A curious instability phenomenon for rounded corners in plasmonic metamaterials

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Coll. with A.-S. Bonnet-Ben Dhia[2], P. Ciarlet[2], C. Carvalho[2], X. Claeys[3], S.A. Nazarov[4]

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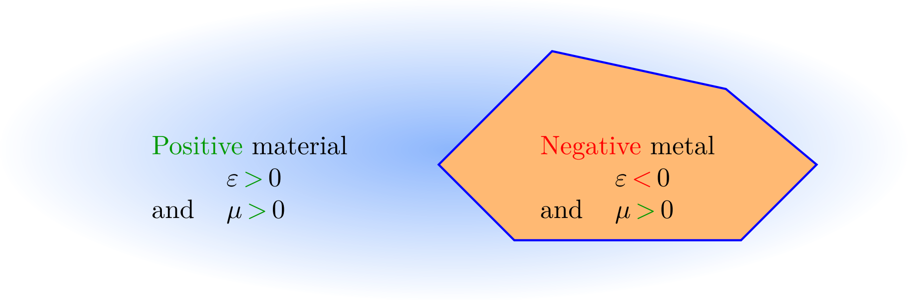
Introduction: general framework

- Scattering by a metal in electromagnetism in time-harmonic regime at optical frequency.
- For metals at optical frequency, $\Re e\,\varepsilon(\omega) < 0$ and $\Im m\,\varepsilon(\omega) << |\Re e\,\varepsilon(\omega)|$.
  $\Rightarrow$ We neglect losses and study the ideal case $\varepsilon(\omega) \in (-\infty; 0)$.



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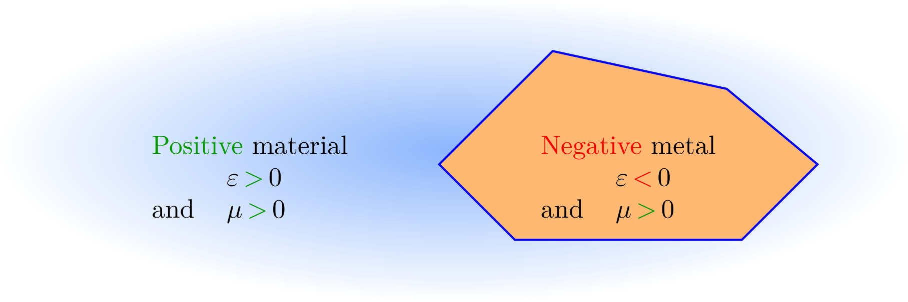
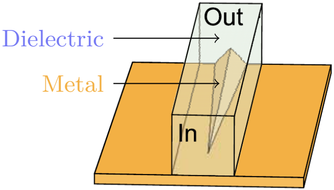
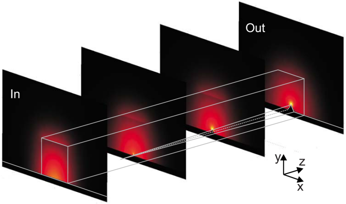


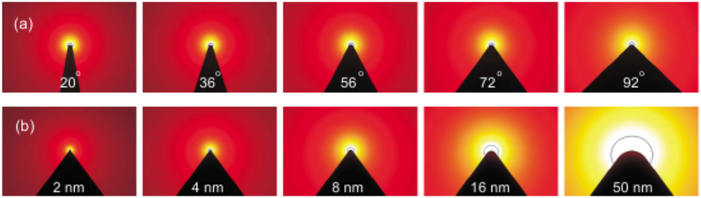
- Waves called Surface Plasmon Polaritons can propagate at the interface between a dielectric and a negative metal.

Introduction: applications

- Surface Plasmons Polaritons can propagate information. Physicists hope to exploit them to reduce the size of computer chips.







Figures from `O'Connor et al., Appl. Phys. Lett. 95, 171112 (2009)`

- In this context, physicists use singular geometries to focus energy. It allows to stock information.

Introduction: in this talk

- We study a scalar model problem set in a bounded domain $\Omega \subset \mathbb{R}^2$:

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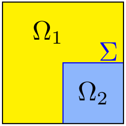

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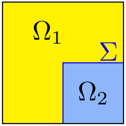

  - $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
  - f is the source term in $\mathrm{H}^{-1}(\Omega)$

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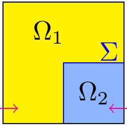

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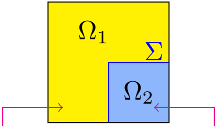

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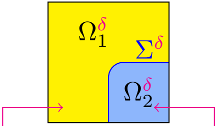

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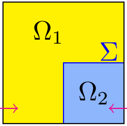

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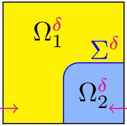

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? What is the behaviour of the sequence $(u^\delta)_\delta$ when δ tends to zero?

Outline of the talk

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To get an intuition, we discretize $\left(\mathscr{P}^{\delta}\right)$ and observe what happens when $\delta \to 0$.

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We present the properties of the limit problem in the geometry with the real corner ($\delta = 0$). Since σ changes sign, original phenomena appear.

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We present the properties of the limit problem in the geometry with the real corner ($\delta = 0$). Since σ changes sign, original phenomena appear.

3. Asymptotic analysis

We prove a curious instability phenomenon: for certain configurations, $(\mathscr{P}^{\delta})$ critically depends on δ.

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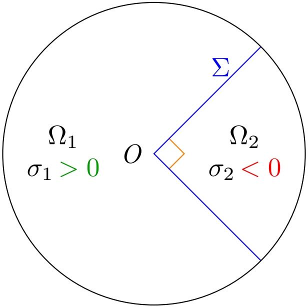
- For the numerical experiments, we round the corner in a particular way



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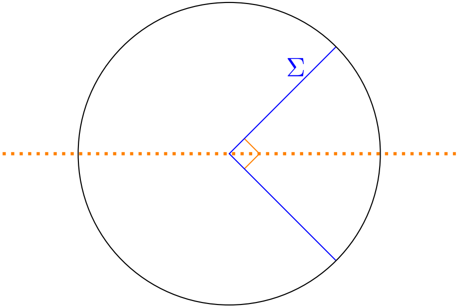

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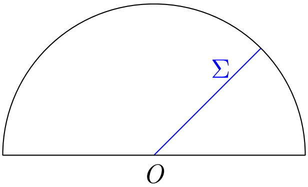

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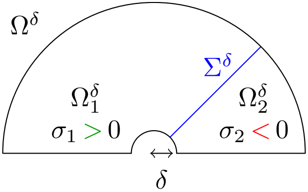

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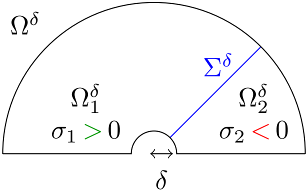

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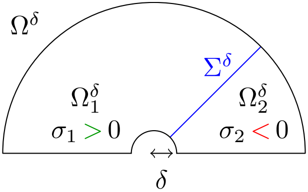

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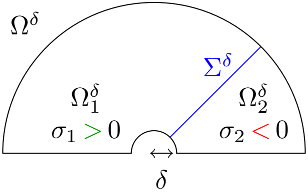

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- We approximate u^δ, *assuming it is well-defined*, by a usual P1 Finite Element Method. We compute the solution u_h^δ of the discretized problem with *FreeFem++*.

We display the behaviour of u_h^δ as $\delta \to 0$.

Numerical experiments 1/2

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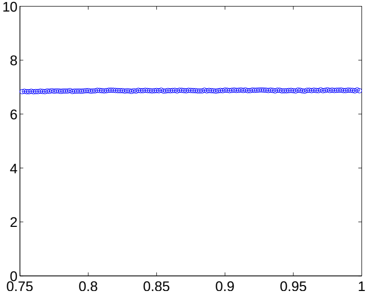


u_h^δ w.r.t. δ

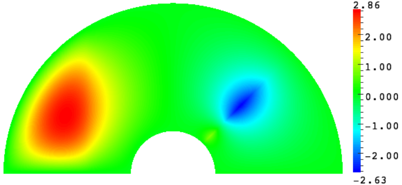
$\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

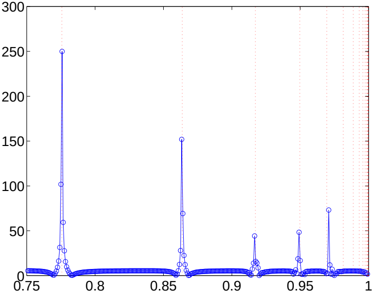
- For positive materials, it is well-known that $(u^\delta)_\delta$ converges to u, the solution in the limit geometry.
- The rate of convergence depends on the regularity of u.
- To avoid to mesh Ω^δ, we can approximate u^δ by u_h.

Numerical experiments 2/2

... and what about for a sign-changing σ???

$$\boxed{\sigma_1 = 1 \text{ and } \sigma_2 = -0.9999}$$





u_h^δ w.r.t. δ $\qquad$ $\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

- For this configuration, u^δ seems to depend critically on δ.

In this talk, our goal is to explain this behaviour.

Mathematical difficulty

- Classical case $\sigma > 0$ everywhere:

$$a(u,u) = \int_\Omega \sigma\, |\nabla u|^2 \geq \min(\sigma)\, \|u\|^2_{\mathrm{H}^1_0(\Omega)} \qquad \text{coercivity}$$

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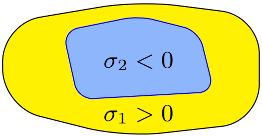
  - When $\sigma_2 = -\sigma_1$, $(\mathscr{P})$ is always ill-posed (`Costabel-Stephan 85`). For a symmetric domain (w.r.t. Σ) we can build a kernel of infinite dimension.

Problems with a sign changing coefficient

$$(\mathscr{P}) \quad \left|\begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\sigma\nabla u) = f \quad \text{in } \Omega. \end{array}\right.$$

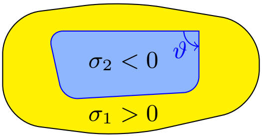
- We have the following properties (see Costabel and Stephan 85, Dauge and Texier 97, Bonnet-Ben Dhia *et al.* 99,10,12,13):

Smooth interface Σ



✔ $(\mathscr{P})$ well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

Interface Σ with a corner



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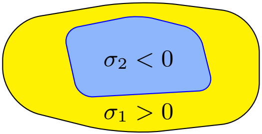

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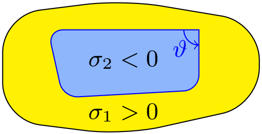

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Well-posedness depends on the smoothness of Σ and on σ.

The problematic of the rounded corner

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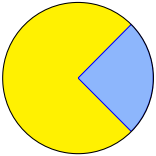

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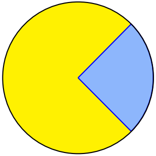

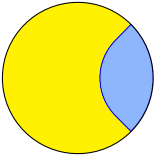

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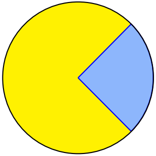
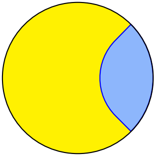
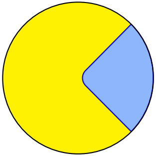




- When the interface is smooth, $(\mathscr{P})$ is well-posed in the Fredholm sense iff $\kappa_\sigma \neq -1$.

What happens for a slightly rounded corner when $\kappa_\sigma \in I_c \setminus \{-1\}$?



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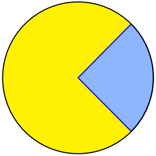

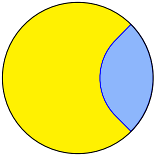

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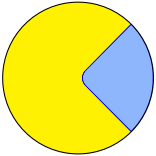

- We need to clarify the properties of $(\mathscr{P})$ when the interface has a corner in the case $\kappa_\sigma \in I_c \backslash \{-1\}$.

Properties of the limit problem inside the critical interval

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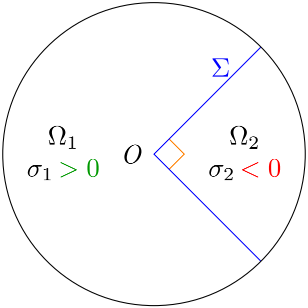
- To simplify the presentation, we work on a particular configuration.



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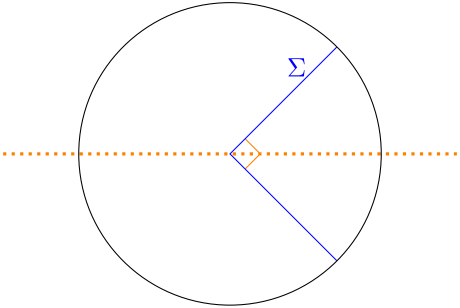
$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\sigma \nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$

- To simplify the presentation, we work on a particular configuration.



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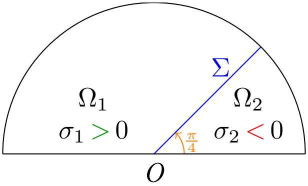

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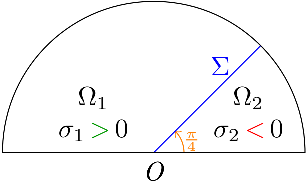

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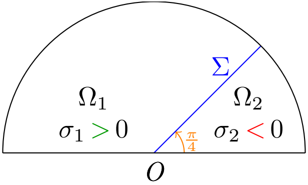

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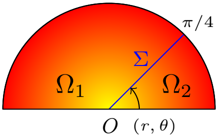
What happens when $\kappa_\sigma \in (-1; -1/3]$?

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

$$\underbrace{-\mathrm{div}(\sigma\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} = f$$

Analogy with a waveguide problem

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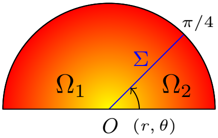

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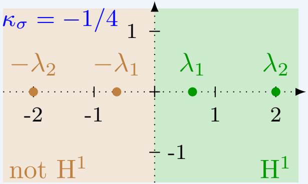
- Singularities in the sector

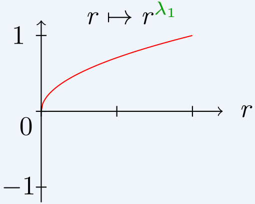
$$s(r, \theta) = r^\lambda \varphi(\theta)$$

Analogy with a waveguide problem

We compute the singularities $s(r,\theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

- Outside the critical interval





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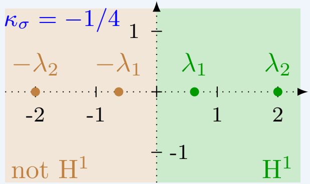

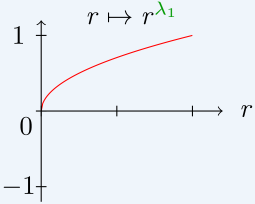

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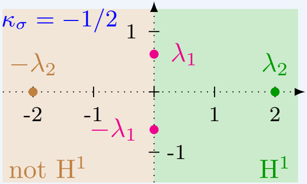

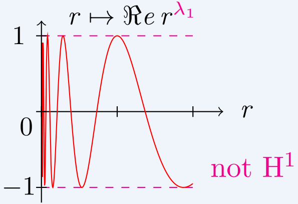

Inside the critical interval: message 1

For a contrast κ_σ inside the critical interval, there are singularities of the form $s(r,\theta) = r^{\pm i\eta}\varphi(\theta)$ with $\eta \in \mathbb{R} \setminus \{0\}$.

- Using these singularities, we can show that the following *a priori* estimate does not hold

$$\cancel{\|u\|_{\mathrm{H}^1_0(\Omega)} \leq C\left(\|Lu\|_{\mathrm{H}^1_0(\Omega)} + \|u\|_{L^2(\Omega)}\right), \qquad \forall u \in \mathrm{H}^1_0(\Omega),}$$

where $L : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is the operator such that

$$(Lu, v)_{\mathrm{H}^1_0(\Omega)} = (\sigma\nabla u, \nabla v)_\Omega, \qquad \forall u, v \in \mathrm{H}^1_0(\Omega).$$

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Proposition. For $\kappa_\sigma \in (-1; -1/3)$, the operator L is not of Fredholm type ($\Im m\, L$ is not closed in $\mathrm{H}^1_0(\Omega)$).

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Let's see how to change the functional framework to recover a well-posed problem ...

Analogy with a waveguide problem

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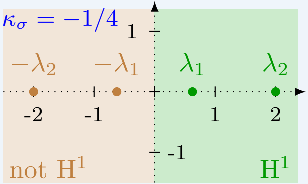

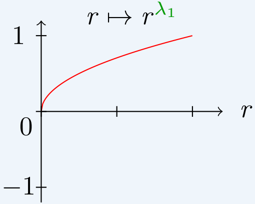

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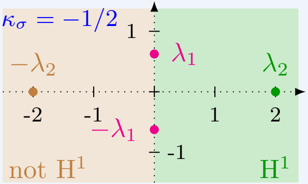

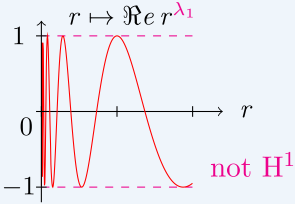

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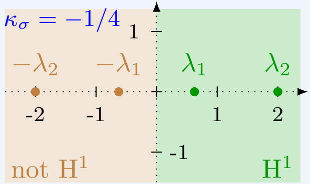

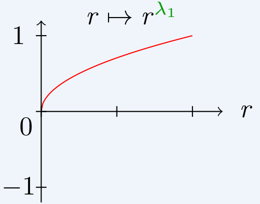

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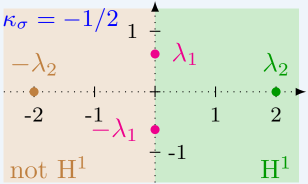
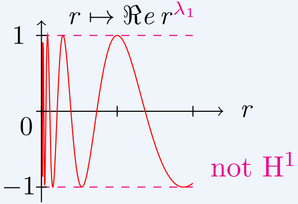
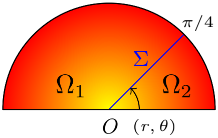




How to deal with the propagative singularities inside the critical interval?

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

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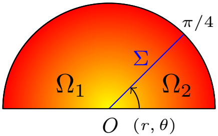
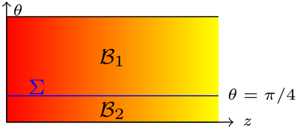


$(z,\theta) = (-\ln r, \theta)$

$(r,\theta) = (e^{-z}, \theta)$

- Half-strip $\mathcal{B}$



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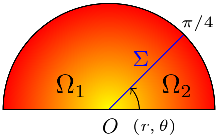

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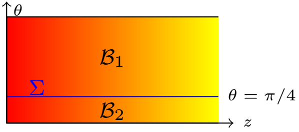

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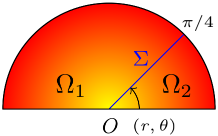

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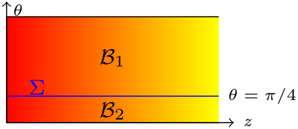

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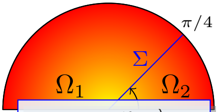
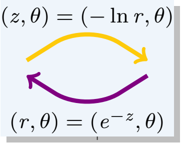
Analogy with a waveguide problem

- Bounded sector Ω





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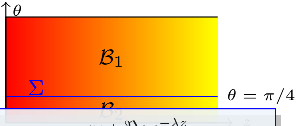

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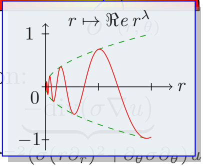

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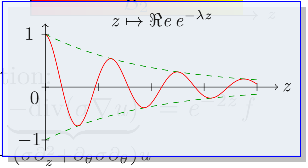

- Singularities in the sector

$s(r,\theta) = r^{\lambda}\varphi(\theta)$

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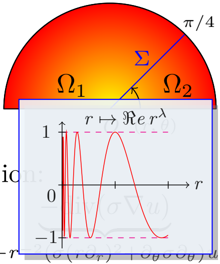
$m(z,\theta) = e^{-\lambda z}\varphi(\theta)$

$s \in \mathrm{H}^1(\Omega)$ $\quad \Re e\,\lambda > 0 \quad$ m is evanescent

Analogy with a waveguide problem

- Bounded sector Ω



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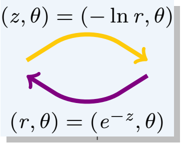

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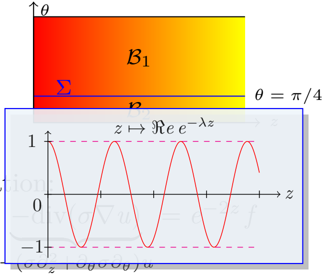

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$$s(r,\theta) = r^{\lambda}\varphi(\theta)$$
$$= \cancel{r^{a}}\,(\cos b \ln r + i \sin b \ln r)\varphi(\theta)$$

- Modes in the strip

$$m(z,\theta) = e^{-\lambda z}\varphi(\theta)$$
$$= \cancel{e^{-az}}\,(\cos bz - i \sin bz)\varphi(\theta)$$

$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

$s \in \mathrm{H}^1(\Omega)$	$\Re e\,\lambda > 0$	m is evanescent
$s \notin \mathrm{H}^1(\Omega)$	$\Re e\,\lambda = 0$	m is propagative

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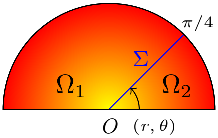

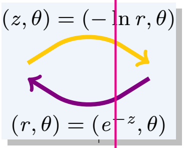

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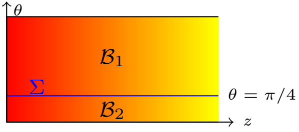

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- Singularities in the sector

$$\begin{aligned} s(r,\theta) &= r^\lambda\varphi(\theta) \\ &= \cancel{r^a}\,(\cos b\ln r + i\sin b\ln r)\varphi(\theta) \end{aligned}$$

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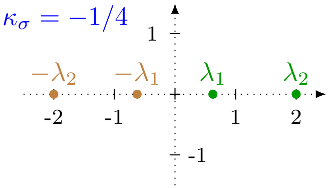
► This encourages us to use modal decomposition in the half-strip.

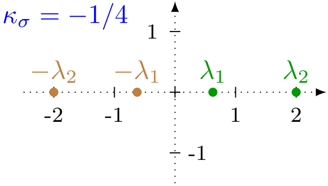
Modal analysis in the waveguide



- Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$

Modal analysis in the waveguide



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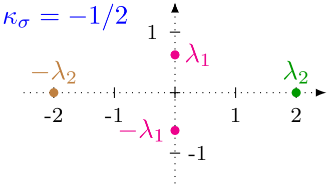
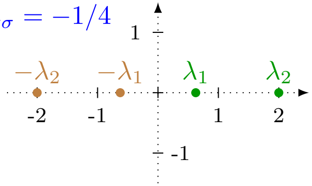


- Inside the critical interval. There are exactly two propagative modes.

Modal analysis in the waveguide

$\kappa_\sigma = -1/4$



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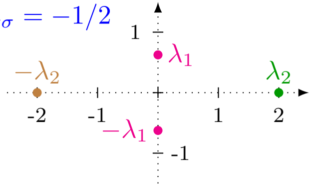

- Inside the critical interval. There are exactly two propagative modes.

$\rightarrow$ The decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c\,\varphi_1\, e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e}_{\text{evanescent part}}.$$

$\boxed{\text{non } \mathrm{H}^1 \text{ framework}}$

Inside the critical interval: message 2

There is a functional framework, different from $\mathrm{H}^1_0(\Omega)$, involving one singularity, where existence and uniqueness of the solution holds.

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$\mathrm{W}_{-\beta} = \{v \mid e^{\beta z} v \in \mathrm{H}^1_0(\mathcal{B})\}$ space of exponentially decaying functions

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THEOREM. Let $\kappa_\sigma \in (-1; -1/3)$ and $0 < \beta < 2$. The operator A^{+} : $\mathrm{div}(\sigma\nabla\cdot)$ from W^{+} to ${\mathrm{W}_{\beta}}^{*}$ is an isomorphism.

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IDEAS OF THE PROOF:

1. $A_{-\beta} : \mathrm{div}(\sigma\nabla\cdot)$ from $\mathrm{W}_{-\beta}$ to ${\mathrm{W}_\beta}^*$ is injective but not surjective.

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4. Limiting absorption principle to select the outgoing mode.

How to numerically approximate the solution in this new framework ?

Naive approximation

- Let us try a usual Finite Element Method (P1 Lagrange Finite Element). We solve the problem

$$\left|\begin{array}{l}\text{Find } u_h \in \mathrm{V}_h \text{ s.t.:}\\ \displaystyle\int_\Omega \sigma \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v \in \mathrm{V}_h,\end{array}\right.$$

where V_h approximates $\mathrm{H}_0^1(\Omega)$ as $h \to 0$ (h is the mesh size).

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- We display u_h as $h \to 0$.

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THE SEQUENCE (u_h) DOES NOT CONVERGE AS $h \to 0$!!!

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- We display u_h as $h \to 0$.

(...) (...)

Contrast $\kappa_\sigma = -0.999 \in (-1; -1/3)$.

Remark

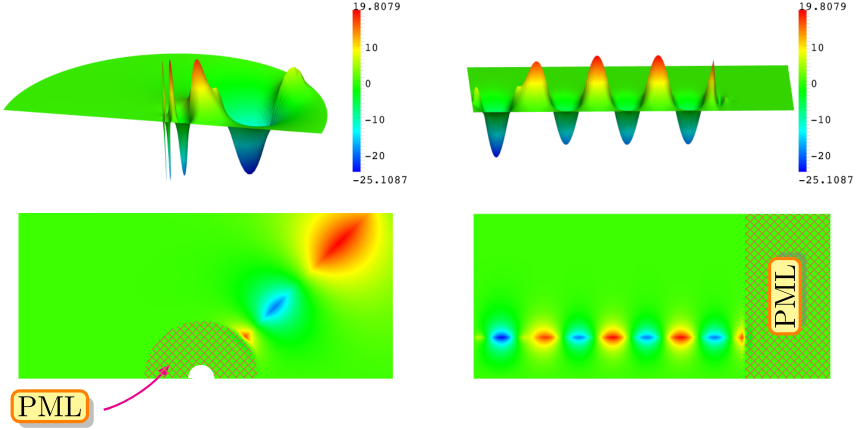
- Outside the critical interval, for the classical approximation method, the sequence (u_h) converges.

$(\dots)$ $(\dots)$

Contrast $\kappa_\sigma = -1.001 \notin (-1; -1/3)$.

How to approximate the solution?

- We use a PML (*Perfectly Matched Layer*) to bound the domain $\mathcal{B}$ + finite elements in the truncated strip ($\kappa_\sigma = -0.999 \in (-1;-1/3)$).



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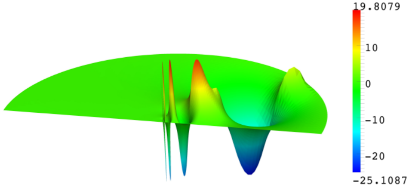
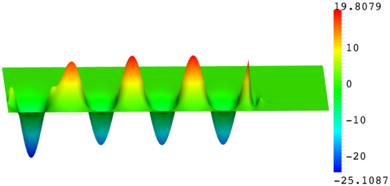
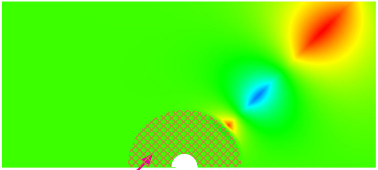
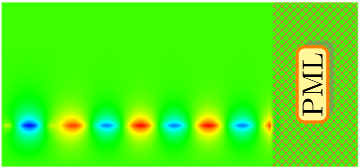








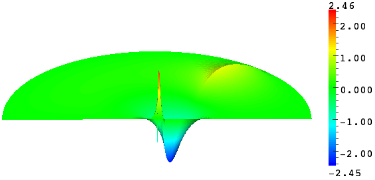
PML

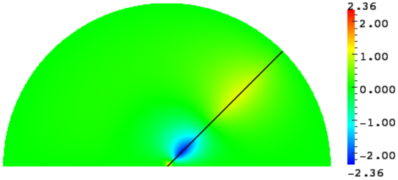
Without the PML, the solution in the truncated strip of length L does not converge when $L \to \infty$. This is what we observed in our numerical experiment for the rounded corner.

A curious black hole phenomenon

- For the Helmholtz equation $\operatorname{div}(\sigma \nabla u) + \omega^2 u = f$, analogously, it is necessary to modify the functional framework to have a well-posed problem.

- In time domain, the solution adopts a curious behaviour.





$$(\boldsymbol{x}, t) \mapsto \Re e\left(u(\boldsymbol{x})e^{-i\omega t}\right) \quad \text{for } \kappa_\sigma = -1/1.3$$

- Everything happens like if a waves was absorbed by the corner point.

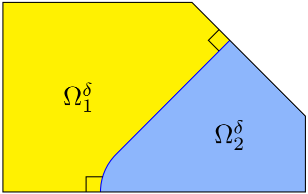
- Analogous phenomena occur in cuspidal domains in the theory of water-waves and in elasticity (`Cardone, Nazarov, Taskinen`).

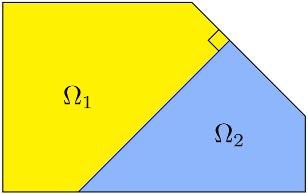
1. Numerical experiments

2. Properties of the limit problem

3. Asymptotic analysis

Source term problem





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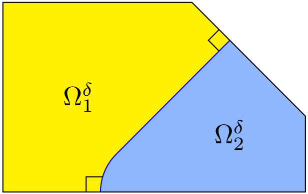

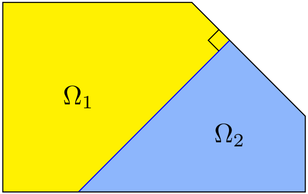

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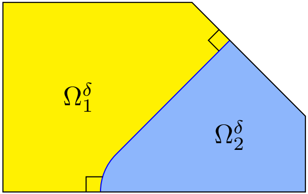

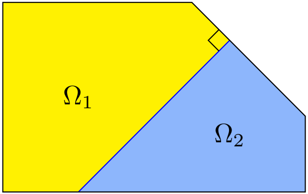

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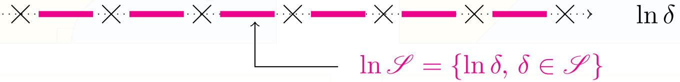
If the limit problem is well-posed only in the exotic framework, then $(\mathscr{P}^\delta)$ critically depends on the value of the rounding parameter δ.

Source term problem

IDEA OF THE APPROACH:

1 We prove the *a priori* estimate $\|u^\delta\|_{\mathrm{H}^1_0(\Omega)} \le c\,|\ln\delta|^{1/2}\|f\|_\Omega$ for all δ in some set $\mathscr{S}$ which excludes a discrete set accumulating in zero (♠ hard part of the proof, `Nazarov's` technique).



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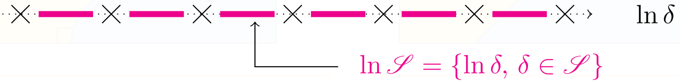

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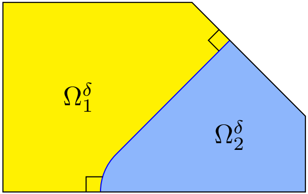
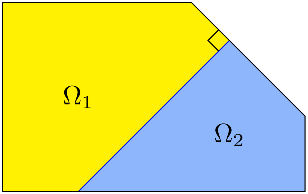
4. Conclusion.

The sequence $(u^\delta)_\delta$ does not converge, even for the L^2-norm!

Source term problem





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Spectral problem

- In the geometry with a rounded corner, we consider the spectral problem

$$\boxed{\left|\begin{array}{l} \text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}^1_0(\Omega) \setminus \{0\}) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{in } \Omega. \end{array}\right.}$$

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Proposition. Assume that $\kappa_\sigma \neq -1$. For $\delta > 0$ (in this case the interface is "smooth"), the operator A^δ is selfadjoint and has compact resolvent. Its spectrum $\mathfrak{S}(\mathrm{A}^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda^\delta_{-n} \le \dots \le \lambda^\delta_{-1} < 0 \le \lambda^\delta_1 \le \lambda^\delta_2 \le \dots \le \lambda^\delta_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

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Proposition. Assume that $\kappa_\sigma \neq -1$. For $\delta > 0$ (in this case the interface is "smooth"), the operator A^δ is selfadjoint and has compact resolvent. Its spectrum $\mathfrak{S}(\mathrm{A}^\delta)$ consists in two sequences of isolated eigenvalues:

$$-\infty \underset{n\to+\infty}{\leftarrow} \dots \lambda^\delta_{-n} \leq \dots \leq \lambda^\delta_{-1} < 0 \leq \lambda^\delta_1 \leq \lambda^\delta_2 \leq \dots \leq \lambda^\delta_n \dots \underset{n\to+\infty}{\rightarrow} +\infty.$$

? For $n \in \mathbb{Z}^*$, what is the behaviour of λ^δ_n when δ tends to zero?

$\Rightarrow$ This depends on the features of the limit operator for $\delta = 0$...

Spectral problem

- Let $\mathrm{A} : D(\mathrm{A}) \to \mathrm{L}^2(\Omega)$ denote the limit operator ($\delta = 0$) such that

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Spectral problem 2/4

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♣ When $\kappa_\sigma \in I_c \setminus \{-1\}$, there holds $D(\mathrm{A}^*) = D(\mathrm{A}) \oplus \mathrm{span}(s_+, s_-)$ where $s_\pm = \zeta r^{\pm i\eta}\varphi(\theta)$ (in particular A is not selfadjoint). Moreover, $\mathfrak{S}(\mathrm{A}) = \mathbb{C}$.

Inside the critical interval:

1. The selfadjoint extensions of A are the operators $\mathrm{A}(\tau) : D(\mathrm{A}(\tau)) \to \mathrm{L}^2(\Omega)$, $\tau \in \mathbb{R}$, such that

$$\left|\begin{array}{l} D(\mathrm{A}(\tau)) = D(\mathrm{A}) \oplus \mathrm{span}(s_+ + e^{i\tau} s_-) \\ \mathrm{A}(\tau) u = \mathrm{div}(\sigma \nabla u). \end{array}\right.$$

Spectral problem 2/4

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CONJECTURE. Assume that $\kappa_\sigma \in I_c \setminus \{-1\}$. There are $a \neq 0$, $b \in \mathbb{R}$, such that $\mathrm{dist}(\mathfrak{S}(\mathrm{A}^\delta), \mathfrak{S}(\mathrm{A}(a \ln \delta + b))) \to 0$ on each compact set of $\mathbb{R}$ as $\delta \to 0$.

Spectral problem 2/4

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3. Conclusion (conjecture).

The spectrum of A^δ does not converge when $\delta \to 0$. Asymptotically, $\mathfrak{S}(\mathrm{A}^\delta)$ is $2\pi/a$–periodic in $\ln \delta$–scale.

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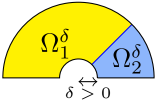
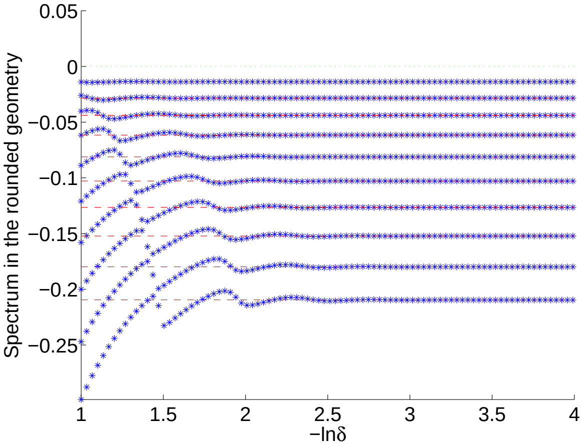
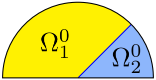
Spectral problem: numerical experiments 3/4

$\kappa_\sigma = -1.0001$ (outside the critical interval)







- $\mathfrak{S}(\mathrm{A}^\delta)$ converges to $\mathfrak{S}(\mathrm{A})$ (A is the limit operator) when $\delta \to 0$.

Spectral problem: numerical experiments 4/4

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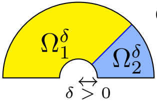

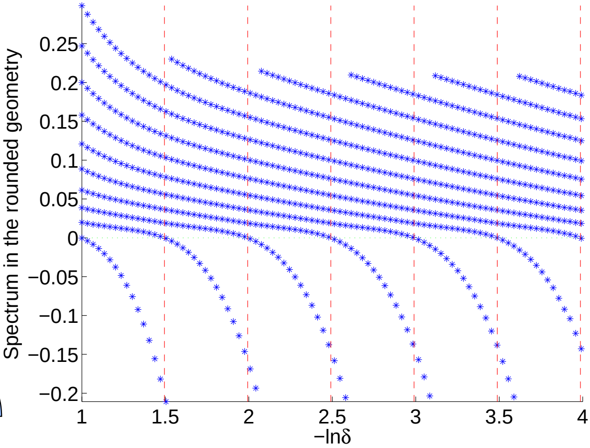

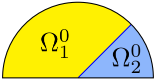

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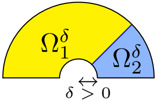

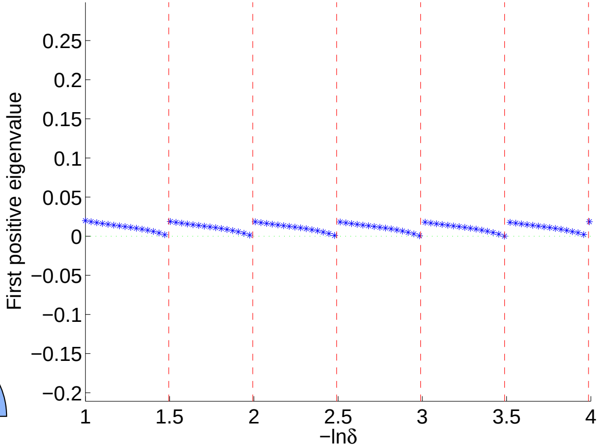

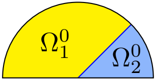

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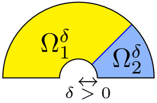

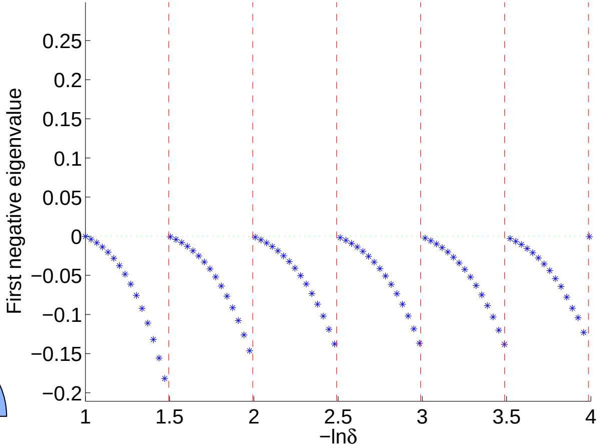

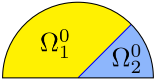

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1. Numerical experiments

2. Properties of the limit problem

3. Asymptotic analysis

Conclusion

Let us remind the initial question:

What is the behaviour of $(u^\delta)_\delta$ when δ tends to zero?

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This depends on the features of the limit problem.

$(\dots)$ $(\dots)$

$\kappa_\sigma = -1.0001 \notin I_c$ $\kappa_\sigma = -0.9999 \in I_c$

When $\kappa_\sigma \in I_c$, $(u^\delta)_\delta$ does not converge, even for the L^2-norm!

In this case, it is impossible to simulate the fields since it is impossible to measure exactly δ. $\Rightarrow$ What happens physically?

Future directions

- Spectral problem in presence of doubly negative materials:

$$\left|\begin{array}{l} \text{Find } (\lambda, u) \in \mathbb{C} \times \mathrm{H}^1(\Omega) \setminus \{0\} \text{ such that} \\ -\mathrm{div}(\mu^{-1}\nabla u) \;=\; \lambda\,\varepsilon\,u \quad \text{in } \Omega; \\ \qquad\qquad\qquad u \;=\; 0 \qquad\;\; \text{on } \partial\Omega. \end{array}\right.$$

$\Rightarrow$ What can we say when both ε and μ change sign (non selfadjoint pb)?

Future directions

Frequency and time dependent models for negative materials

- The physical parameters ε and μ depend on the frequency. For metals at optical frequencies, the Drude model gives

$$\varepsilon(\omega) \approx \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

where ε_0 and ω_p are given.

$\Rightarrow$ What can we say of the non linear spectral problem

$$\boxed{\left|\begin{array}{ll} \text{Find } (\omega, u) \in \mathbb{C} \times \mathrm{H}^1(\Omega) \setminus \{0\} \text{ such that} & \\ -\mathrm{div}(\varepsilon^{-1}(\omega)\,\nabla u) = \omega^2\, u & \text{in } \Omega; \\ \vec{n} \cdot \varepsilon^{-1}(\omega) \nabla u = 0 & \text{on } \partial\Omega \qquad ? \end{array}\right.}$$

- First results have been obtained in time domain for a flat interface.

$\Rightarrow$ Can we prove the limiting amplitude principle for the black-hole?

Thank you for your attention!!!

- ANR project Metamath coordinated by S. Fliss.

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, 46, 1363–1387, 2012.

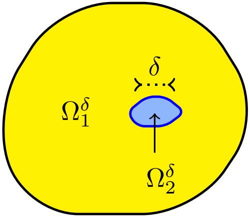
A.-S. Bonnet-Ben Dhia, L. Chesnel, X. Claeys, *Radiation condition for a non-smooth interface between a dielectric and a metamaterial*, M3AS, 23, 9:1629–1662, 2013.

L. Chesnel, X. Claeys, S.A. Nazarov, *A curious instability phenomenon for a rounded corner in presence of a negative material*, Asymp. Anal., 88, 1-2:43–74, 2014.

Spectrum for a small inclusion: setting

- Let Ω, Ξ be smooth domains of $\mathbb{R}^3$ such that $O \in \Xi$, $\overline{\Xi} \subset \Omega$.
- For $\delta \in (0;1]$, we consider the spectral problem

$$\left| \begin{array}{l} \text{Find } (\lambda^\delta, u^\delta) \in \mathbb{C} \times (\mathrm{H}^1_0(\Omega) \setminus \{0\}) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = \lambda^\delta u^\delta \quad \text{ in } \Omega, \end{array} \right.$$



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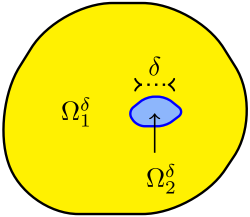

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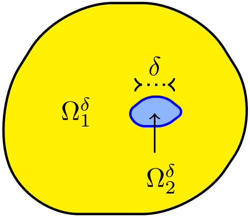

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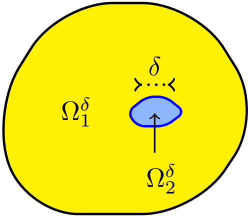

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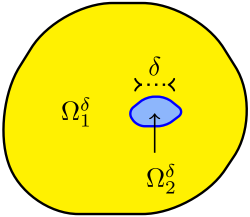

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Limit operators

- As $\delta \to 0$, the small inclusion of negative material disappears. We introduce the far field operator A^0 such that

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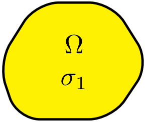

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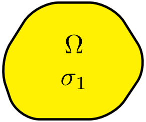

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- Introduce the rapid coordinate $\boldsymbol{\xi} := \delta^{-1}\boldsymbol{x}$ and let $\delta \to 0$. Define the near field operator B^∞ such that

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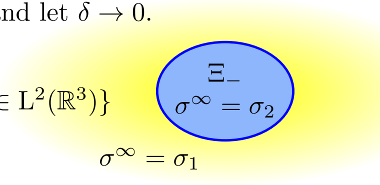

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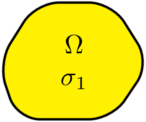
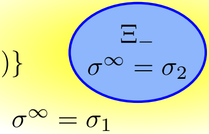


There holds $\mathfrak{S}(\mathrm{A}^0) = \{\mu_n\}_{n\geq 1}$ with $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_n \ldots \underset{n\to+\infty}{\to} +\infty$.

- Introduce the rapid coordinate $\boldsymbol{\xi} := \delta^{-1}\boldsymbol{x}$ and let $\delta \to 0$. Define the near field operator B^∞ such that

$$\left|\begin{array}{lcl} D(\mathrm{B}^\infty) & := & \{w \in \mathrm{H}^1(\mathbb{R}^3) \,|\, \mathrm{div}\,(\sigma^\infty \nabla w) \in \mathrm{L}^2(\mathbb{R}^3)\} \\ \mathrm{B}^\infty w & = & -\mathrm{div}\,(\sigma^\infty \nabla w). \end{array}\right.$$



Proposition. Assume that $\kappa_\sigma \neq -1$. The continuous spectrum of B^∞ is equal to $[0; +\infty)$ while its discrete spectrum is a sequence of eigenvalues:

$$\mathfrak{S}(\mathrm{B}^\infty) \setminus \overline{\mathbb{R}_+} = \{\mu_{-n}\}_{n\geq 1} \quad \text{with} \quad 0 > \mu_{-1} \geq \cdots \geq \mu_{-n} \ldots \underset{n\to+\infty}{\to} -\infty.$$

Spectrum for a small inclusion: results

Assume that $\kappa_\sigma \neq -1$ and that B^∞ is injective. For $n \in \mathbb{N}^*$, we denote $\lambda^\delta_{\pm n}$, μ^δ_n, μ^δ_{-n} the eigenvalues of A^δ, A^0, B^∞ as in the previous slides.

Theorem. (Positive spectrum) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

$$|\lambda^\delta_n - \mu_n| \leq C\,\delta^{3/2-\varepsilon}, \qquad \forall \delta \in (0;\delta_0].$$

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THEOREM. (POSITIVE SPECTRUM) For all $n \in \mathbb{N}^*$, $\varepsilon \in (0;1)$, there exist constants $C, \delta_0 > 0$ depending on n, ε but independent of δ, such that

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$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C\exp(-\gamma/\delta), \qquad \forall \delta \in (0;\delta_0].$$

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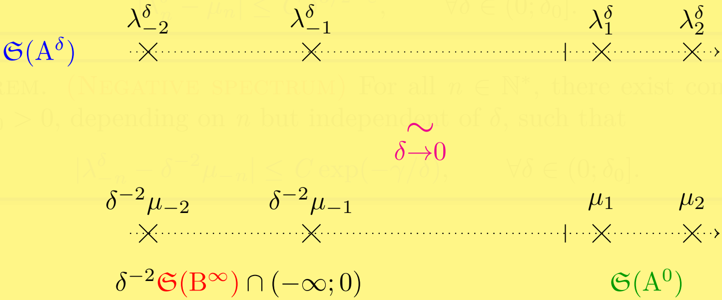
Theorem. (Negative spectrum) For all $n \in \mathbb{N}^*$, there exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$|\lambda^\delta_{-n} - \delta^{-2}\mu_{-n}| \leq C\exp(-\gamma/\delta), \qquad \forall \delta \in (0;\delta_0].$$

Schematically, we have:



Spectrum for a small inclusion: results

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Spectrum for a small inclusion: results

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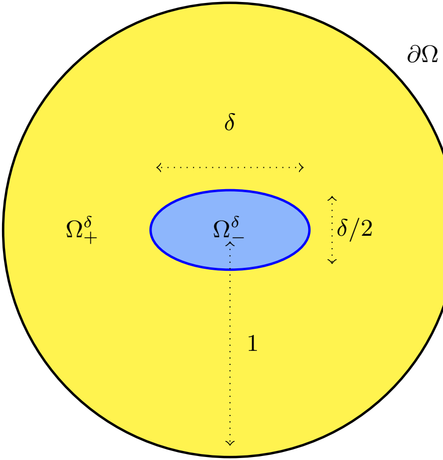
PROPOSITION. (LOCALIZATION EFFECT) For all $n \in \mathbb{N}^*$, let u^δ_{-n} be an eigenfunction corresponding to the negative eigenvalue λ^δ_{-n}. There exist constants $C, \gamma, \delta_0 > 0$, depending on n but independent of δ, such that

$$\int_\Omega (|u^\delta_{-n}|^2 + |\nabla u^\delta_{-n}|^2)\, e^{\gamma x/\delta}\, d\boldsymbol{x} \leq C\,\|u^\delta_{-n}\|_\Omega, \qquad \forall \delta \in (0;\delta_0].$$

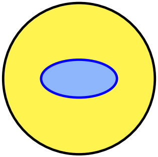
Numerical experiments for the small inclusion

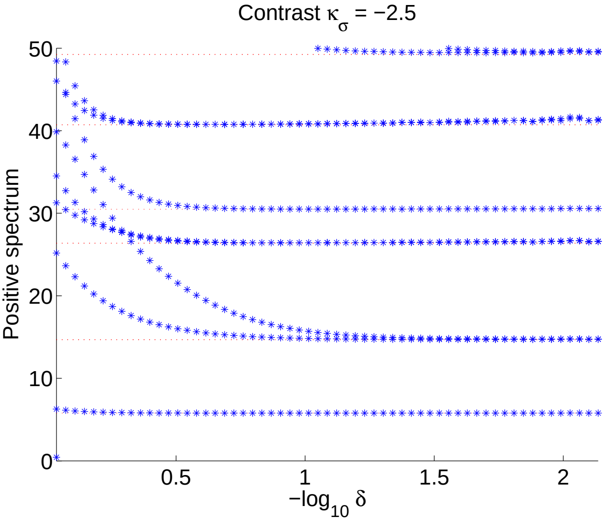
- We approximate numerically the spectrum of A^δ using a usual P1 Finite Element Method and we make δ goes to zero.

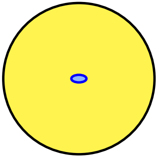
- We consider the following 2D geometry:



Numerical experiments for the small inclusion

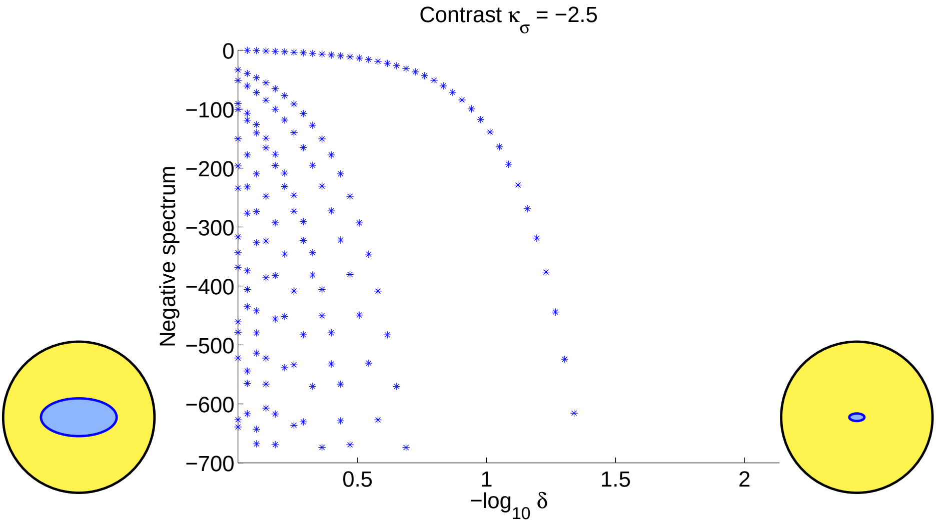






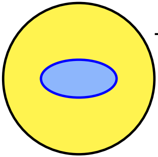
- The positive part of $\mathfrak{S}(\mathrm{A}^\delta)$ converges to $\mathfrak{S}(\mathrm{A}^0)$ when $\delta \to 0$.

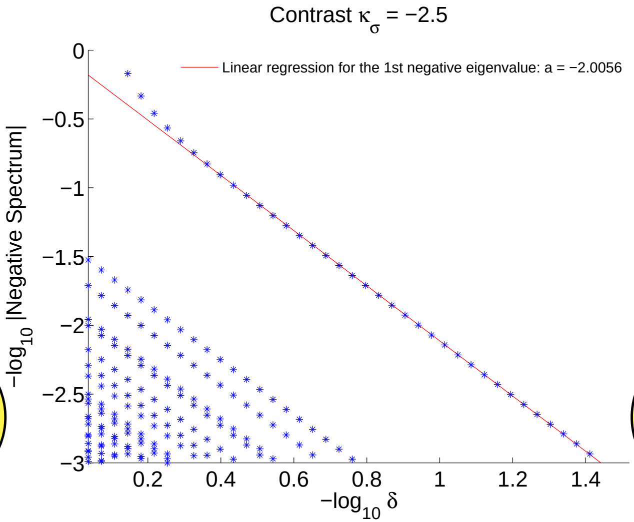
Numerical experiments for the small inclusion

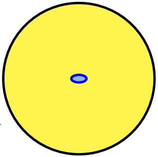


► The negative part of $\mathfrak{S}(\mathrm{A}^\delta)$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathfrak{S}(\mathrm{B}^\infty)$ when $\delta \to 0$.

Numerical experiments for the small inclusion







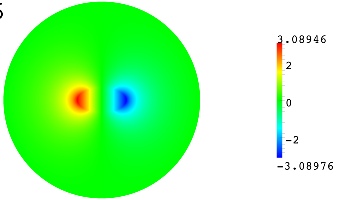
- The negative part of $\mathfrak{S}(\mathrm{A}^{\delta})$ is asymptotically equivalent to the negative part of $\delta^{-2}\mathfrak{S}(\mathrm{B}^{\infty})$ when $\delta \to 0$.

Localization effect

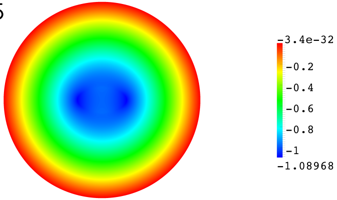
Eigenfunction associated to the first negative eigenvalue

Eigenfunction associated to the first positive eigenvalue

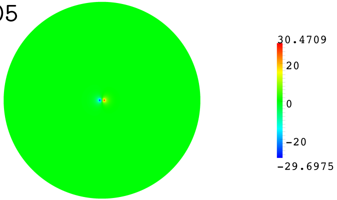
δ=0.5



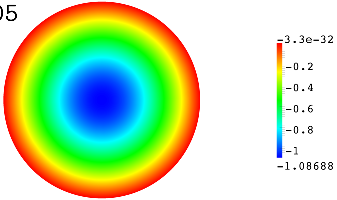
δ=0.5



δ=0.05



δ=0.05



- The eigenfunctions corresponding to the negative eigenvalues are localized around the small inclusion. Here, $\kappa_\sigma = -2.5$.