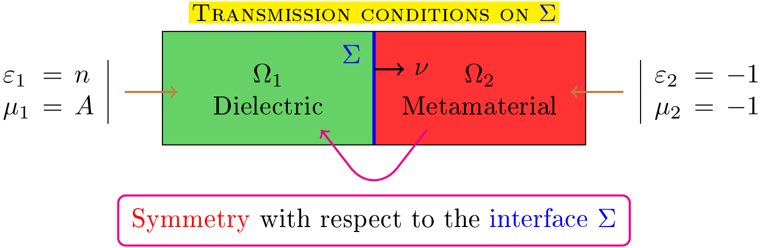
- DMTEP in the domain Ω:



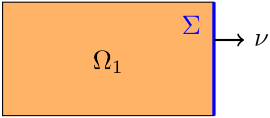
Symmetry with respect to the interface Σ

Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:

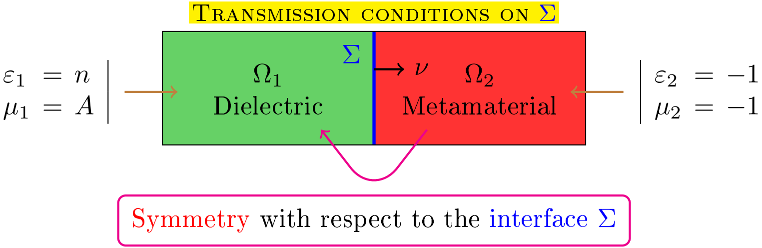


- We obtain a problem analogous to the ITEP in Ω_1:

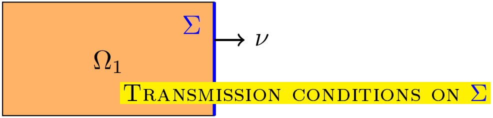


Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:

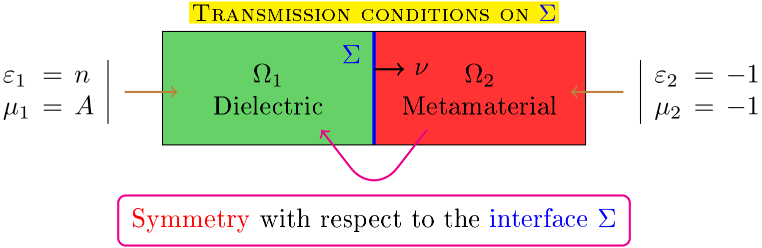


- We obtain a problem analogous to the ITEP in Ω_1:

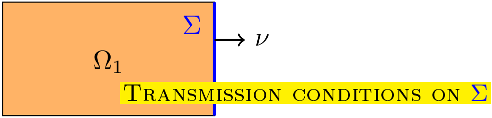


Equivalence DMTEP/ITEP

- DMTEP in the domain Ω:



- We obtain a problem analogous to the ITEP in Ω_1:



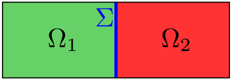
- The interface Σ in the DMTEP plays the role of the boundary ∂D in the ITEP.

Outline of the talk: three steps

1. An analogy between two transmission problems
2. The $\mathtt{T}$-coercivity method for the Dielectric/Metamaterial Transmission Problem
3. The $\mathtt{T}$-coercivity method for the Interior Transmission Problem

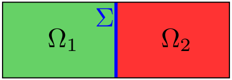
Study of the DMTP

- Problem for E_z in a symmetric 2D domain:



Study of the DMTP

- Problem for E_z in a symmetric 2D domain:

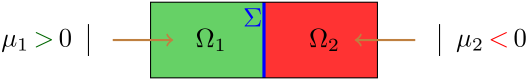


- We focus on the principal part:

$$(\mathscr{P}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1}\, \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:



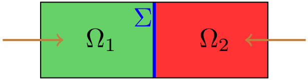
(μ_1 and μ_2 constant to simplify)

- We focus on the principal part:

$$(\mathscr{P}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1}\, \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:

$\mu_1 > 0$ |  | $\mu_2 < 0$

(μ_1 and μ_2 constant to simplify)

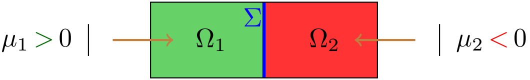
- We focus on the principal part:

$$(\mathscr{P}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1}\, \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Definition. We will say that the problem $(\mathscr{P}_V)$ is well-posed if the operator $\operatorname{div}(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:



(μ_1 and μ_2 constant to simplify)

- We focus on the principal part:

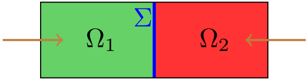
$$(\mathscr{P}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1}\, \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Definition. We will say that the problem $(\mathscr{P}_V)$ is well-posed if the operator $\operatorname{div}(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

- The form a is not coercive.

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:

$\mu_1 > 0$ |  | $\mu_2 < 0$

(μ_1 and μ_2 constant to simplify)

- We focus on the principal part:

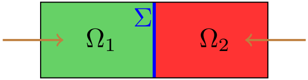
$$(\mathscr{P}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}_0^1(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1} \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}_0^1(\Omega). \end{array} \right.$$

Definition. We will say that the problem $(\mathscr{P}_V)$ is well-posed if the operator $\operatorname{div}(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}_0^1(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

- The form a is not coercive.
- For $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension to $(\mathscr{P}_V)$.

Study of the DMTP

- Problem for E_z in a symmetric 2D domain:

$\mu_1 > 0$ |  | $\mu_2 < 0$

(μ_1 and μ_2 constant to simplify)

- We focus on the principal part:

$$(\mathscr{P}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \underbrace{\int_\Omega \mu^{-1} \nabla v \cdot \nabla v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_\Omega}_{l(v')}, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Definition. We will say that the problem $(\mathscr{P}_V)$ is well-posed if the operator $\operatorname{div}(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$.

- The form a is not coercive.
- For $\mu_2 = -\mu_1$, we can build a kernel of infinite dimension to $(\mathscr{P}_V)$.

Idea 2: Use the T-coercivity approach to deal with problem $(\mathscr{P}_V)$.

Idea of the T-coercivity 1/2

Let T be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$$(\mathscr{P}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(v, v') = l(v'),\ \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Idea of the T-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$(\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}})$ $\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ a(v, \mathtt{T}v') = l(\mathtt{T}v'),\ \forall v' \in \mathrm{H}^1_0(\Omega).\end{array}\right.$

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$(\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}})$ | Find $v \in \mathrm{H}^1_0(\Omega)$ such that: $a(v, \mathtt{T}v') = l(\mathtt{T}v'),\ \forall v' \in \mathrm{H}^1_0(\Omega)$.

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla v \cdot \nabla(\mathtt{T}v) \geq C\,\|v\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

Idea of the $\mathtt{T}$-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

$(\mathscr{P}_V) \;\Leftrightarrow (\mathscr{P}_V^{\mathtt{T}})$ | Find $v \in \mathrm{H}^1_0(\Omega)$ such that: $a(v, \mathtt{T}v') = l(\mathtt{T}v'),\ \forall v' \in \mathrm{H}^1_0(\Omega)$.

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla v \cdot \nabla(\mathtt{T}v) \geq C\,\|v\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow (\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

1. Define $\mathtt{T}_1\, v = \begin{cases} v_1 & \text{in } \Omega_1 \\ -v_2 + \ldots & \text{in } \Omega_2 \end{cases}$

Idea of the T-coercivity 1/2

Let $\mathtt{T}$ be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

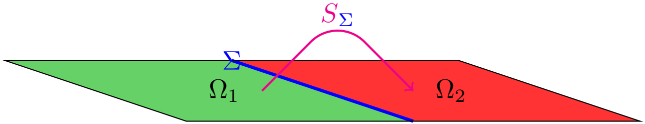
$$(\mathscr{P}_V) \;\Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(v, \mathtt{T}v') = l(\mathtt{T}v'),\ \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla v \cdot \nabla(\mathtt{T}v) \geq C\,\|v\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

1. Define $\mathtt{T}_1\, v = \left| \begin{array}{ll} v_1 & \text{in } \Omega_1 \\ -v_2 + 2S_\Sigma v_1 & \text{in } \Omega_2 \end{array} \right.$, where S_Σ is the symmetry.



Idea of the T-coercivity 1/2

Let T be an isomorphism of $\mathrm{H}^1_0(\Omega)$.

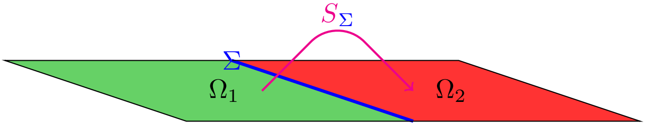
$$(\mathscr{P}_V) \Leftrightarrow (\mathscr{P}_V^{\mathtt{T}}) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ a(v, \mathtt{T}v') = l(\mathtt{T}v'),\ \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

Goal: Find T such that a is T-coercive: $\displaystyle\int_\Omega \mu^{-1}\,\nabla v \cdot \nabla(\mathtt{T}v) \geq C\,\|v\|^2_{\mathrm{H}^1_0(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

---

1. Define $\mathtt{T}_1\, v = \left| \begin{array}{ll} v_1 & \text{in } \Omega_1 \\ -v_2 + 2S_\Sigma v_1 & \text{in } \Omega_2 \end{array} \right.$, where S_Σ is the symmetry.



2. $\mathtt{T}_1 \circ \mathtt{T}_1 = Id$ so $\mathtt{T}_1$ is an isomorphism of $\mathrm{H}^1_0(\Omega)$

Idea of the T-coercivity 2/2

3. One has $a(v, \mathtt{T}_1 v) = \displaystyle\int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma\, v_1)$

Idea of the $\mathtt{T}$-coercivity 2/2

3 One has $a(v, \mathtt{T}_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1)$

Young's inequality + $\|S_\Sigma\| = 1 \Rightarrow$ a is $\mathtt{T}$-coercive when $|\mu_2| > \mu_1$.

Idea of the T-coercivity 2/2

3 One has $a(v, \mathtt{T}_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla(S_\Sigma v_1)$

Young's inequality + $\|S_\Sigma\| = 1 \Rightarrow$ a is T-coercive when $|\mu_2| > \mu_1$.

4 With $\mathtt{T}_2 v = \left| \begin{array}{ll} v_1 - 2 S_\Sigma v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{array} \right.$, a is T-coercive when $\mu_1 > |\mu_2|$.

Idea of the $\mathtt{T}$-coercivity 2/2

3. One has $a(v, \mathtt{T}_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1)$

Young's inequality + $\|S_\Sigma\| = 1 \Rightarrow$ a is $\mathtt{T}$-coercive when $|\mu_2| > \mu_1$.

4. With $\mathtt{T}_2 v = \left| \begin{array}{ll} v_1 - 2 S_\Sigma v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{array} \right.$, a is $\mathtt{T}$-coercive when $\mu_1 > |\mu_2|$.

5. Conclusion:

THEOREM. The operator $\mathrm{div}\,(\mu^{-1} \nabla \cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$ if and only if the contrast $\kappa_\mu = \mu_1 / \mu_2$ satisfies $\kappa_\mu \neq -1$.

Idea of the $\mathtt{T}$-coercivity 2/2

3. One has $a(v, \mathtt{T}_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla (S_\Sigma v_1)$

Young's inequality + $\|S_\Sigma\| = 1 \Rightarrow$ a is $\mathtt{T}$-coercive when $|\mu_2| > \mu_1$.

4. With $\mathtt{T}_2 v = \left| \begin{array}{ll} v_1 - 2 S_\Sigma v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{array} \right.$, a is $\mathtt{T}$-coercive when $\mu_1 > |\mu_2|$.

5. Conclusion:

THEOREM. The operator $\mathrm{div}\,(\mu^{-1} \nabla \cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$ if and only if the contrast $\kappa_\mu = \mu_1/\mu_2$ satisfies $\kappa_\mu \neq -1$.

- By a localization process, when μ_1 and μ_2 are not constant, we can prove that $\mathrm{div}\,(\mu^{-1} \nabla \cdot)$ is of Fredholm type when

$$\inf_{\Omega_1 \cap \mathcal{V}} \mu_1 / \inf_{\Omega_2 \cap \mathcal{V}} \mu_2 < -1 \qquad \text{or} \qquad \sup_{\Omega_1 \cap \mathcal{V}} \mu_1 / \sup_{\Omega_2 \cap \mathcal{V}} \mu_2 > -1$$

where $\mathcal{V}$ is a neighbourhood of Σ.

Idea of the T-coercivity 2/2

3. One has $a(v, \mathtt{T}_1 v) = \int_\Omega |\mu|^{-1} |\nabla v|^2 - 2 \int_{\Omega_2} \mu_2^{-1} \nabla v \cdot \nabla(S_\Sigma v_1)$

Young's inequality + $\|S_\Sigma\| = 1 \Rightarrow$ a is T-coercive when $|\mu_2| > \mu_1$.

4. With $\mathtt{T}_2 v = \left| \begin{array}{ll} v_1 - 2S_\Sigma v_2 & \text{in } \Omega_1 \\ -v_2 & \text{in } \Omega_2 \end{array} \right.$, a is T-coercive when $\mu_1 > |\mu_2|$.

5. Conclusion:

Theorem. The operator $\text{div}\,(\mu^{-1}\nabla\cdot)$ is an isomorphism from $\mathrm{H}^1_0(\Omega)$ to $\mathrm{H}^{-1}(\Omega)$ if and only if the contrast $\kappa_\mu = \mu_1/\mu_2$ satisfies $\kappa_\mu \neq -1$.

- By a localization process, when μ_1 and μ_2 are not constant, we can prove that $\text{div}\,(\mu^{-1}\nabla\cdot)$ is of Fredholm type when

$$\inf_{\Omega_1\cap\mathcal{V}} \mu_1 / \inf_{\Omega_2\cap\mathcal{V}} \mu_2 < -1 \qquad \text{or} \qquad \sup_{\Omega_1\cap\mathcal{V}} \mu_1 / \sup_{\Omega_2\cap\mathcal{V}} \mu_2 > -1$$

where $\mathcal{V}$ is a neighbourhood of Σ.

- This technique also allows to deal with non symmetric configurations.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.

Study of the ITEP

- Define on $\mathrm{X}\times\mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u\cdot\overline{\nabla u'} - \nabla w\cdot\overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w)\in \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\,|\,u-w\in\mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u-2w,-w)$.

- For $k\in\mathbb{R}i\backslash\{0\}$, $A > Id$ and $n>1$, one finds

$$\Re e\, a((u,w),\mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}),\quad \forall (u,w)\in\mathrm{X}.$$

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.

- For $k \in \mathbb{R}i \backslash \{0\}$, $A > Id$ and $n > 1$, one finds

$$\Re e\, a((u,w), \mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}), \quad \forall (u,w) \in \mathrm{X}.$$

- Using the analytic Fredholm theorem, one deduces the

> **Proposition.** Suppose that $A > Id$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.

Study of the ITEP

- Define on $\mathrm{X} \times \mathrm{X}$ the sesquilinear form

$$a((u,w),(u',w')) = \int_\Omega A\nabla u \cdot \overline{\nabla u'} - \nabla w \cdot \overline{\nabla w'} - k^2(nu\overline{u'} - w\overline{w'}),$$

with $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \,|\, u - w \in \mathrm{H}^1_0(\Omega)\}$.

- Introduce the isomorphism $\mathtt{T}(u,w) = (u - 2w, -w)$.

- For $k \in \mathbb{R}i \setminus \{0\}$, $A > Id$ and $n > 1$, one finds

$$\Re e\, a((u,w), \mathtt{T}(u,w)) \geq C\,(\|u\|^2_{\mathrm{H}^1(\Omega)} + \|w\|^2_{\mathrm{H}^1(\Omega)}), \quad \forall (u,w) \in \mathrm{X}.$$

- Using the analytic Fredholm theorem, one deduces the

PROPOSITION. Suppose that $A > Id$ and $n > 1$. Then the set of transmission eigenvalues is discrete and countable.

- This result can be extended to situations where $A - Id$ and $n - 1$ change sign in Ω working with $\mathtt{T}(u,w) = (u - 2\chi w, w)$).

ITEP when $A = Id$

- When $A = Id$, the ITP is not of Fredholm type in X likewise the DMTP is not of Fredholm type in $H_0^1(\Omega)$ when $\mu_1 = -\mu_2$.

ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in \mathrm{H}_0^2(D)$: k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}_0^2(D) \backslash \{0\}$ such that, for all $v' \in \mathrm{H}_0^2(D)$,

$$\int_D \frac{1}{1-n}(\Delta v + k^2 n v)(\Delta v' + k^2 v') = 0.$$

- We focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}_0^2(D) \text{ such that:} \\ \underbrace{\int_D \frac{1}{1-n} \Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_D}_{l(v')}, \quad \forall v' \in \mathrm{H}_0^2(D). \end{array} \right.$$

ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in \mathrm{H}^2_0(D)$: k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}^2_0(D)\backslash\{0\}$ such that, for all $v' \in \mathrm{H}^2_0(D)$,

$$\int_D \frac{1}{1-n}(\Delta v + k^2 n v)(\Delta v' + k^2 v') = 0.$$

- We focus on the principal part:

$$(\mathscr{F}_V) \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(D) \text{ such that:} \\ \underbrace{\int_D \frac{1}{1-n}\Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v'\rangle_D}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(D). \end{array} \right.$$

Transmission problem with a possible sign-changing coefficient

ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in \mathrm{H}^2_0(D)$: k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}^2_0(D)\backslash\{0\}$ such that, for all $v' \in \mathrm{H}^2_0(D)$,

$$\int_D \frac{1}{1-n}(\Delta v + k^2 n v)(\Delta v' + k^2 v') = 0.$$

- We focus on the principal part:

$$(\mathscr{F}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(D) \text{ such that:} \\ \underbrace{\int_D \frac{1}{1-n}\Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v' \rangle_D}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(D). \end{array} \right.$$

Transmission problem with a possible sign-changing coefficient

Idea 3: This transmission problem is very different from DMTP.

ITEP when $A = Id$

- We change the functional framework working on the difference $v := u - w \in \mathrm{H}^2_0(D)$: k is a transmission eigenvalue if and only if there exists $v \in \mathrm{H}^2_0(D)\backslash\{0\}$ such that, for all $v' \in \mathrm{H}^2_0(D)$,

$$\int_D \frac{1}{1-n}(\Delta v + k^2 n v)(\Delta v' + k^2 v') = 0.$$

- We focus on the principal part:

$$(\mathscr{F}_V) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(D) \text{ such that:} \\ \underbrace{\int_D \frac{1}{1-n}\Delta v \Delta v'}_{a(v,v')} = \underbrace{\langle f, v'\rangle_D}_{l(v')}, \quad \forall v' \in \mathrm{H}^2_0(D). \end{array} \right.$$

TRANSMISSION PROBLEM WITH A POSSIBLE SIGN-CHANGING COEFFICIENT

Idea 3: This transmission problem is very different from DMTP.

THEOREM. The problem $(\mathscr{F}_V)$ is well-posed in the Fredholm sense as soon as $1-n$ does not change sign in a neighbourhood of ∂D.

- Proof: T-coercivity or see J. Sylvester's work for a more precise study.

Generalizations

✔ T-coercivity approach can be used for non-constant coefficients (L^∞) and other problems (Maxwell's equations, elasticity, ...).

✔ It allows to justify the convergence of standard finite element methods.

♠ What happens when $A - Id$ change sign in a neighbourhood of the boundary?

☛ For the equivalent DMTP, strong singularities appear at the interface and H^1 is no longer the appropriate functional framework. We observe a black hole phenomenon (joint work with X. Claeys).

♠ We are not able to use the T-coercivity technique to prove existence of transmission eigenvalues.
⇒ T-coercivity gives positivity but operators are no longer symmetric.

Thank you for your attention.

- A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, to appear, 2012.
- A.-S. Bonnet-Ben Dhia, L. Chesnel, H. Haddar, *On the use of T-coercivity to study the interior transmission eigenvalue problem*, C. R. Acad. Sci. Paris, Ser. I, 349:647–651, 2011.
- A.-S. Bonnet-Ben Dhia, P. Ciarlet Jr., C.M. Zwölf, *Time harmonic wave diffraction problems in materials with sign-shifting coefficients*, J. Comput. Appl. Math, 234:1912–1919, 2010, Corrigendum *J. Comput. Appl. Math.*, 234:2616, 2010.
- F. Cakoni, H. Haddar, *Transmission eigenvalues in inverse scattering theory*, submitted, 2012.
- L. Chesnel, *Interior transmission eigenvalue problem for Maxwell's equations: the T-coercivity as an alternative approach*, Inverse problems, to appear, 2012.