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Conférence en l'honneur de Martin Costabel pour ses 65 ans

Bilaplacian problems with a sign-changing coefficient

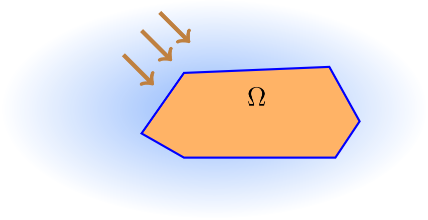
Lucas Chesnel[†], Jérémy Firozaly[‡]

[†]Inverse Problems Research Group, Aalto University, Helsinki, Finland
[‡]POems team, Ensta, Paris, France

University of Rennes 1, August 26th-30th, 2013

Introduction: the ITEP

- Scattering in time-harmonic regime by a penetrable inclusion Ω (coefficient n) in $\mathbb{R}^2$: we look for an incident wave that does not scatter.



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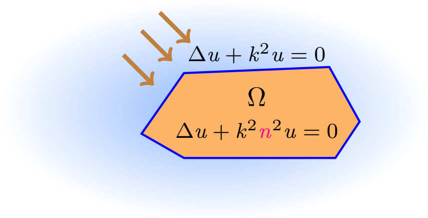

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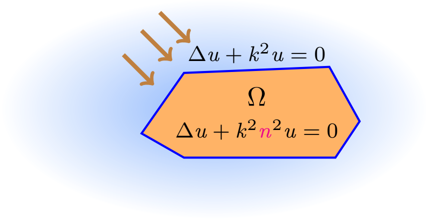

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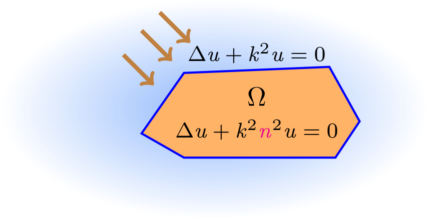

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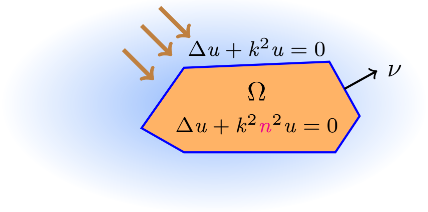

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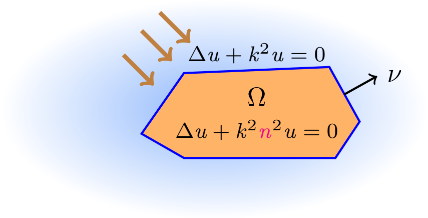

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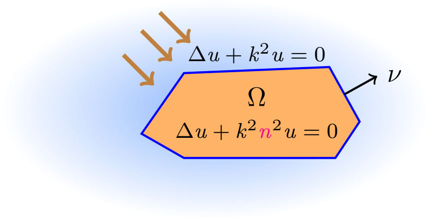

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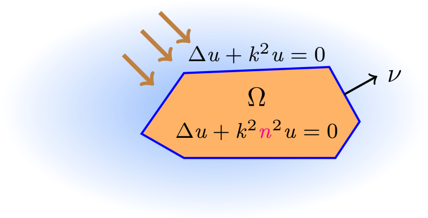

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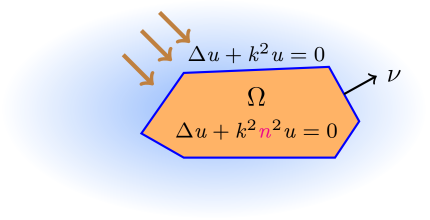

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DEFINITION. Values of $k \in \mathbb{C}$ for which this problem has a nontrivial solution (u, w) are called transmission eigenvalues.

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- This implies

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- This problem has been widely studied since 1986-1988 (`Bellis, Cakoni, Colton, Gintides, Guzina, Haddar, Kirsch, Kress, Monk, Päivärinta, Rynne, Sleeman, Sylvester...`) when $n > 1$ on Ω or $n < 1$ on Ω.

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What happens when $1 - n^2$ changes sign?

- We define $\sigma = (1 - n^2)^{-1}$ and we focus on the principal part:

$$\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \Delta v \Delta v' = \langle f, v' \rangle_\Omega, \quad \forall v' \in \mathrm{H}^2_0(\Omega).\end{array}\right.$$

Outline of the talk

- ... and more generally, we study the problem:

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We study $(\mathscr{P})$ with $\mathbf{X} = \mathbf{H}^1_0(\boldsymbol{\Delta}) := \{v \in \mathbf{H}^1_0(\boldsymbol{\Omega}) \,|\, \boldsymbol{\Delta} v \in \mathbf{L}^2(\boldsymbol{\Omega})\}$.

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3. A bilaplacian problem with Dirichlet boundary conditions

   We study $(\mathscr{P})$ with $\mathbf{X} = \mathbf{H}^2_0(\boldsymbol{\Omega})$.

Reminder: properties of div $(\sigma\nabla\cdot)$

- In the fields of plasmonic and negative metamaterials, we study:

$$(\mathscr{F}) \quad \left| \begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \nabla v \cdot \nabla v' = \langle f, v' \rangle_\Omega \,, \quad \forall v' \in \mathrm{H}^1_0(\Omega). \end{array} \right.$$

- Ω is partitioned into two domains Ω_1 and Ω_2. We assume that $\sigma_1 := \sigma|_{\Omega_1}$ and $\sigma_2 := \sigma|_{\Omega_2}$ are constants.

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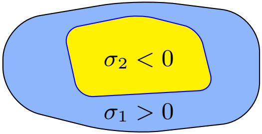
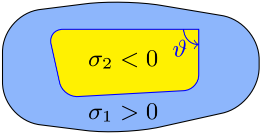
Smooth interface



✔ $(\mathscr{F})$ well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

Interface with a corner



✔ $(\mathscr{F})$ well-posed in the Fredholm sense iff $\kappa_\sigma \notin [-I; -1/I]$, $I = (2\pi - \vartheta)/\vartheta$.

Well-posedness depends on the smoothness of the interface and on σ (c.f. work of M. Costabel, E. Stephan 1985).

Mixed Boundary Conditions I

Let T be an isomorphism of X.

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Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \sigma\, \Delta u \Delta(\mathtt{T}u) \geq C\, \|u\|_{\mathrm{X}}^2$.

In this case, Lax-Milgram $\Rightarrow$ $(\mathscr{P}^{\mathtt{T}})$ (and so $(\mathscr{P})$) well-posed.

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Theorem. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Then, the operator $A : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ associated with $(\mathscr{P})$ is an isomorphism.

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In this section, $\mathrm{X} = \mathrm{H}^1_0(\Delta)$.

1. Define $\mathtt{T}u \in \mathrm{H}^1_0(\Omega)$ the function such that $\Delta(\mathtt{T}u) = \sigma^{-1}\Delta u$.
2. $\mathtt{T}$ is an isomorphism of $\mathrm{H}^1_0(\Delta)$.
3. One obtains $a(u, \mathtt{T}u) = \displaystyle\int_\Omega \sigma\, \Delta u \Delta(\mathtt{T}u) =$

The change of sign of σ is not a problem!

Theorem. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Then, the operator $A : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ associated with $(\mathscr{P})$ is an isomorphism.

Mixed Boundary Conditions II

In this section, $\mathrm{X} = \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$.

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma \, \Delta u \Delta v = l(v), \, \forall v \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega). \end{array} \right.$$

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THEOREM. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Assume that Ω is convex or of class $\mathscr{C}^2$. Then, the operator $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ associated with $(\mathscr{P})$ is an isomorphism.

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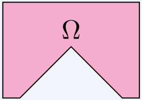
Theorem. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Assume that Ω is convex or of class $\mathscr{C}^2$. Then, the operator $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ associated with $(\mathscr{P})$ is an isomorphism.

What happens if Ω has a reentrant corner ?

Polygonal $\partial\Omega$ with one reentrant corner

Reminder



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Polygonal $\partial\Omega$ with one reentrant corner

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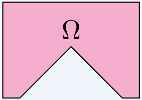

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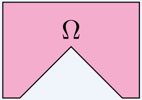

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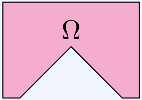

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(c.f. Sapongyan paradox S.A. Nazarov, G. Sweers)

Polygonal $\partial\Omega$ with one reentrant corner

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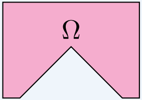

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We want $\mathtt{T}u \in H^2(\Omega)$

Polygonal $\partial\Omega$ with one reentrant corner

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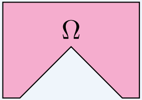

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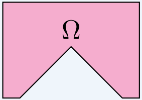

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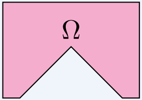

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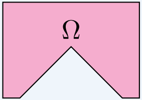

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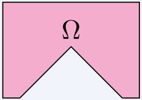

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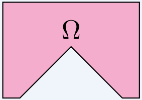

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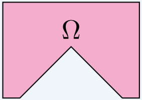
Polygonal $\partial\Omega$ with one reentrant corner

Reminder



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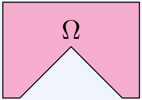
Polygonal $\partial\Omega$ with one reentrant corner



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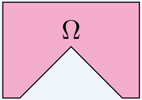

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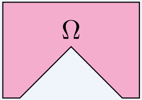

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- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then A is an isomorphism.

Polygonal $\partial\Omega$ with one reentrant corner

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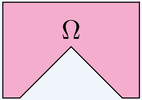

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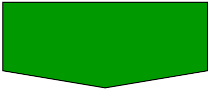
THEOREM. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Introduce $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ the operator associated with $(\mathscr{P})$.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, then A is an isomorphism.
- If $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$, then A is Fredholm of index zero and $\dim\ker A = 1$.

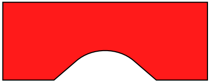
Summary of the results when $\mathrm{X} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma\, \Delta u \Delta v = l(v),\ \forall v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega). \end{array} \right.$$

- We introduce the operator $A : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that $(\Delta(Au), \Delta v)_\Omega = (\sigma \Delta u, \Delta v)_\Omega$ for all $u,\ v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$.



✔ A is an isomorphism.



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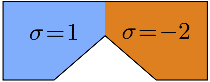
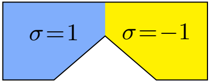


✔ A is an isomorphism because $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$.



✔ A is a Fredholm operator of index 0 and $\dim \ker A = 1$ because $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$.

A bilaplacian problem with Dirichlet boundary conditions

- In this section, $\mathrm{X} = \mathrm{H}_0^2(\Omega)$.

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}_0^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma\, \Delta u \Delta v = l(v), \quad \forall v \in \mathrm{H}_0^2(\Omega). \end{array} \right.$$

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Message: The operators $\Delta(\sigma\Delta\cdot) : \mathrm{H}_0^2(\Omega) \to \mathrm{H}^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}^{-1}(\Omega)$ have very different properties.

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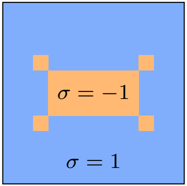
THEOREM. The problem $(\mathscr{P})$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian problem with Dirichlet boundary conditions

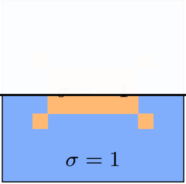
Ideas of the proof: We have

$$a(v,v) = (\sigma \Delta v, \Delta v)_\Omega.$$

We would like to build $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$

so that $$a(v, \mathtt{T}v) = (\sigma\Delta v, \Delta(\mathtt{T}v))_\Omega = (\Delta v, \Delta v)_\Omega.$$

Fredholm



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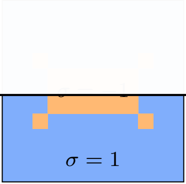

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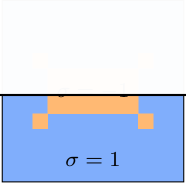
1. Let $w \in \mathrm{H}^1_0(\Omega)$ such that $\Delta w = \sigma^{-1}\Delta v$.



A bilaplacian problem with Dirichlet boundary conditions

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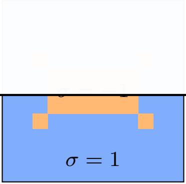

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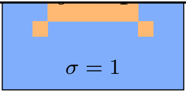
   where $K : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact.



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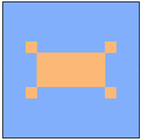

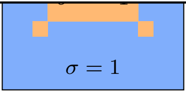

A bilaplacian problem with Dirichlet boundary conditions

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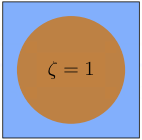

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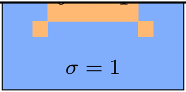


A bilaplacian problem with Dirichlet boundary conditions

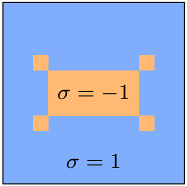
- In this section, $X = H_0^2(\Omega)$.

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in H_0^2(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \sigma\, \Delta u \Delta v = l(v), \quad \forall v \in H_0^2(\Omega). \end{array} \right.$$

Message: The operators $\Delta(\sigma\Delta\cdot) : H_0^2(\Omega) \to H^{-2}(\Omega)$ and $\operatorname{div}(\sigma\nabla\cdot) : H_0^1(\Omega) \to H^{-1}(\Omega)$ have very different properties.

THEOREM. The problem $(\mathscr{P})$ is well-posed in the Fredholm sense as soon as σ does not change sign in a neighbourhood of $\partial\Omega$.

Fredholm



A bilaplacian problem with Dirichlet boundary conditions

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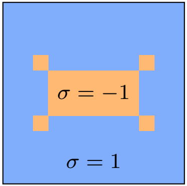
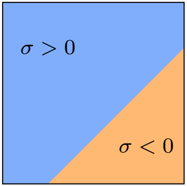
... but $(\mathscr{P})$ can be ill-posed (not Fredholm) when σ changes sign "on $\partial\Omega$"
$\Rightarrow$ work with `J. Firozaly`.

Fredholm



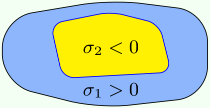


Not always Fredholm

Conclusion

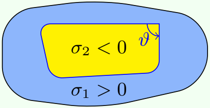
$$\text{Find } v \in \mathrm{H}^1_0(\Omega) \text{ s.t.}, \forall v' \in \mathrm{H}^1_0(\Omega), \quad \int_\Omega \sigma \nabla v \cdot \nabla v' = \ell(v').$$

♠ Smooth interface



Well-posed in the Fredholm sense iff $\kappa_\sigma = \sigma_2/\sigma_1 \neq -1$.

♠ Interface with a corner



Well-posed in the Fredholm sense iff $\kappa_\sigma \notin [-I; -1/I]$, $I = (2\pi - \vartheta)/\vartheta$.

$$\text{Find } v \in \mathrm{X} \text{ s.t.}, \forall v' \in \mathrm{X}, \quad \int_\Omega \sigma \Delta v \Delta v' = \ell(v').$$

We assume $\sigma \in \mathrm{L}^\infty(\Omega)$, $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$.

♠ If $\mathrm{X} = \mathrm{H}^1_0(\Delta)$: Well-posed.

♠ If $\mathrm{X} = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$:

- Well-posed when Ω is convex or of class $\mathscr{C}^2$.
- When Ω has one reentrant corner, it can occur a kernel of dimension 1.

♠ If $\mathrm{X} = \mathrm{H}^2_0(\Omega)$:

- Well-posed in the Fredholm sense when σ does not change sign on a neighbourhood of $\partial\Omega$.
- When σ changes sign on $\partial\Omega$, Fredholmness can be lost.

Thank you for your attention!!!