Séminaire Idefix

Model reduction for the Dirichlet Laplacian in thin domains

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Collaboration with S.A. Nazarov[2].

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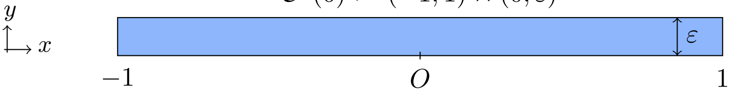
Palaiseau, 12/03/2026

- Motivated by the study of quantum waveguides, we want to investigate the properties of the Dirichlet Laplacian in thin domains.

- For $\varepsilon > 0$ small, define the **thin straight** strip

$$\mathcal{S}^{\varepsilon}(0) := (-1;1) \times (0;\varepsilon)$$



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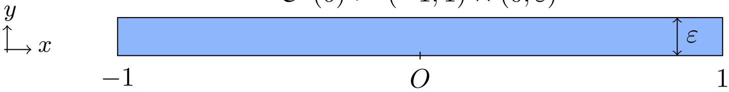

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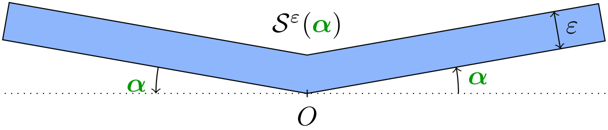

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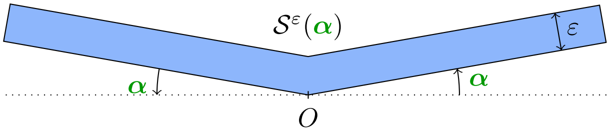
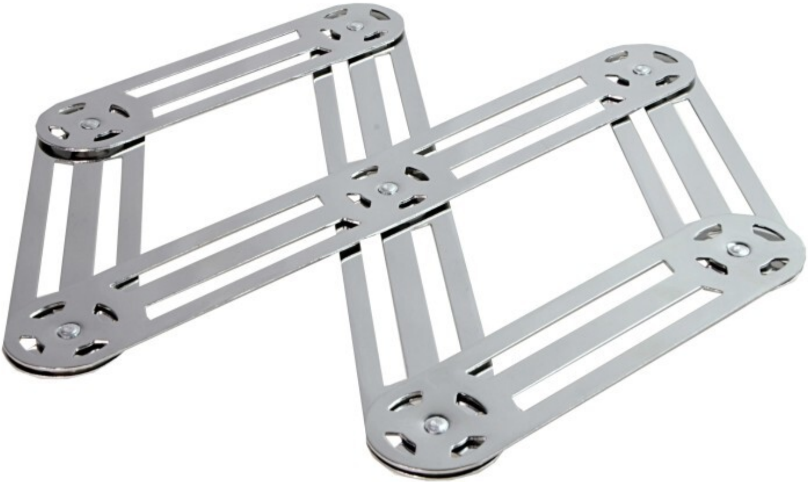


- Consider the spectral problem for the Dirichlet Laplacian

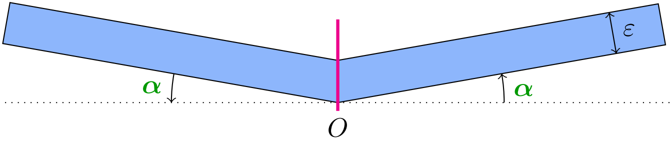
$$\left| \begin{array}{rcll} -\Delta u & = & \lambda^{\varepsilon} u & \text{in } \mathcal{S}^{\varepsilon}(\boldsymbol{\alpha}) \\ u & = & 0 & \text{on } \partial \mathcal{S}^{\varepsilon}(\boldsymbol{\alpha}). \end{array} \right.$$

- Another important example of application:



Introduction

- Let us exploit symmetry wrt to the line $x = 0$



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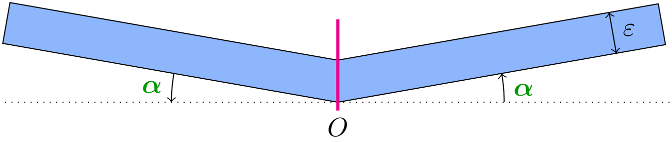

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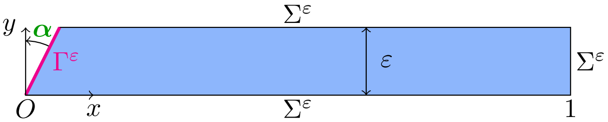
- This leads to consider in the trapezoid T^ε



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$$(\mathscr{P}^\varepsilon) \left| \begin{array}{rcll} -\Delta u &=& \lambda^\varepsilon u & \text{in } \mathrm{T}^\varepsilon \\ u &=& 0 & \text{on } \Sigma^\varepsilon \\ \partial_\nu u &=& 0 & \text{on } \boldsymbol{\Gamma^\varepsilon}. \end{array} \right.$$

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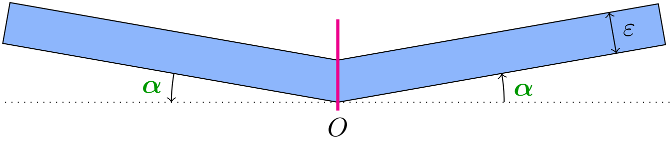

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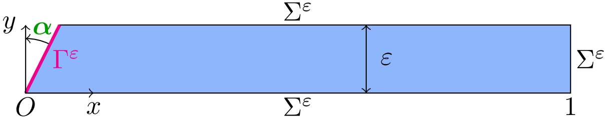

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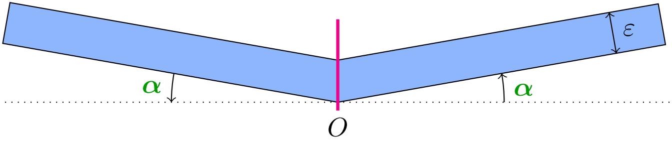

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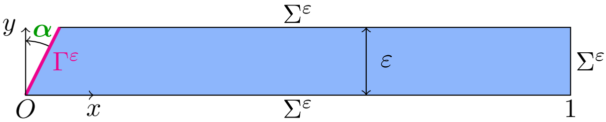

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- Its spectrum forms a sequence of normal eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \ldots \quad \to +\infty.$$

Introduction 3/4

- For $\boldsymbol{\alpha = 0}$, T^ε is a rectangle $\Rightarrow$ **explicit computations** can be done.

For the first eigenvalues, one obtains, for $n \in \mathbb{N}$,

$$\lambda_n^\varepsilon = \frac{\boldsymbol{\pi^2}}{\boldsymbol{\varepsilon^2}} + (n + 1/2)^2\pi^2, \qquad u_n^\varepsilon(x,y) = \cos(\pi(n+1/2)x)\sin(\pi y/\varepsilon).$$

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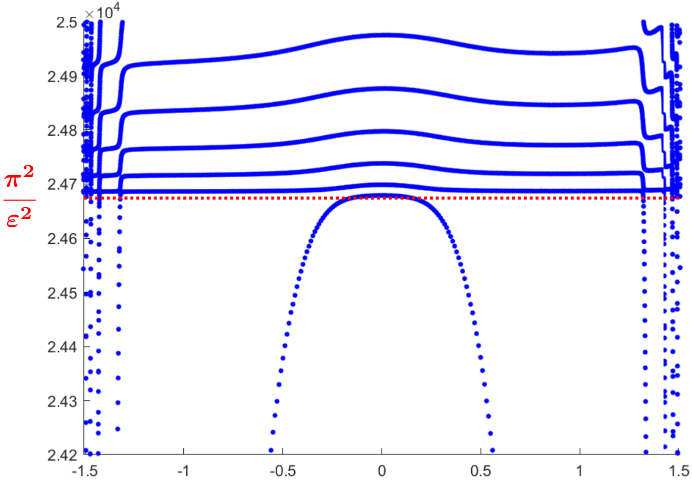
- For $\alpha \neq 0$, let us do **numerics** with `Freefem++`.

- Spectrum of $(\mathscr{P}^\varepsilon)$ with respect to $\alpha \in (-\pi/2;\pi/2)$ for $\varepsilon = 0.02$:



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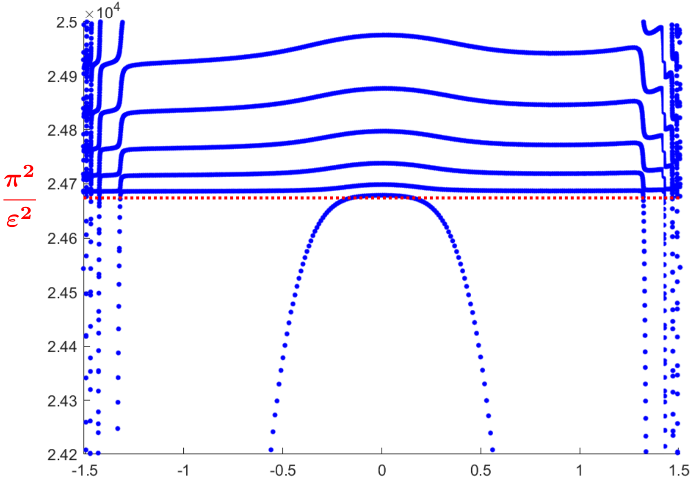

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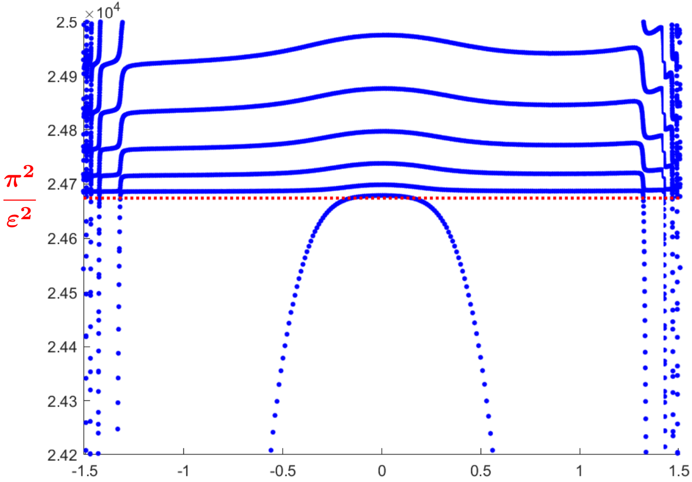

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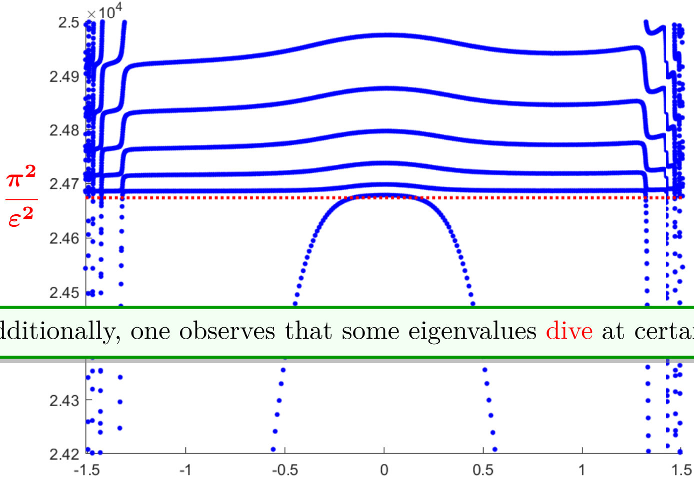

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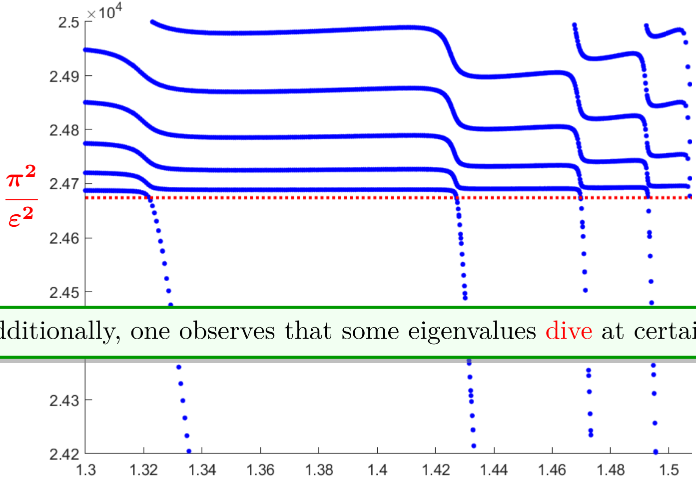

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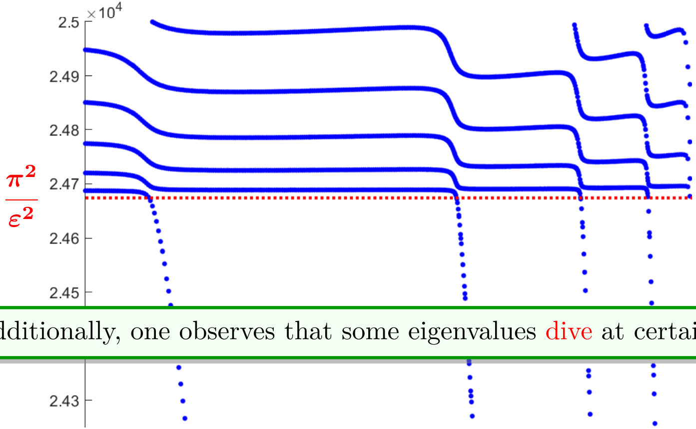

Additionally, one observes that some eigenvalues dive at certain α.

Goal of the talk

We wish to explain this phenomenon of diving eigenvalue and characterize the angles at which it occurs.

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Outline of the talk

1. Preparatory work

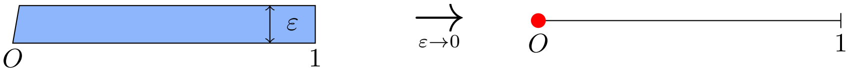
2. Asymptotic analysis at a fixed α

3. Model problems at the critical angles

4. Spectral breathing in periodic waveguides

Asymptotic analysis - general picture

- Roughly speaking, at the limit $\varepsilon \to 0$, we will obtain a 1D model problem, with an **explicit dependence on α**, on the segment $I := (0;1)$.



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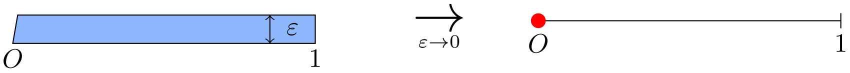

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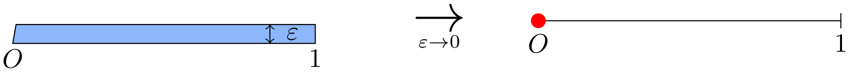

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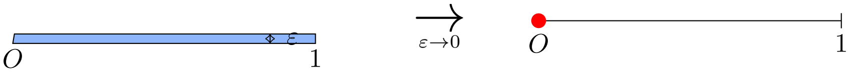

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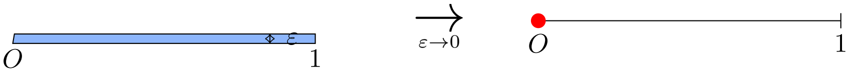

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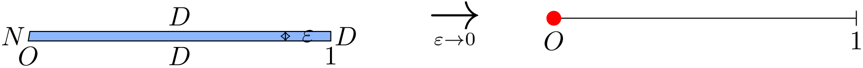

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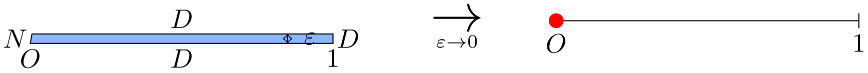

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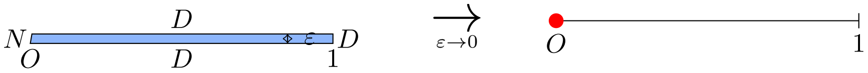

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- Classically, we will consider different expansions far from O and in a neighbourhood of O that we will match in some intermediate regions.

Near field problem 1/2

► To capture the rapid variations of the eigenfunctions close to O, introduce the variables

$$(X, Y) = (x/\varepsilon, y/\varepsilon).$$

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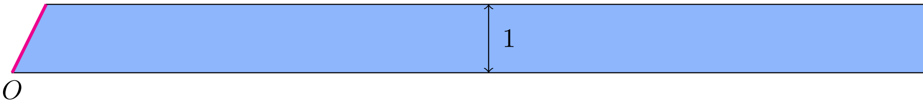

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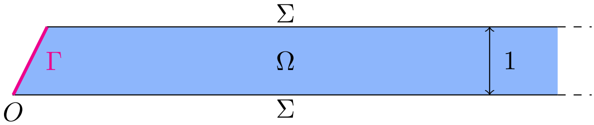

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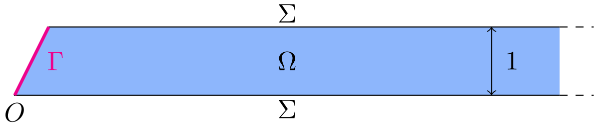

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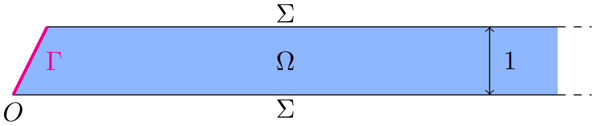

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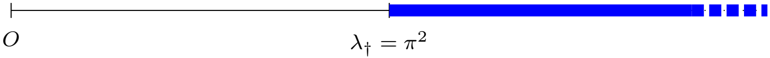
Denote by A^Ω the unbounded operator of $\mathrm{L}^2(\Omega)$ such that

$$\left|\begin{array}{rcl} \mathcal{D}(A^\Omega) &:=& \{v \in \mathrm{H}^1(\Omega) \,|\, \Delta v \in \mathrm{L}^2(\Omega),\, v = 0 \text{ on } \Sigma,\, \partial_\nu v = 0 \text{ on } \Gamma\} \\ A^\Omega v &=& -\Delta v. \end{array}\right.$$

Near field problem 2/2

Spectrum of A^Ω:



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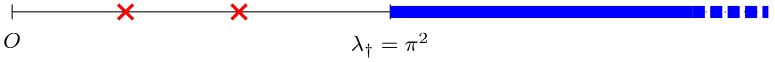

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- **Discrete spectrum** depends on α. There holds $\left|\begin{array}{ll} \sigma_{\mathrm{d}}(A^\Omega) = \emptyset & \text{if } \alpha = 0 \\ \sigma_{\mathrm{d}}(A^\Omega) \neq \emptyset & \text{if } \alpha \neq 0 \end{array}\right.$ (Avishai *et al.* 91).

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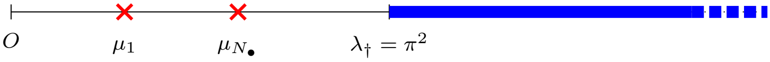

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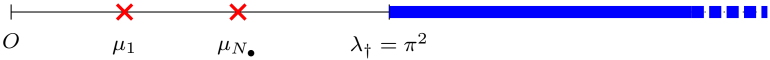

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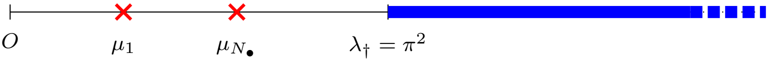

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Eigenfunctions associated with two eigenvalues of the discrete spectrum:

Near field problem at the threshold 1/2

- In the sequel, the properties of the problem

$$(\mathscr{P}_\dagger)\left|\begin{array}{rcll}\Delta W+\pi^2 W&=&0&\text{in }\Omega\\ W&=&0&\text{on }\Sigma\\ \partial_\nu W&=&0&\text{on }\Gamma\end{array}\right.$$

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- For $(\mathscr{P}_\dagger)$, we have the **degenerated** waves (their flux of energy through a transverse section is zero)

$$
w_0(x,y) = \sin(\pi y), \qquad w_1(x,y) = x\sin(\pi y)
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- For $(\mathscr{P}_\dagger)$, we have the **degenerated** waves (their flux of energy through a transverse section is zero)

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Near field problem at the threshold 1/2

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The **threshold scattering matrix** $\mathbb{S}$ (a scalar here) is **unitary**.

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- For all $\alpha \in [0; \pi/2)$, we have $|\mathbb{S}| = |\mathbb{S}(\alpha)| = 1$. But in general $\mathbb{S}(\alpha) \neq -1$.

For most angles α, there holds $\mathrm{X}_\dagger = \{0\}$.

First main results

Pick $\alpha \in [0;\pi/2)$. To simplify, assume absence of trapped modes for $(\mathscr{P}_\dagger)$.

THEOREM: There are constants C_n, $\delta_n > 0$ such that as $\varepsilon \to 0$, we have:

For $n = 1, \dots, N_\bullet$:

$$\left|\lambda_n^\varepsilon - \varepsilon^{-2}\mu_n\right| \leq C_n\, e^{-(1+\delta_n)\sqrt{\pi^2-\mu_n}/\varepsilon};$$

For $n = N_\bullet + m$, $m \in \mathbb{N}^*$:

$i)$ if $\mathrm{X}_\dagger = \{0\}$, $\quad \left|\lambda_n^\varepsilon - \left(\varepsilon^{-2}\pi^2 + m^2\pi^2\right)\right| \leq C_n\, \varepsilon^{\delta_n};$

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1. The asymptotics of the first $N_\bullet$ eigenvalues is dictated by $\sigma_{\mathrm{d}}(A^\Omega)$.

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COMMENTS:

1. The asymptotics of the first $N_\bullet$ eigenvalues is dictated by $\sigma_{\mathrm{d}}(A^\Omega)$.

2. The behavior of the next eigenvalues depends on the existence or not of almost stabilizing solutions.

Elements of proof – first $N_\bullet$ eigenvalues

For $1 \leq n \leq N_\bullet$, let $\mu_n \in (0;\pi^2)$, v be an eigenpair of the discrete spectrum of A^Ω:

Inserting $(\varepsilon^{-2}\mu_n, v(\cdot/\varepsilon))$ in $(\mathscr{P}^\varepsilon)$ only leaves a small discrepancy at $x = 1$ because v is exponentially decaying at infinity.

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- Let λ_n^ε, u^ε be the n-th eigenpair of $(\mathscr{P}^\varepsilon)$. As $\varepsilon \to 0$, we can prove the asymptotic expansion

$$\lambda_n^\varepsilon = \varepsilon^{-2}\mu_n + \ldots, \qquad u^\varepsilon(z) = v(z/\varepsilon) + \ldots$$

where $z := (x, y)$.

Elements of proof – higher eigenvalues

For $n > N_\bullet$, let u^ε be an eigenfunction associated with λ_n^ε.

FAR-FIELD EXPANSION. Far from O, consider the ansätze, as $\varepsilon \to 0$,

$$\lambda_n^\varepsilon = \varepsilon^{-2}\pi^2 + \boldsymbol{\kappa} + \dots, \quad u^\varepsilon(z) = \gamma(x)\,\sin(\pi y/\varepsilon) + \dots \text{ for } x \in I.$$

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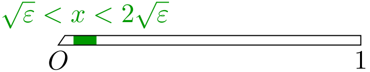
Take $U^0 = c^0\,W$, $U' = c'\,W$ where $c^0, c' \in \mathbb{R}$ and W is the one above.

Elements of proof – higher eigenvalues

- Let us match the two representations of u^ε in some intermediate region where both $x \to 0$ and $x/\varepsilon \to +\infty$:



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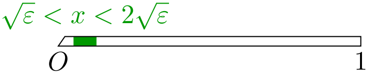

Far-field expansion at zero. As $x \to 0$, one has

$$\begin{aligned} \gamma(x)\,\sin(\pi y/\varepsilon) &= (\gamma(0) + \gamma'(0)x + \dots)\,\sin(\pi y/\varepsilon) \\ &= (\gamma(0) + \varepsilon\gamma'(0)\,\tfrac{x}{\varepsilon} + \dots)\,\sin(\pi y/\varepsilon). \end{aligned}$$

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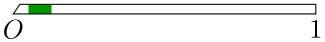

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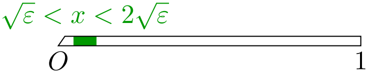

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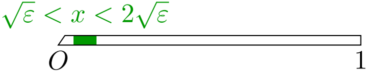
Elements of proof – higher eigenvalues

- Let us match the two representations of u^ε in some intermediate region where both $x \to 0$ and $x/\varepsilon \to +\infty$:



Far-field expansion at zero. As $x \to 0$, one has

$$\begin{aligned}\gamma(x)\,\sin(\pi y/\varepsilon) &= (\gamma(0) + \gamma'(0)x + \dots)\,\sin(\pi y/\varepsilon)\\ &= (\gamma(0) + \varepsilon\gamma'(0)\,\tfrac{x}{\varepsilon} + \dots)\,\sin(\pi y/\varepsilon).\end{aligned}$$

Inner-field expansion at infinity. As $x/\varepsilon \to +\infty$, one has

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- We conclude that we must impose

$$\gamma(0) = c^0 i\,(1-\mathbb{S}), \qquad 0 = c^0\,(1+\mathbb{S}) \qquad \text{and} \qquad \gamma'(0) = c'\,(1+\mathbb{S}).$$

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- Now the study depends on the value of $\mathbb{S}$.

Case $\mathbb{S} \neq -1 \;\Leftrightarrow\; \mathbf{X_\dagger = \{0\}}$**.**

We take $c^0 = 0$ and impose $\gamma(0) = 0$, *i.e.* Dirichlet at O in $(\mathscr{P}_{1\mathrm{D}})$.

Solving $(\mathscr{P}_{1\mathrm{D}})$, we get

$$\left|\begin{array}{l} \lambda_n^\varepsilon = \varepsilon^{-2}\pi^2 + m^2\pi^2 + \dots \\ u^\varepsilon(z) = \sin(m\pi x)\,\sin(\pi y/\varepsilon) + \dots \end{array}\right.$$
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We set $\gamma'(0) = 0$, *i.e.* Neumann at O in $(\mathscr{P}_{1\mathrm{D}})$, and take $c^0 = -i\gamma(0)/2$.

Solving $(\mathscr{P}_{1\mathrm{D}})$, we get

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far-field expansion.

Critical angles

Are there α such that

$$\mathbb{S}(\alpha) = -1 \quad \Leftrightarrow \quad \dim \mathrm{X}_\dagger = 1 \quad \Leftrightarrow \quad \left| \begin{array}{l} (\mathscr{P}_\dagger) \text{ admits a bounded solution} \\ \text{which does not decay at infinity} \end{array} \right.$$

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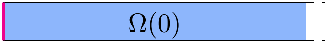
✓ Yes, this is true for $\boldsymbol{\alpha = 0}$ because



$$W(x,y) = \sin(\pi y) \quad \text{satisfies} \quad \left| \begin{array}{rcll} \Delta W + \pi^2 W & = & 0 & \text{in } \Omega(0) \\ W & = & 0 & \text{on } \Sigma(0) \\ \partial_\nu W & = & 0 & \text{on } \Gamma(0). \end{array} \right.$$

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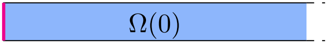

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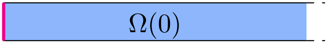

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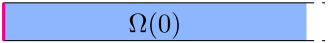

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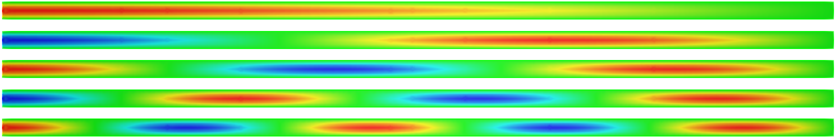
Numerically, we find non zero $\alpha_k^\star$ but proving their existence is an open pb.

Numerics

- Eigenfunctions of $(\mathscr{P}^\varepsilon)$ associated with the 5 first eigenvalues for $\varepsilon = 0.02$:

$\alpha = 0$



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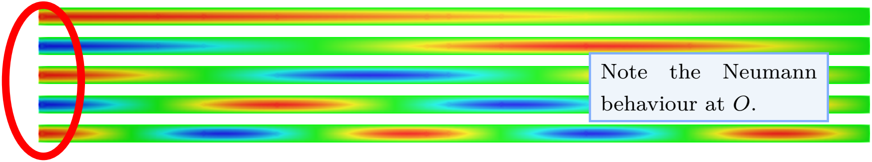

Note the Neumann behaviour at O.

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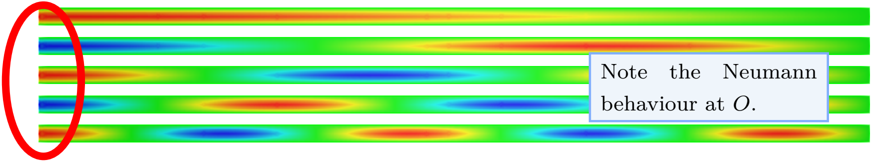
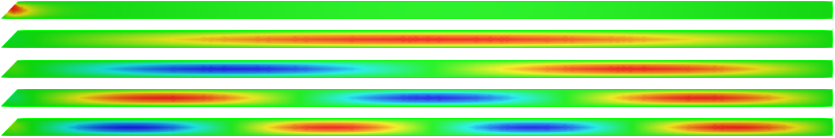


$\alpha = \pi/4$



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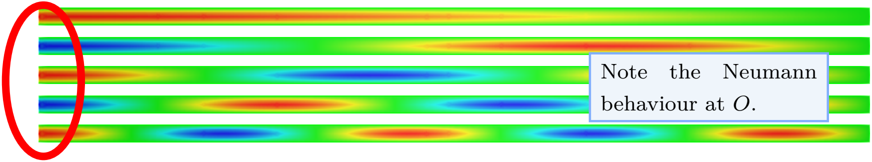

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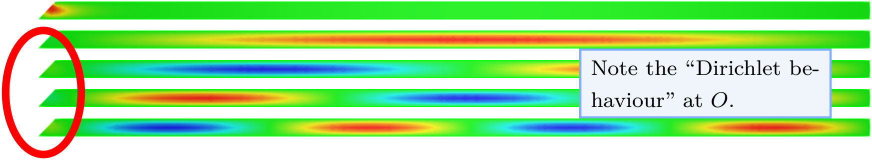

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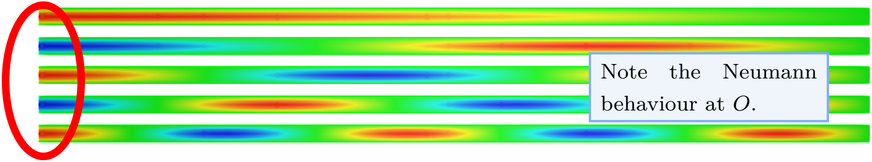

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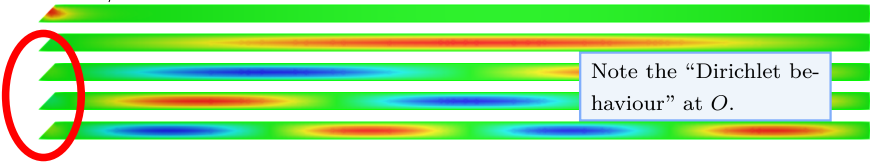
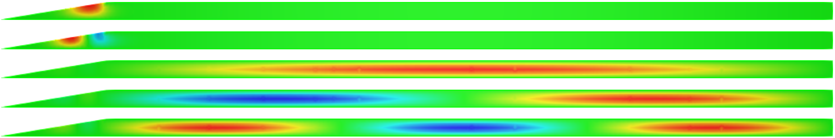


$\alpha = 0.45\pi$



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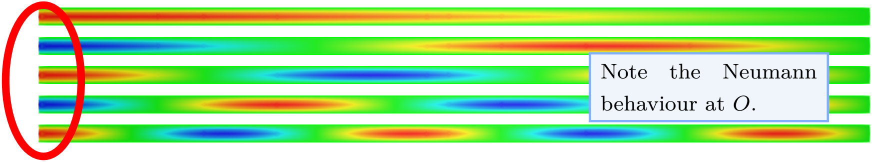

Note the Neumann behaviour at O.

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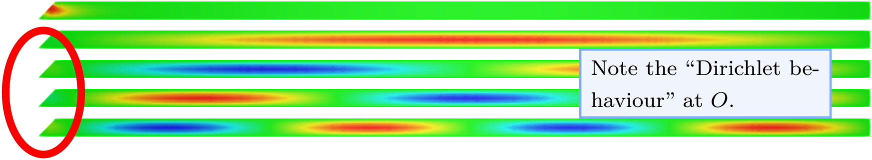

Note the “Dirichlet behaviour” at O.

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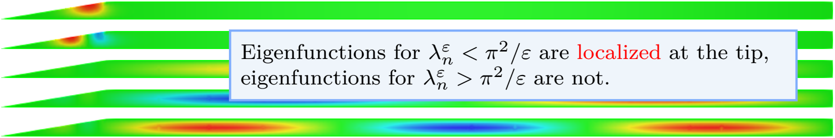

Eigenfunctions for $\lambda_n^\varepsilon < \pi^2/\varepsilon$ are localized at the tip, eigenfunctions for $\lambda_n^\varepsilon > \pi^2/\varepsilon$ are not.

Numerics

- Numerically, to compute the critical angles, we need to approximate

$$W = W_- + \mathbb{S}\,W_+ + \tilde{W} \qquad \text{satisfying} \qquad \left|\begin{array}{rcll} \Delta W + \pi^2 W & = & 0 & \text{in } \Omega \\ W & = & 0 & \text{on } \Sigma \\ \partial_\nu W & = & 0 & \text{on } \Gamma. \end{array}\right.$$

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$$\left|\begin{array}{rcll} \Delta \hat{W} + \pi^2 \hat{W} & = & 0 & \text{in } \Omega_L := \{z \in \Omega \,|\, x < L\} \\ \hat{W} & = & 0 & \text{on } \Sigma \cap \partial\Omega_L \\ \partial_\nu \hat{W} & = & 0 & \text{on } \Gamma \\ \partial_\nu(\hat{W} - W_-) & = & \dfrac{1}{L-i}(\hat{W} - W_-) & \text{at } x = L. \end{array}\right.$$

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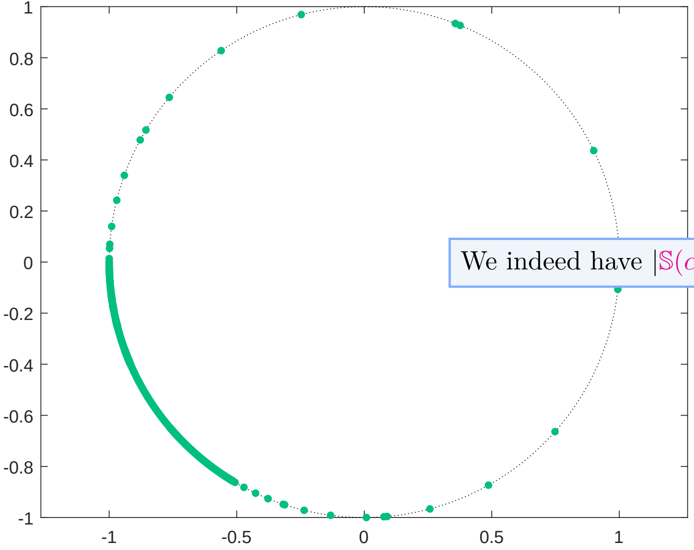
- Finally, from $\hat{W}$ we obtain an approximation of $\mathbb{S}$.

Numerics

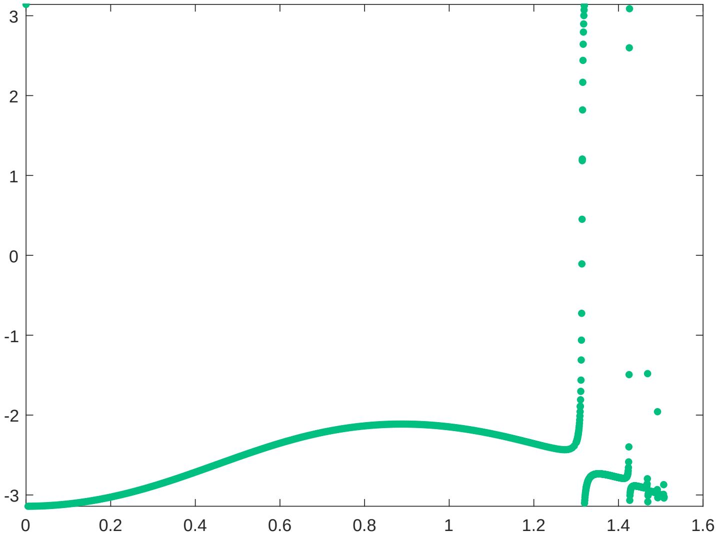
- $\mathbb{S}(\alpha)$ for 400 values of $\alpha \in [0; 0.48\pi)$ in the complex plane



We indeed have $|\mathbb{S}(\alpha)| = 1$.

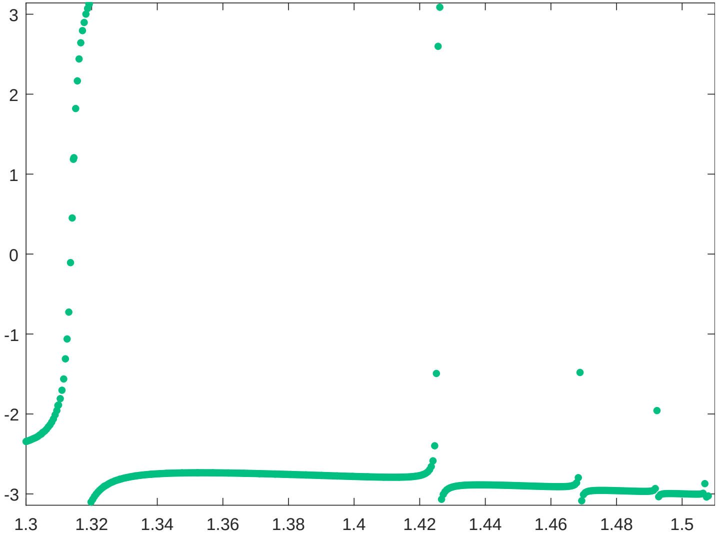
Numerics

- Phase of $\mathbb{S}(\alpha)$ for $\alpha \in [0; 0.48\pi)$



Numerics

- Phase of $\mathbb{S}(\alpha)$ for $\alpha \in [1.3; 0.48\pi)$



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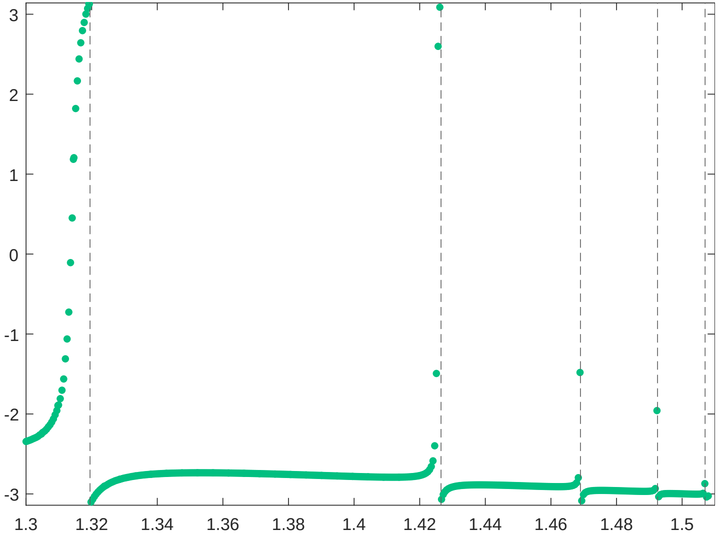
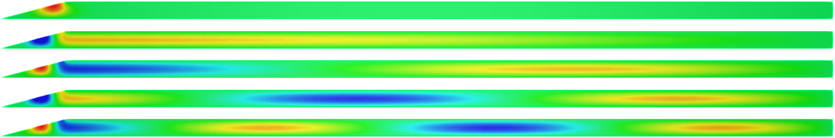


Numerically, we obtain a sequence $(\alpha_k^\star)$ which accumulates at $\pi/2$.

Numerics at a positive critical angle

- First eigenfunctions of $(\mathscr{P}^{\varepsilon})$ for $\varepsilon = 0.02$ and $\alpha = \alpha_1^{\star} \approx 1.321$:



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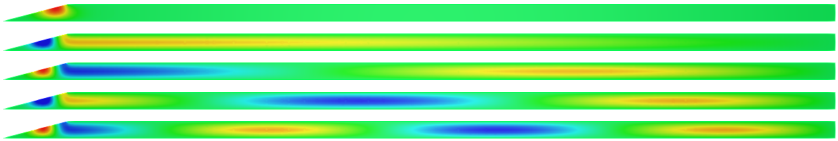
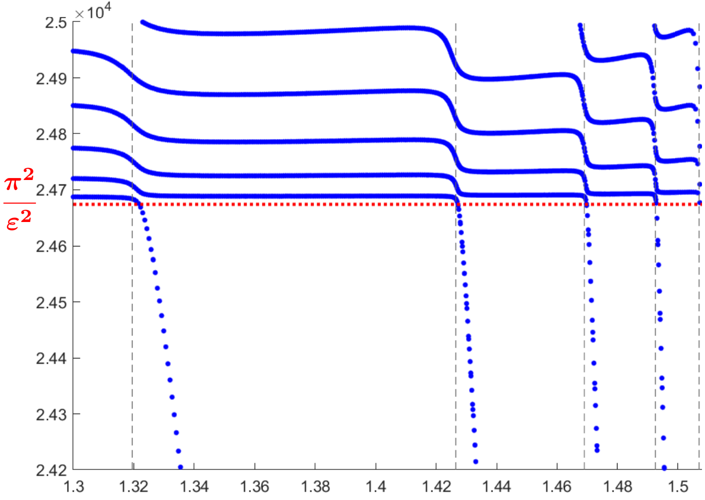


We indeed have a "**Neumann behaviour**" at O for the far-field of the eigenfunctions associated to the $\lambda_n^{\varepsilon} > \pi^2/\varepsilon^2$.

Initial picture

- Eigenvalues of $(\mathscr{P}^\varepsilon)$ wrt $\alpha \in (1.3; 0.48\pi)$



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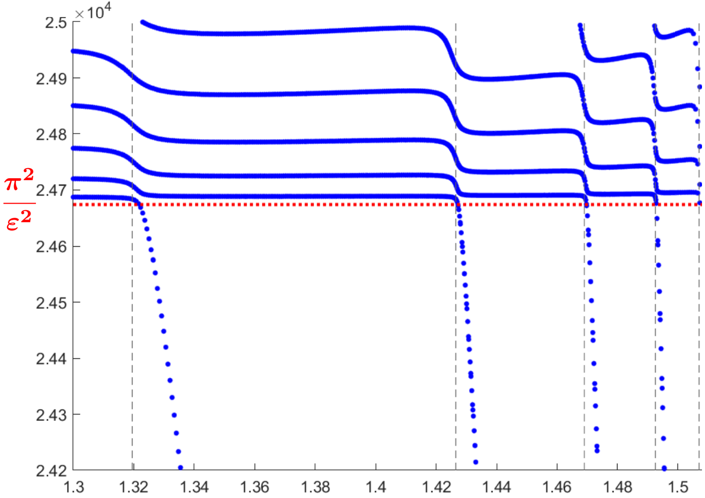

✓ We have identified the **critical angles** where an eigenvalue dives;
✗ We have not explained the phenomenon of diving eigenvalue.

Setting

Goal now:

To describe the **transition** between the model with **Dirichlet** at O valid for $\alpha \neq \alpha_k^\star$ and the one with **Neumann** at O valid for $\alpha = \alpha_k^\star$.

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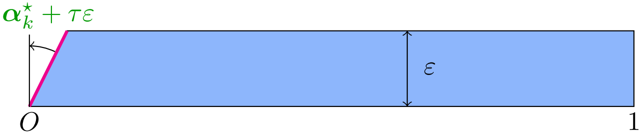

- For $\tau \in \mathbb{R}$ and $k \in \mathbb{N}$, define the trapezoid

$$\mathrm{T}_\tau^\varepsilon := \mathrm{T}^\varepsilon(\alpha_k^\star + \tau\varepsilon).$$

Both the small side and the angle of the tip depend on ε.

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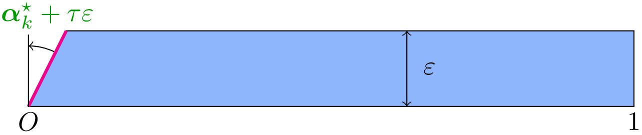

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- We denote with a subscript τ all the quantities introduced previously:

$$0 < \lambda_{\tau,1}^\varepsilon < \lambda_{\tau,2}^\varepsilon \leq \cdots \leq \lambda_{\tau,n}^\varepsilon \leq \ldots \quad \to +\infty.$$

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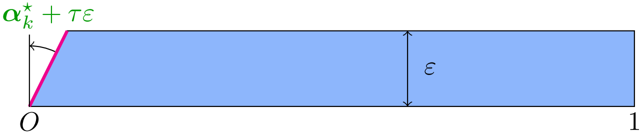

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- We wish to obtain an asymptotic expansion of the $\lambda_{\tau,n}^\varepsilon$ as $\varepsilon \to 0$.

Asymptotic analysis

- For an eigenpair $(\lambda_\tau^\varepsilon, u_\tau^\varepsilon)$ close to π^2/ε^2, consider the ansätze

$$\left|\begin{array}{lcll} \lambda_\tau^\varepsilon & = & \varepsilon^{-2}\pi^2 + \eta_\tau + \dots & \\ u_\tau^\varepsilon(z) & = & \gamma_\tau(x)\sin(\pi y/\varepsilon) + \dots & \text{(far-field expansion)} \\ u_\tau^\varepsilon(z) & = & U_\tau^0(z/\varepsilon) + \varepsilon\, U_\tau'(z/\varepsilon) + \dots & \text{(near-field expansion)} \end{array}\right.$$

where the constants η_τ and the functions γ_τ, U_τ^0, U_τ' are to be determined.

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where the constants η_τ and the functions γ_τ, U_τ^0, U_τ' are to be determined.

- Inserting it in $(\mathscr{P}^\varepsilon)$, we obtain

$$\left|\begin{array}{rcll} \partial_x^2\gamma_\tau + \eta_\tau\gamma_\tau & = & 0 & \text{in } I \\ \gamma_\tau(1) & = & 0 & \end{array}\right. \qquad \left|\begin{array}{rcll} \Delta U_\tau^0 + \pi^2 U_\tau^0 & = & 0 & \text{in } \Omega(\alpha_k^\star) \\ U_\tau^0 & = & 0 & \text{on } \Sigma(\alpha_k^\star) \\ \partial_\nu U_\tau^0 & = & 0 & \text{on } \Gamma(\alpha_k^\star) \end{array}\right.$$

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Asymptotic analysis

- From the Taylor series

$$\begin{aligned}\gamma_\tau(x)\,\sin(\pi y/\varepsilon) &= (\gamma_\tau(0) + \partial_x\gamma_\tau(0)x + \dots)\,\sin(\pi y/\varepsilon)\\ &= (\gamma_\tau(0) + \;\varepsilon\partial_x\gamma_\tau(0)\,\tfrac{x}{\varepsilon}\; + \dots)\,\sin(\pi y/\varepsilon),\end{aligned}$$

we set $\boxed{U_\tau^0 = \gamma_\tau(0)\,W}$ $\left(\begin{array}{l} W = \sin(\pi Y) + \tilde{W} \text{ is the solution} \\ \text{of } (\mathscr{P}_\dagger) \text{ introduced before.}\end{array}\right)$.

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$$U'_\tau(Z) = (C_0 + C_1 X)\sin(\pi Y) + \tilde{U}'_\tau(Z)$$

where C_0 (not unique), $C_1 \in \mathbb{R}$ and $\tilde{U}'_\tau \in \mathrm{H}^1(\Omega(\alpha^\star_k))$.

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- Matching the terms in x at order ε, we conclude that we must impose

$$\boxed{\partial_x\gamma_\tau(0)=-2\tau B\gamma_\tau(0)}.$$

Model problem at positive critical angles

- Finally, for γ_τ, we obtain the model problem with Robin BC at O

$$(\mathscr{P}^{\tau}_{\mathrm{1D}}) \quad \left| \begin{array}{rcll} \partial_x^2\gamma_\tau + \eta_\tau\gamma_\tau & = & 0 & \text{in } I \\ \gamma_\tau(1) & = & 0 & \\ \partial_x\gamma_\tau(0) & = & -2\tau B\gamma_\tau(0). & \end{array} \right.$$

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THEOREM: Consider a **positive** critical angle $\alpha_k^\star$ and some $\tau \in \mathbb{R}$.
For $n = N_\bullet + m$, $m \in \mathbb{N}^*$, there are constants C, $\delta > 0$, s.t. as $\varepsilon \to 0$,

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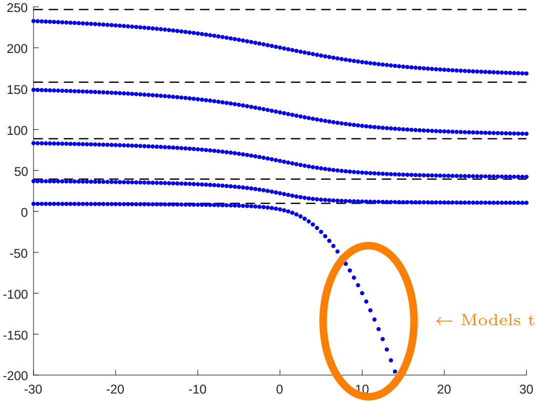

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$$\eta_{1,\tau} \underset{\tau\to\pm\infty}{\sim} -4D^2\tau^4 \qquad \text{and for } n \geq 2, \qquad \lim_{\tau\to\pm\infty} \eta_{n,\tau} = (n-1)^2\pi^2.$$

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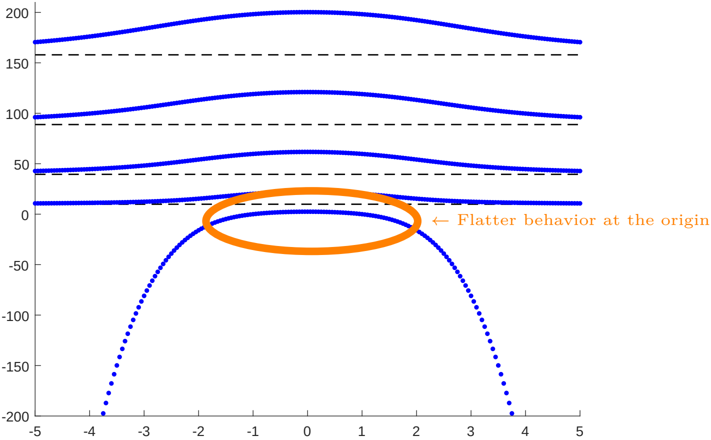

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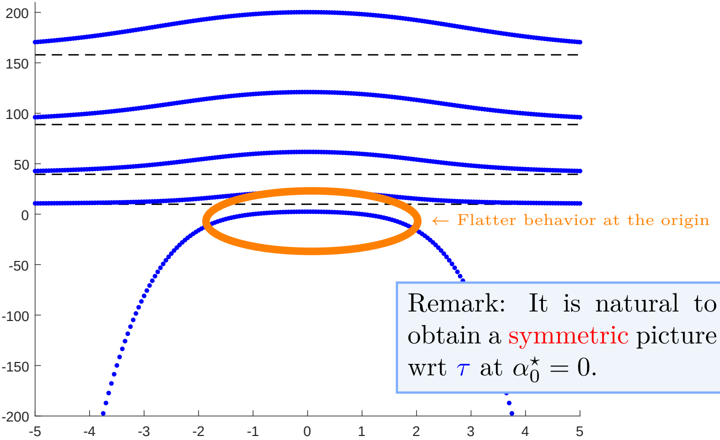

Remark: It is natural to obtain a symmetric picture wrt τ at $\alpha_0^\star = 0$.

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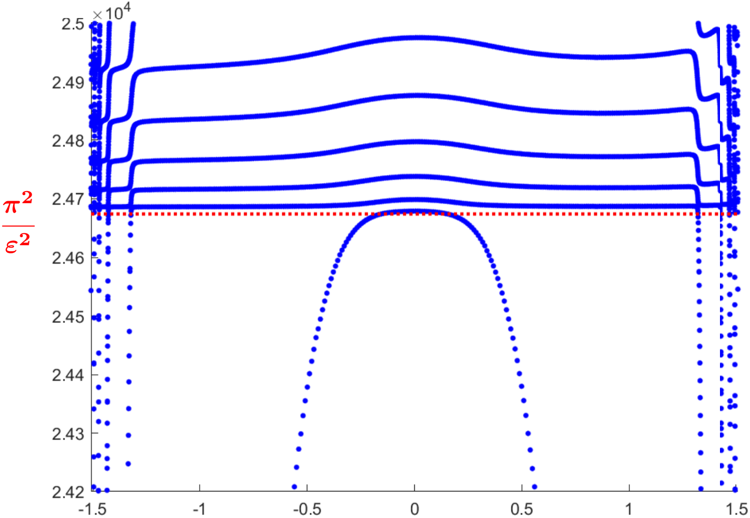
Numerics

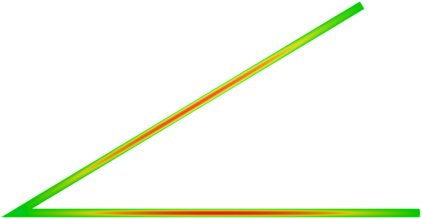
- Spectrum of $(\mathscr{P}^\varepsilon)$ with respect to $\alpha \in (-\pi/2;\pi/2)$ for $\varepsilon = 0.02$:



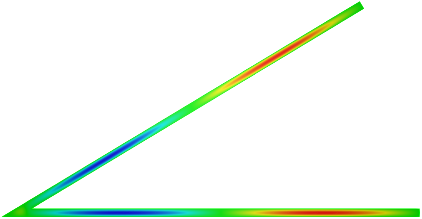
This explains the mild behavior of the diving eigenvalue at $\alpha = 0$.

Numerics in the broken strip

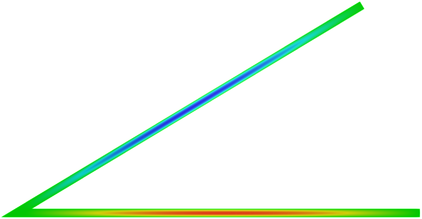
- Eigenfunctions for α varying around $\alpha_1^\star$ and $\varepsilon = 0.02$:



Eigenfunction 2



Eigenfunction 4



Eigenfunction 3

And for the Neumann Laplacian?

- Consider the problem with Neumann BCs

$$\left| \begin{array}{rcll} -\Delta u &=& \lambda^\varepsilon u & \text{in } \mathrm{T}^\varepsilon(\alpha) \\ \partial_\nu u &=& 0 & \text{on } \partial\mathrm{T}^\varepsilon(\alpha). \end{array} \right.$$

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No large variation of the behavior of eigenvalues when varying α.

⇒

Conclusion

What we did

- ♠ We studied the asymptotics of the spectrum of the Dirichlet Laplacian in thin broken strips.
  - All eigenvalues go to $+\infty$ as $O(\varepsilon^{-2})$;
  - The behavior of the first eigenvalues depends on the discrete spectrum of the near field operator;
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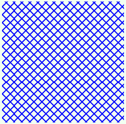
Possible extensions and open questions



1) We can work similarly in quasi 1D periodic waveguides.
2) Can one find examples of Ω such that $\dim \mathrm{X}_\dagger = 2$?
3) Can one work with other models, *i.e.* $\Delta\Delta u - k^4 u = 0$+Dirichlet BC?

Thank you!

- P. Cacciapuoti, P. Exner, *Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide*, J. Phys. A, Math. Theor., vol. 40, 26:511-523, 2007.
- L. Chesnel, S.A. Nazarov, *Eigenvalue falls in thin broken quantum strips*, arXiv:2508.08403, 2026.
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- S.A. Nazarov, *Transmission conditions in one-dimensional model of a rectangular lattice of thin quantum waveguides*, J. Math. Sci., vol. 219, 6:994-1015, 2016.
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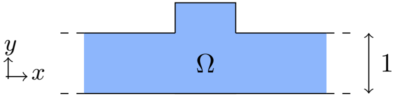
Setting

- We consider the propagation of waves in a 2D thin periodic quantum waveguide Π^ε.

- Start with some domain $\Omega \subset \mathbb{R}^2$ which coincides with the strip $\mathbb{R} \times (0;1)$ outside of a bounded region (the resonator).



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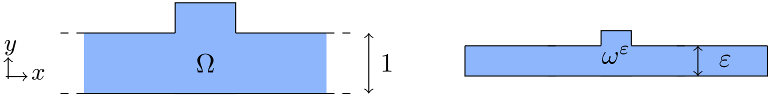

- Shrink Ω by a small factor $\varepsilon > 0$ to create the unit cell

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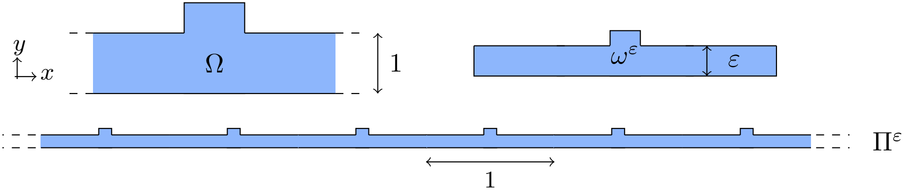

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- Define the periodic waveguide

$$\Pi^\varepsilon := \{z \in \mathbb{R}^2 \mid (x-m, y) \in \omega^\varepsilon,\ m \in \mathbb{Z}\}.$$

- In Π^ε we consider the spectral problem for the Dirichlet Laplacian

$$(\mathscr{P}^\varepsilon)\left|\begin{array}{rcll} -\Delta u^\varepsilon & = & \lambda^\varepsilon\, u^\varepsilon & \text{in } \Pi^\varepsilon \\ u^\varepsilon & = & 0 & \text{on } \partial\Pi^\varepsilon. \end{array}\right.$$

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Setting 2/2

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Goal of the section

We wish to study the lower part of $\sigma(A^\varepsilon)$, the spectrum of A^ε, as $\varepsilon \to 0$.

Reduction to a problem in the unit cell 1/2

- The Floquet Bloch transform

$$u^\varepsilon(z) \mapsto U^\varepsilon(z,\eta) = \frac{1}{(2\pi)^{1/2}} \sum_{j\in\mathbb{Z}} e^{i\eta j} u^\varepsilon(x+j,y), \quad \eta \in \mathbb{R},$$

converts $(\mathscr{P}^\varepsilon)$ into a spectral pb set in ω^ε with quasi-periodic BCs at $x = \pm 1/2$

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$$(\mathscr{P}^\varepsilon(\eta)) \left| \begin{array}{rcll} -\Delta U^\varepsilon(z,\eta) &=& \lambda^\varepsilon(\eta)\, U^\varepsilon(z,\eta) & z \in \omega^\varepsilon \\ U^\varepsilon(z,\eta) &=& 0 & z \in \partial\omega^\varepsilon \cap \partial\Pi^\varepsilon \\ U^\varepsilon(-1/2,y,\eta) &=& e^{i\eta} U^\varepsilon(+1/2,y,\eta) & y \in (-\varepsilon/2;\varepsilon/2) \\ \partial_x U^\varepsilon(-1/2,y,\eta) &=& e^{i\eta} \partial_x U^\varepsilon(+1/2,y,\eta) & y \in (-\varepsilon/2;\varepsilon/2). \end{array} \right.$$

- It suffices to study $(\mathscr{P}^\varepsilon(\eta))$ for $\eta \in [0;2\pi)$.

Reduction to a problem in the unit cell 1/2

- The Floquet Bloch transform

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- It suffices to study $(\mathscr{P}^\varepsilon(\eta))$ for $\eta \in [0;2\pi)$.

- For $\eta \in [0;2\pi)$, the spectrum of $(\mathscr{P}^\varepsilon(\eta))$ is discrete, made of the unbounded sequence of real eigenvalues

$$0 < \lambda_1^\varepsilon(\eta) \le \lambda_2^\varepsilon(\eta) \le \cdots \le \lambda_p^\varepsilon(\eta) \le \ldots.$$

Reduction to a problem in the unit cell 2/2

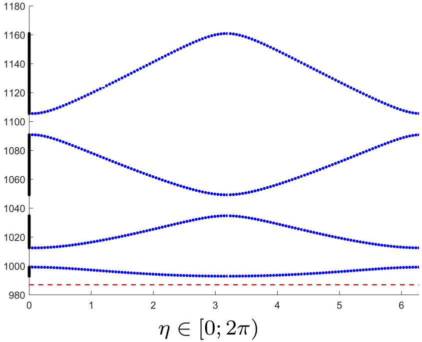
- The functions

$$\eta \mapsto \lambda_p^\varepsilon(\eta)$$

are continuous so that the spectral bands

$$\boxed{\Upsilon_p^\varepsilon := \{\lambda_p^\varepsilon(\eta),\ \eta \in [0;2\pi)\}}$$

are compact segments in $[0;+\infty)$.



- Finally, we have $\boxed{\sigma(A^\varepsilon) = \bigcup_{p\in\mathbb{N}^*:=\{1,2,\dots\}} \Upsilon_p^\varepsilon.}$

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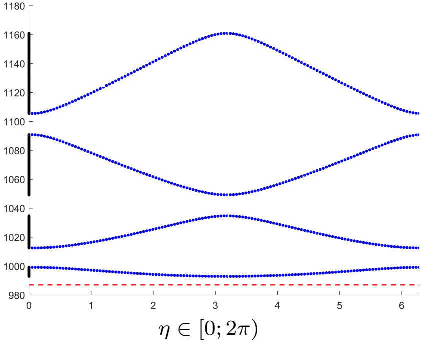

- Finally, we have $\boxed{\sigma(A^\varepsilon) = \bigcup_{p\in\mathbb{N}^*:=\{1,2,\dots\}} \Upsilon_p^\varepsilon.}$

To study the behaviour of $\sigma(A^\varepsilon)$ as $\varepsilon \to 0$, we have to consider the asymptotics of $\lambda_p^\varepsilon(\eta)$ as $\varepsilon \to 0$.

Asymptotic analysis

- For $p \in \mathbb{N}^*$, denote by $a^\varepsilon_{p-} \leq a^\varepsilon_{p+}$ the bounds of the spectral band Υ^ε_p:

$$\Upsilon^\varepsilon_p = [a^\varepsilon_{p-}; a^\varepsilon_{p+}].$$

- Working as in the first section, we can establish the

THEOREM: There are constants $c_{p-} < c_{p+}$, $C_p, \delta_p > 0$ s.t. as $\varepsilon \to 0$,

For $p = 1, \dots, N_\bullet$:

$$\left| a^\varepsilon_{p\pm} - \left(\varepsilon^{-2}\mu_p + \varepsilon^{-2} e^{-\sqrt{\pi^2-\mu_p}/\varepsilon} c_{p\pm} \right) \right| \leq C_p \, e^{-(1+\delta_p)\sqrt{\pi^2-\mu_p}/\varepsilon};$$

For $p = N_\bullet + m$, $m \in \mathbb{N}^*$:

$i)$ if $\mathrm{X}_\dagger = \{0\}$, $\quad \left| a^\varepsilon_{p\pm} - \left(\varepsilon^{-2}\pi^2 + m^2\pi^2 + \varepsilon c_{p\pm} \right) \right| \leq C_p \, \varepsilon^{1+\delta_p};$

$ii)$ if $\dim \mathrm{X}_\dagger = 1$, $\quad \left| a^\varepsilon_{p\pm} - \left(\varepsilon^{-2}\pi^2 + c_{p\pm} \right) \right| \leq C_p \, \varepsilon^{\delta_p};$

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When **$\dim \mathbf{X}_\dagger \neq 1$**, the Υ_p^ε are of length $O(\varepsilon)$. Moreover, between Υ_p^ε and $\Upsilon_{p+1}^\varepsilon$, there is a gap whose length tends to $(2m+1)\pi^2$.

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Generically, the propagation of waves in Π^ε is hampered and occurs only for very narrow intervals of frequencies.

When $\mathbf{dim}\,\mathbf{X}_\dagger = \mathbf{1}$, the situation is very different because asymptotically the Υ_p^ε are of length $c_{p+} - c_{p-}$, with in general $c_{p+} > c_{p-}$.

For particular Ω, waves can propagate in Π^ε for much larger intervals of frequencies than above.

Spectral breathing

Then we can describe the change of $\sigma(A^\varepsilon)$ when perturbing the inner field geometry around a $\Omega_\star$ where $\dim \mathrm{X}_\dagger = 1$.

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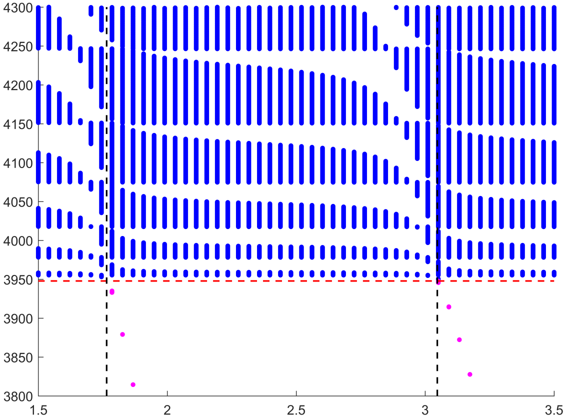

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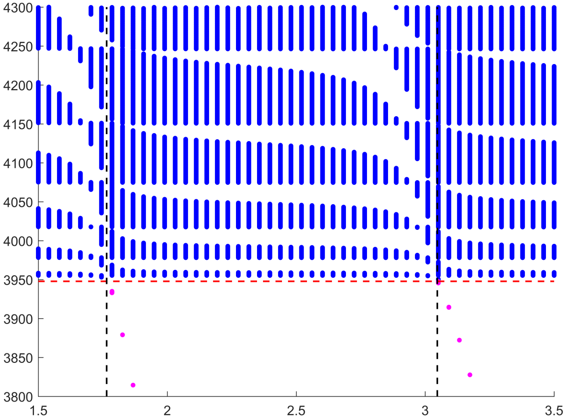


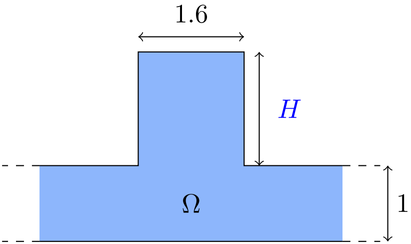
- By deriving model problems as in the previous section, one shows:

- When perturbing the near field geometry around $\Omega_\star$, the spectral bands expand and shrink $\Rightarrow$ breathing phenomenon of $\sigma(A^\varepsilon)$.

- In the process, a band **dives** below π^2/ε^2, stops breathing and becomes extremely short.

Numerical illustration

- We start from the inner field geometry



that we shrink by a factor ε, cut at $x = \pm 1/2$ and periodize.

- We use `Freefem++` to compute the spectrum of $(\mathscr{P}^{\varepsilon}(\eta))$ in the corresponding unit cell for different values of H.

Numerical illustration