A bilaplacian problem with a sign-changing coefficient

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Introduction: the ITEP

- Scattering in time-harmonic regime by a penetrable inclusion $\Omega$ (coefficient $n$) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.
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\[ \Omega \]

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- This leads to study the Interior Transmission Eigenvalue Problem:
  - $u$ is the total field in $\Omega$
  - $\Delta u + k^2 n^2 u = 0$ in $\Omega$
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- This leads to study the **Interior Transmission Eigenvalue Problem**:
  - $u$ is the total field in $\Omega$
  - $w$ is the incident field in $\Omega$

\[
\begin{align*}
\Delta u + k^2 n^2 u &= 0 \quad \text{in } \Omega \\
\Delta w + k^2 w &= 0 \quad \text{in } \Omega
\end{align*}
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- This leads to study the Interior Transmission Eigenvalue Problem:

\[ u \text{ is the total field in } \Omega \]
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\begin{align*}
\Delta u + k^2 n^2 u &= 0 \quad \text{in } \Omega \\
\Delta w + k^2 w &= 0 \quad \text{in } \Omega \\
[u] &= 0 \quad \text{on } \partial \Omega \\
[n \cdot \nabla u] &= 0 \quad \text{on } \partial \Omega \\
\end{align*}
\]

BCs?

\[ u = w + 0 \text{ in } \mathbb{R}^2 \setminus \Omega. \]
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\begin{align*}
[u] &= 0 & \text{on } \partial \Omega \\
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Definition. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution $(u, w)$ are called transmission eigenvalues.
Introduction: the ITEP

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\[ \Delta u + k^2 n^2 u = 0 \quad \text{in } \Omega \]

- This leads to study the Interior Transmission Eigenvalue Problem:

  $u$ is the total field in $\Omega$ \quad $w$ is the incident field in $\Omega$

\[
\begin{align*}
\Delta u + k^2 n^2 u &= 0 \quad \text{in } \Omega \\
\Delta w + k^2 w &= 0 \quad \text{in } \Omega \\
\quad u - w &= 0 \quad \text{on } \partial \Omega \\
\nu \cdot \nabla u - \nu \cdot \nabla w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
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**Definition.** Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution $(u, w)$ are called transmission eigenvalues.
Introduction: a bilaplacian problem

- Introducing \( v = u - w \) the scattered field inside \( \Omega \)

- One of the goals is to prove that the set of transmission eigenvalues is at most discrete.

- This problem has been widely studied since 1986-1988 (Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Païvärinta, Rynne, Sleeman, Sylvester...)

- What happens when \( 1 - n^2 \) changes sign?

- Transmission problem with a sign-changing coefficient

- We define \( \sigma = (1 - n^2)^{-1} \) and we focus on the principal part:

\[
\int_{\Omega} \sigma \Delta v \Delta v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H_{20}(\Omega).
\]
Introducing $v = u - w$ the scattered field inside $\Omega$

- There holds $\Delta u + k^2 n^2 u = 0$ and $\Delta w + k^2 w = 0$ in $\Omega$. 

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What happens when $1 - n^2$ changes sign?

Transmission problem with a sign-changing coefficient

We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

Find $v \in H^2_0(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta v \Delta v' = \langle f, v' \rangle_{\Omega}, \quad \forall \ v' \in H^2_0(\Omega).$$
Introduction: a bilaplacian problem

Introducing \( v = u - w \) the scattered field inside \( \Omega \)

- There holds \( \Delta u + k^2 n^2 u = 0 \) and \( \Delta w + k^2 w = 0 \) in \( \Omega \).
- We deduce \( \Delta v + k^2 n^2 v = k^2 (1 - n^2) w \) in \( \Omega \).

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- Introducing $v = u - w$ the scattered field inside $\Omega$

- There holds $\Delta u + k^2 n^2 u = 0$ and $\Delta w + k^2 w = 0$ in $\Omega$.
- We deduce $\Delta v + k^2 n^2 v = k^2 (1 - n^2) w$ in $\Omega$.
- This implies

\[
(\Delta + k^2) \left( \frac{1}{1 - n^2} (\Delta v + k^2 n^2 v) \right) = 0 \quad \text{in } \Omega
\]
\[
v = \nu \cdot \nabla v = 0 \quad \text{on } \partial \Omega.
\]
Introducing \( v = u - w \) the scattered field inside \( \Omega \), we can write an equivalent formulation:

Find \((k, v) \in \mathbb{C} \times H^2_0(\Omega) \setminus \{0\}\) such that:

\[
\int_{\Omega} \frac{1}{1 - n^2} (\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \quad \forall v' \in H^2_0(\Omega).
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\int_{\Omega} \frac{1}{1 - n^2} (\Delta v + k^2 n^2 v) (\Delta v' + k^2 v') = 0, \quad \forall v' \in H_0^2(\Omega).
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- We define \( \sigma = (1 - n^2)^{-1} \) and we focus on the principal part:

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\begin{align*}
\text{Find} \ v \in H^2_0(\Omega) \text{ such that:} \\
\int_{\Omega} \sigma \Delta v \Delta v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^2_0(\Omega).
\end{align*}
\]
... and more generally, we study the problem:

\[
(P) \quad \text{Find } v \in X \text{ such that: }
\begin{align*}
\int_{\Omega} \sigma \Delta v \Delta v' &= \langle f, v' \rangle_{\Omega}, \\
\underbrace{a(v,v')}_{a(v,v')} + l(v') &= \forall v' \in X.
\end{align*}
\]

The form \( a \) is not coercive. Does well-posedness hold for this problem?
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The form \( a \) is not coercive. Does well-posedness hold for this problem?

1. A bilaplacian problem with mixed boundary conditions I

We study \((P)\) with \( X = H^1_0(\Delta) := \{ v \in H^1_0(\Omega) \mid \Delta v \in L^2(\Omega) \} \).
... and more generally, we study the problem:

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1. **A bilaplacian problem with mixed boundary conditions I**
   
   We study \((P)\) with \(X = \mathcal{H}^1_0(\Delta) := \{v \in \mathcal{H}^1_0(\Omega) \mid \Delta v \in L^2(\Omega)\}\).

2. **A bilaplacian problem with mixed boundary conditions II**
   
   We study \((P)\) with \(X = \mathcal{H}^1_0(\Omega) \cap \mathcal{H}^2(\Omega)\).
Outline of the talk

... and more generally, we study the problem:

\[ \text{(P)} \quad \text{Find } v \in X \text{ such that:} \]
\[ \int_{\Omega} \sigma \Delta v \Delta v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in X. \]

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1. A bilaplacian problem with mixed boundary conditions I
   We study (P) with \( X = H^1_0(\Delta) := \{ v \in H^1_0(\Omega) \mid \Delta v \in L^2(\Omega) \} \).

2. A bilaplacian problem with mixed boundary conditions II
   We study (P) with \( X = H^1_0(\Omega) \cap H^2(\Omega) \).

3. A bilaplacian problem with Dirichlet boundary conditions
   We study (P) with \( X = H^2_0(\Omega) \).
Reminder: properties of $\text{div}(\sigma \nabla \cdot)$

- In the fields of plasmonic and negative metamaterials, we study:

  $$\mathcal{F} \quad \text{Find } v \in H^1_0(\Omega) \text{ such that:}$$
  $$\int_{\Omega} \sigma \nabla v \cdot \nabla v' = \langle f, v' \rangle_\Omega, \quad \forall v' \in H^1_0(\Omega).$$

- $\Omega$ is partitioned into two domains $\Omega_1$ and $\Omega_2$. We assume that $\sigma_1 := \sigma|_{\Omega_1}$ and $\sigma_2 := \sigma|_{\Omega_2}$ are constants.
Reminder: properties of $\text{div} (\sigma \nabla \cdot)$

- In the fields of **plasmonic** and **negative metamaterials**, we study:

  (F) Find $v \in H^1_0(\Omega)$ such that:

  $$
  \int_{\Omega} \sigma \nabla v \cdot \nabla v' = \langle f, v' \rangle_{\Omega}, \quad \forall v' \in H^1_0(\Omega).
  $$

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  ![Smooth interface](image1)

  ![Interface with a corner](image2)

  ✅ (F) well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

  ✅ (F) well-posed in the Fredholm sense iff $\kappa_\sigma \notin [-I; -1/I]$, $I = (2\pi - \vartheta)/\vartheta$. 

Well-posedness depends on the smoothness of the interface and on $\sigma$ (c.f. talks given by X. Claeys and A.-S. Bonnet-Ben Dhia).
Reminder: properties of $\text{div} (\sigma \nabla \cdot)$

- In the fields of plasmonic and negative metamaterials, we study:

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Smooth interface

$$\sigma_2 < 0 \quad \sigma_1 > 0$$

Interface with a corner

$$\sigma_2 < 0 \quad \sigma_1 > 0$$

✓ $\mathcal{F}$ well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

✓ $\mathcal{F}$ well-posed in the Fredholm sense iff $\kappa_\sigma \notin [-I; -1/I]$, $I = (2\pi - \theta)/\theta$.

Well-posedness depends on the smoothness of the interface and on $\sigma$ (c.f. talks given by X. Claeys and A.-S. Bonnet-Ben Dhia).
1 A bilaplacian problem with mixed boundary conditions I

2 A bilaplacian problem with mixed boundary conditions II

3 A bilaplacian problem with Dirichlet boundary conditions
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

Find $u \in X$ such that:

\[
(P) \quad a(u, v) = l(v), \quad \forall v \in X.
\]

**Theorem.** Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Then, the operator $A: H_{10}(\Delta) \to H_{10}(\Delta)$ associated with $(P)$ is an isomorphism.

The change of sign of $\sigma$ is not a problem!
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

\[(P) \iff (P^T)\]

Find $u \in X$ such that:

$$a(u, Tv) = l(Tv), \ \forall v \in X.$$
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

\[(\mathcal{P}) \iff (\mathcal{P}^T) \mid \text{Find } u \in X \text{ such that: } a(u, Tv) = l(Tv), \forall v \in X.\]

Goal: Find $T$ such that $a$ is $T$-coercive: $\int_{\Omega} \sigma \Delta u \Delta (Tu) \geq C \| u \|^2_X$.

In this case, Lax-Milgram $\Rightarrow (\mathcal{P}^T)$ (and so $(\mathcal{P})$) well-posed.
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

\[(\mathcal{P}) \iff (\mathcal{P}^T)\mid \text{Find } u \in X \text{ such that: } a(u, T v) = l(T v), \forall v \in X.\]

Goal: Find $T$ such that $a$ is $T$-coercive: 
\[\int_{\Omega} \sigma \Delta u \Delta (T u) \geq C \|u\|_X^2.\]

In this case, Lax-Milgram $\Rightarrow$ $(\mathcal{P}^T)$ (and so $(\mathcal{P})$) well-posed.

In this section, $X = H^1_0(\Delta)$.

1. Define $T u \in H^1_0(\Omega)$ the function such that $\Delta(T u) = \sigma^{-1}\Delta u$. 

Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

$$(P) \Leftrightarrow (P^T) \bigg| \begin{align*}
\text{Find } u \in X \text{ such that: } \\
a(u,Tv) &= l(Tv), \forall v \in X.
\end{align*}$$

Goal: Find $T$ such that $a$ is $T$-coercive:
$$\int_{\Omega} \sigma \Delta u \Delta (Tu) \geq C \|u\|^2_X.$$

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In this section, $X = H^1_0(\Delta)$.

1. Define $Tu \in H^1_0(\Omega)$ the function such that $\Delta(Tu) = \sigma^{-1}\Delta u$.
2. $T$ is an isomorphism of $H^1_0(\Delta)$. 
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

\[(\mathcal{P}) \iff (\mathcal{P}^T) \quad \text{Find } u \in X \text{ such that:} \]
\[a(u, Tv) = l(Tv), \forall v \in X.\]

**Goal:** Find $T$ such that $a$ is $T$-coercive:
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\int_{\Omega} \sigma \Delta u \Delta (Tu) \geq C \|u\|_X^2.
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3. One obtains $a(u, Tu) = \int_{\Omega} \sigma \Delta u \Delta (Tu)$.
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Let $T$ be an isomorphism of $X$.

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Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

$(P) \iff (P^T)$

Find $u \in X$ such that:

$$a(u, Tv) = l(Tv), \forall v \in X.$$

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In this case, Lax-Milgram $\Rightarrow (P^T)$ (and so $(P)$) well-posed.

In this section, $X = H_0^1(\Delta)$.

1. Define $Tu \in H_0^1(\Omega)$ the function such that $\Delta(Tu) = \sigma^{-1}\Delta u$.
2. $T$ is an isomorphism of $H_0^1(\Delta)$.
3. One obtains $a(u, Tu) = \int_{\Omega} \sigma \Delta u \Delta(Tu) = \|\Delta u\|^2_{\Omega}$.

**Theorem.** Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Then, the operator $A : H_0^1(\Delta) \to H_0^1(\Delta)$ associated with $(P)$ is an isomorphism.
Mixed Boundary Conditions I

Let $T$ be an isomorphism of $X$.

$$(\mathcal{P}) \iff (\mathcal{P}^T) \quad\text{Find } u \in X \text{ such that: } a(u, Tv) = l(Tv), \forall v \in X.$$ 

Goal: Find $T$ such that $a$ is $T$-coercive: 

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1. Define $Tu \in H^1_0(\Omega)$ the function such that $\Delta(Tu) = \sigma^{-1} \Delta u$.

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**Theorem.** Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Then, the operator $A : H^1_0(\Delta) \to H^1_0(\Delta)$ associated with $(\mathcal{P})$ is an isomorphism.
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3 A bilaplacian problem with Dirichlet boundary conditions
Mixed Boundary Conditions II

In this section, \( X = H_0^1(\Omega) \cap H^2(\Omega) \).

\[
(\mathcal{P}) \quad \text{Find } u \in H_0^1(\Omega) \cap H^2(\Omega) \text{ such that:}
\]

\[
\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega).
\]
Mixed Boundary Conditions II

In this section, $X = H^1_0(\Omega) \cap H^2(\Omega)$.

\[(\mathcal{P}) \iff (\mathcal{P}^T)\]

Find $u \in H^1_0(\Omega) \cap H^2(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta(Tv) = l(Tv), \forall v \in H^1_0(\Omega) \cap H^2(\Omega).$$
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(P) \iff (P^T) \quad \text{Find } u \in H^1_0(\Omega) \cap H^2(\Omega) \text{ such that: }
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In this section, \( X = H^1_0(\Omega) \cap H^2(\Omega) \).

\[
(P) \iff (P^T) \quad \text{Find } u \in H^1_0(\Omega) \cap H^2(\Omega) \text{ such that: }
\int_{\Omega} \sigma \Delta u \Delta(Tv) = l(Tv), \forall v \in H^1_0(\Omega) \cap H^2(\Omega).
\]

1. Define \( Tu \in H^1_0(\Omega) \) the function such that \( \Delta(Tu) = \sigma^{-1}\Delta u \).

2. Assume that \( \Omega \) is convex or of class \( C^2 \).
Mixed Boundary Conditions II

In this section, \( X = H^1_0(\Omega) \cap H^2(\Omega) \).

\[(\mathcal{P}) \iff (\mathcal{P}^T) \quad \text{Find } u \in H^1_0(\Omega) \cap H^2(\Omega) \text{ such that:} \]

\[
\int_{\Omega} \sigma \Delta u \Delta(Tv) = l(Tv), \quad \forall v \in H^1_0(\Omega) \cap H^2(\Omega).
\]

1. Define \( Tu \in H^1_0(\Omega) \) the function such that \( \Delta(Tu) = \sigma^{-1} \Delta u \).
2. Assume that \( \Omega \) is convex or of class \( C^2 \). Then, \( T \) is an isomorphism of \( H^1_0(\Omega) \cap H^2(\Omega) \).
3. One obtains \( a(u, Tu) = \int_{\Omega} \sigma \Delta u \Delta(Tu) = \| \Delta u \|_{\Omega}^2 \).

**Theorem.** Assume that \( \sigma \in L^\infty(\Omega) \) is such that \( \sigma^{-1} \in L^\infty(\Omega) \). Assume that \( \Omega \) is convex or of class \( C^2 \). Then, the operator \( A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega) \) associated with \( (\mathcal{P}) \) is an isomorphism.
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**Theorem.** Assume that \( \sigma \in L^\infty(\Omega) \) is such that \( \sigma^{-1} \in L^\infty(\Omega) \). Assume that \( \Omega \) is convex or of class \( \mathcal{C}^2 \). Then, the operator \( A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega) \) associated with \( (\mathcal{P}) \) is an isomorphism.

What happens if \( \Omega \) has a reentrant corner?
i) The space of functions $\psi \in L^2(\Omega)$ s.t $\Delta \psi = 0$ in $\Omega$ and $\psi = 0$ on $\partial \Omega$, is of dimension 1, spanned by some $\zeta$. 

**Remark 1**

Define $T u \in H^1_0(\Omega)$ the function such that $\Delta(T u) = \sigma - 1 (\Delta u - a \zeta)$ with $a = (\sigma - 1 \Delta u, \zeta)_{\Omega} / (\sigma - 1 \zeta, \zeta)_{\Omega}$ (c.f. Sapongyan paradox S.A. Nazarov, G. Sweers).

**Remark 2**

One can prove that $T$ is an isomorphism of $H^1_0(\Omega) \setminus H^2(\Omega)$.

**Remark 3**

One obtains $a(u, Tu) = \int_\Omega \sigma \Delta u \Delta(T u) = \int_\Omega \Delta u (\Delta u - a \zeta) = \| \Delta u \|^2_{\Omega}$. 

**Theorem.**

Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma - 1 \in L^\infty(\Omega)$. Introduce $A : H^1_0(\Omega) \setminus H^2(\Omega) \to H^1_0(\Omega) \setminus H^2(\Omega)$ the operator associated with $(P)$. If $(\sigma - 1 \zeta, \zeta)_{\Omega} \neq 0$, then $A$ is an isomorphism. If $(\sigma - 1 \zeta, \zeta)_{\Omega} = 0$, then $A$ is Fredholm of index zero and $\text{dim ker } A = 1$. 

**Remark 4**

Polygonal $\partial \Omega$ with one reentrant corner.
i) The space of functions $\psi \in L^2(\Omega)$ s.t $\Delta \psi = 0$ in $\Omega$ and $\psi = 0$ on $\partial \Omega$, is of dimension 1, spanned by some $\zeta$.

ii) $\varphi \in H^1_0(\Omega)$ s.t. $\Delta \varphi \in L^2(\Omega)$ is in $H^2(\Omega)$ iff $(\Delta \varphi, \zeta)_\Omega = 0$. 

(Theorem)

Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma - 1 \in L^\infty(\Omega)$. Introduce $A : H^1_0(\Omega) \setminus H^2(\Omega) \rightarrow H^1_0(\Omega) \setminus H^2(\Omega)$ the operator associated with $(P)$.

If $(\sigma - 1 \zeta, \zeta)_\Omega \neq 0$, then $A$ is an isomorphism.

If $(\sigma - 1 \zeta, \zeta)_\Omega = 0$, then $A$ is Fredholm of index zero and $\dim \ker A = 1$. 

Remark 1: Define $T u \in H^1_0(\Omega)$ the function such that $\Delta(T u) = \sigma - 1 (\Delta u - a \zeta)$ with $a = (\sigma - 1 \Delta u, \zeta)_\Omega / (\sigma - 1 \zeta, \zeta)_\Omega$.

(c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)

Remark 2: One can prove that $T$ is an isomorphism of $H^1_0(\Omega)$.

Remark 3: One obtains $a (u, T u) = \int_\Omega \sigma \Delta u \Delta(T u) = \int_\Omega \Delta u (\Delta u - a \zeta) = \|\Delta u\|_\Omega^2$. 

Theorem.
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   (c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)
Polygonal \( \partial \Omega \) with one reentrant corner

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Polygonal $\partial \Omega$ with one reentrant corner

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We want $Tu \in H^2(\Omega) \Leftrightarrow (\Delta(Tu), \zeta)_\Omega = 0$
Polygonal $\partial \Omega$ with one reentrant corner

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$\iff (\sigma^{-1}(\Delta u - a\zeta), \zeta)_\Omega = 0$
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Define $Tu \in H^1_0(\Omega)$ the function such that $\Delta(Tu) = \sigma^{-1}(\Delta u - a\zeta)$ with

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Polygonal $\partial \Omega$ with one reentrant corner

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Polygonal $\partial \Omega$ with one reentrant corner

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1. Define $T u \in H^1_0(\Omega)$ the function such that $\Delta(T u) = \sigma^{-1}(\Delta u - a \zeta)$ with $a = (\sigma^{-1} \Delta u, \zeta)_\Omega / (\sigma^{-1} \zeta, \zeta)_\Omega$. (c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)

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Theorem. Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Introduce $A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega)$ the operator associated with $(\mathcal{P})$.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then $A$ is an isomorphism.
Polygonal $\partial \Omega$ with one reentrant corner

1. The space of functions $\psi \in L^2(\Omega)$ s.t $\Delta \psi = 0$ in $\Omega$ and $\psi = 0$ on $\partial \Omega$, is of dimension 1, spanned by some $\zeta$.

2. $\varphi \in H^1_0(\Omega)$ s.t. $\Delta \varphi \in L^2(\Omega)$ is in $H^2(\Omega)$ iff $(\Delta \varphi, \zeta)_\Omega = 0$.

1. Define $T u \in H^1_0(\Omega)$ the function such that $\Delta(T u) = \sigma^{-1}(\Delta u - a\zeta)$ with $a = (\sigma^{-1}\Delta u, \zeta)_\Omega/(\sigma^{-1}\zeta, \zeta)_\Omega$. (c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)

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### Theorem
Assume that $\sigma \in L^\infty(\Omega)$ is such that $\sigma^{-1} \in L^\infty(\Omega)$. Introduce $A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega)$ the operator associated with $\mathcal{P}$.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then $A$ is an isomorphism.
- If $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$, then $A$ is Fredholm of index zero and $\dim \ker A = 1$. 
Summary of the results when $X = H^1_0(\Omega) \cap H^2(\Omega)$

Find $u \in H^1_0(\Omega) \cap H^2(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta v = l(v), \forall v \in H^1_0(\Omega) \cap H^2(\Omega).$$

We introduce the operator $A : H^1_0(\Omega) \cap H^2(\Omega) \to H^1_0(\Omega) \cap H^2(\Omega)$ such that $(\Delta(Au), \Delta v)_\Omega = (\sigma \Delta u, \Delta v)_\Omega$ for all $u, v \in H^1_0(\Omega) \cap H^2(\Omega)$.

- $A$ is an isomorphism.
- $A$ is an isomorphism.
- $A$ is an isomorphism because $(\sigma^{-1} \zeta, \zeta)_\Omega \neq 0$.
- $A$ is a Fredholm operator of index 0 and $\dim \ker A = 1$ because $(\sigma^{-1} \zeta, \zeta)_\Omega = 0$. 

σ = 1  
σ = −1  
σ = 1  
σ = −2
1. A bilaplacian problem with mixed boundary conditions I

2. A bilaplacian problem with mixed boundary conditions II

3. A bilaplacian problem with Dirichlet boundary conditions
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

Find $u \in H^2_0(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).$$

Theorem. The problem $(\mathcal{P})$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$.

$\sigma = -1$ $\sigma = 1$

Fredholm... but $(\mathcal{P})$ can be ill-posed (not Fredholm) when $\sigma$ changes sign "on $\partial \Omega$".

⇒ work with J. Firozaly.
In this section, \( X = H_0^2(\Omega) \).

\[
\begin{align*}
\mathcal{P} & \quad \text{Find } u \in H_0^2(\Omega) \text{ such that:} \\
& \int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H_0^2(\Omega).
\end{align*}
\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H_0^2(\Omega) \to H^{-2}(\Omega) \) and \( \text{div} (\sigma \nabla \cdot) : H_0^1(\Omega) \to H^{-1}(\Omega) \) have very different properties.
A bilaplacian problem with Dirichlet boundary conditions

▶ In this section, \( X = H_0^2(\Omega) \).

\[(\mathcal{P}) \quad \begin{array}{l}
\text{Find } u \in H_0^2(\Omega) \text{ such that:} \\
\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H_0^2(\Omega).
\end{array}\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H_0^2(\Omega) \to H^{-2}(\Omega) \) and \( \text{div}(\sigma \nabla \cdot) : H_0^1(\Omega) \to H^{-1}(\Omega) \) have very different properties.

**THEOREM.** The problem \((\mathcal{P})\) is well-posed in the Fredholm sense as soon as \( \sigma \) does not change sign in a neighbourhood of \( \partial \Omega \).
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

Find $u \in H^2_0(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).$$

Message: The operators $\Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega)$ and $\text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega)$ have very different properties.

Theorem. The problem $(\mathcal{P})$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$.

Ideas of the proof: We have

$$a(v, u) = (\sigma \Delta u, \Delta u)_\Omega.$$

We would like to build $T : H^2_0(\Omega) \to H^2_0(\Omega)$ such that $\Delta(T v) = \sigma^{-1} \Delta v$

so that

$$a(v, T v) = (\sigma \Delta v, \Delta(T v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

Theorem: The problem $(\mathcal{P})$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$.

Fredholm

$\sigma = 1$
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

Find $u \in H^2_0(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).$$

**Ideas of the proof:** We have

$$a(v, u) = (\sigma \Delta v, \Delta v)_\Omega.$$

We would like to build $T : H^2_0(\Omega) \to H^2_0(\Omega)$ such that

$$\Delta(Tv) = \sigma^{-1} \Delta v$$

so that

$$a(v, Tv) = (\sigma \Delta v, \Delta(Tv))_\Omega = (\Delta v, \Delta v)_\Omega.$$

**Theorem.** The problem $(\mathcal{P})$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$.

Message: The operators $\Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega)$ and $\text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega)$ have very different properties.

Not simple!
A bilaplacian problem with Dirichlet boundary conditions

In this section, \( X = H^2_0(\Omega) \).

Find \( u \in H^2_0(\Omega) \) such that:
\[
\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).
\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega) \) and \( \text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega) \) have very different properties.

Theorem. The problem \((P)\) is well-posed in the Fredholm sense as soon as \( \sigma \) does not change sign in a neighborhood of \( \partial \Omega \).

Ideas of the proof: We have
\[
a(v, v) = (\sigma \Delta v, \Delta v)_\Omega.
\]
We would like to build \( T : H^2_0(\Omega) \to H^2_0(\Omega) \) such that \( \Delta(Tv) = \sigma^{-1} \Delta v \)
so that
\[
a(v, Tv) = (\sigma \Delta v, \Delta(Tv))_\Omega = (\Delta v, \Delta v)_\Omega.
\]

Let \( w \in H^1_0(\Omega) \) such that \( \Delta w = \sigma^{-1} \Delta v \).

1. \( \sigma = 1 \)
A bilaplacian problem with Dirichlet boundary conditions

In this section, \( X = H^2_0(\Omega) \).

\((P)\)

Find \( u \in H^2_0(\Omega) \) such that:

\[
\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).
\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega) \) and \( \text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega) \) have very different properties.

Theorem. The problem \((P)\) is well-posed in the Fredholm sense as soon as \( \sigma \) does not change sign in a neighbourhood of \( \partial \Omega \).

Ideas of the proof: We have

\[
a(v, v) = (\sigma \Delta v, \Delta v)_\Omega.
\]

We would like to build \( T : H^2_0(\Omega) \to H^2_0(\Omega) \) such that \( \Delta(Tv) = \sigma^{-1} \Delta v \)

so that

\[
a(v, Tv) = (\sigma \Delta v, \Delta(Tv))_\Omega = (\Delta v, \Delta v)_\Omega.
\]

1. Let \( w \in H^1_0(\Omega) \) such that \( \Delta w = \sigma^{-1} \Delta v \).

2. Let \( \zeta \in C_\infty_0(\Omega) \). Define \( Tv = \zeta w + (1 - \zeta)v \in H^2_0(\Omega) \).

Fredholm

\[
\sigma = 1
\]

Not simple!
A bilaplacian problem with Dirichlet boundary conditions

In this section, \( X = H^2_0(\Omega) \).

\([P]\) Find \( u \in H^2_0(\Omega) \) such that:

\[
\int_\Omega \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).
\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}_0(\Omega) \) and \( \text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}_0(\Omega) \) have very different properties.

Theorem. The problem \([P]\) is well-posed in the Fredholm sense as soon as \( \sigma \) does not change sign in a neighbourhood of \( \partial \Omega \).

1. Let \( w \in H^1_0(\Omega) \) such that \( \Delta w = \sigma^{-1}\Delta v \).

2. Let \( \zeta \in \mathcal{C}_0^\infty(\Omega) \). Define \( T v = \zeta w + (1 - \zeta)v \in H^2_0(\Omega) \).

3. We find

\[
a(v, T v) = ([\zeta + \sigma(1 - \zeta)]\Delta v, \Delta v)_\Omega + (Kv, v)_{H^2_0(\Omega)}
\]

where \( K : H^2_0(\Omega) \to H^2_0(\Omega) \) is compact.

Fredholm

\( \sigma = 1 \)
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

Find $u \in H^2_0(\Omega)$ such that:

$$\int_\Omega \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).$$

The operators $\Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega)$ and $\text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega)$ have very different properties.

Theorem. The problem $(P)$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$.

Ideas of the proof: We have

$$a(v, v) = (\sigma \Delta v, \Delta v)_\Omega.$$ 

We would like to build $T : H^2_0(\Omega) \to H^2_0(\Omega)$ such that $\Delta(Tv) = \sigma^{-1} \Delta v$ so that

$$a(v, Tv) = (\sigma \Delta v, \Delta(Tv))_\Omega = (\Delta v, \Delta v)_\Omega.$$

1. Let $w \in H^1_0(\Omega)$ such that $\Delta w = \sigma^{-1} \Delta v$.

2. Let $\zeta \in \mathcal{C}_0^\infty(\Omega)$. Define $Tv = \zeta w + (1 - \zeta)v \in H^2_0(\Omega)$.

3. We find

$$a(v, Tv) = (\zeta + \sigma(1 - \zeta)) \Delta v, \Delta v)_\Omega + (Kv, v)_{H^2_0(\Omega)}$$

where $K : H^2_0(\Omega) \to H^2_0(\Omega)$ is compact.

Fredholm

$$\sigma = 1$$

Not simple!
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

\[ (\mathcal{P}) \quad \int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega). \]

Message: The operators $\Delta(\sigma \Delta \cdot) : H^2_0(\Omega) \to H^{-2}(\Omega)$ and $\text{div}(\sigma \nabla \cdot) : H^1_0(\Omega) \to H^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathcal{P})$ is well-posed in the Fredholm sense as soon as $\sigma$ does not change sign in a neighbourhood of $\partial \Omega$. 
A bilaplacian problem with Dirichlet boundary conditions

In this section, \( X = H_0^2(\Omega) \).

\[
\begin{align*}
\text{(P)} & \quad \text{Find } u \in H_0^2(\Omega) \text{ such that:} \\
& \quad \int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H_0^2(\Omega).
\end{align*}
\]

Message: The operators \( \Delta(\sigma \Delta \cdot) : H_0^2(\Omega) \to H^{-2}(\Omega) \) and \( \text{div} (\sigma \nabla \cdot) : H_0^1(\Omega) \to H^{-1}(\Omega) \) have very different properties.

**Theorem.** The problem \( (\mathcal{P}) \) is well-posed in the Fredholm sense as soon as \( \sigma \) does not change sign in a neighbourhood of \( \partial \Omega \).
A bilaplacian problem with Dirichlet boundary conditions

In this section, $X = H^2_0(\Omega)$.

Find $u \in H^2_0(\Omega)$ such that:

$$\int_{\Omega} \sigma \Delta u \Delta v = l(v), \quad \forall v \in H^2_0(\Omega).$$

Message: The operators $\Delta (\sigma \Delta \cdot): H^2_0(\Omega) \to H^{-2}(\Omega)$ and $\text{div} (\sigma \nabla \cdot): H^1_0(\Omega) \to H^{-1}(\Omega)$ have very different properties.

... but $(P)$ can be ill-posed (not Fredholm) when $\sigma$ changes sign “on $\partial \Omega$”

$\Rightarrow$ work with J. Firozaly.
1. A bilaplacian problem with mixed boundary conditions I

2. A bilaplacian problem with mixed boundary conditions II

3. A bilaplacian problem with Dirichlet boundary conditions
Find \( v \in H^1_0(\Omega) \) s.t., \( \forall v' \in H^1_0(\Omega), \)
\[
\int_{\Omega} \sigma \nabla v \cdot \nabla v' = \ell(v').
\]

\begin{itemize}
  \item Smooth interface
  \begin{itemize}
    \item \( \sigma_2 < 0 \)
    \item \( \sigma_1 > 0 \)
  \end{itemize}
  Well-posed in the Fredholm sense iff \( \kappa_\sigma = \sigma_2/\sigma_1 \neq -1 \).
  \begin{itemize}
    \item Interface with a corner
    \begin{itemize}
      \item \( \sigma_2 < 0 \)
      \item \( \sigma_1 > 0 \)
    \end{itemize}
    Well-posed in the Fredholm sense iff \( \kappa_\sigma \notin [-I; -1/I], I = (2\pi - \vartheta)/\vartheta \).
  \end{itemize}
\end{itemize}

Find \( v \in X \) s.t., \( \forall v' \in X, \)
\[
\int_{\Omega} \sigma \Delta v \Delta v' = \ell(v').
\]

We assume \( \sigma \in L^\infty(\Omega), \sigma^{-1} \in L^\infty(\Omega) \).

\begin{itemize}
  \item If \( X = H^1_0(\Delta) \): Well-posed.
  \item If \( X = H^1_0(\Omega) \cap H^2(\Omega) \):
    \begin{itemize}
      \item Well-posed when \( \Omega \) is convex or of class \( C^2 \).
      \item When \( \Omega \) has one reentrant corner, it can occur a kernel of dimension 1.
    \end{itemize}
  \item If \( X = H^2_0(\Omega) \):
    \begin{itemize}
      \item Well-posed in the Fredholm sense when \( \sigma \) does not change sign on a neighbourhood of \( \partial \Omega \).
      \item When \( \sigma \) changes sign on \( \partial \Omega \), Fredholmness can be lost.
    \end{itemize}
\end{itemize}
Thank you for your attention!!!