Construction of invisible defects for acoustic problems with a finite number of measurements

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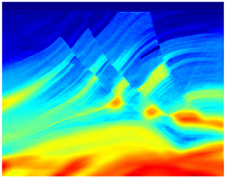
General setting

- We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

- General principle of the methods:
  i) send waves in the medium;
  ii) measure the scattered field;
  iii) deduce information on the structure.





- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Goal of the talk

- The goal of imaging techniques is to find features of the structure from the knowledge of measurements.

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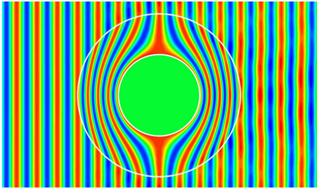
  - Also relevant for applications.



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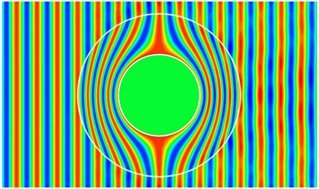
  1. Scattering in free space
  2. Scattering in waveguides



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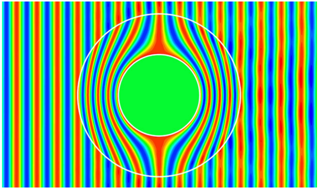

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- At least two reasons to study invisibility questions:
  - We can wish to hide objects.
  - It allows to understand limits of imaging techniques.

Outline of the talk

1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments

2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
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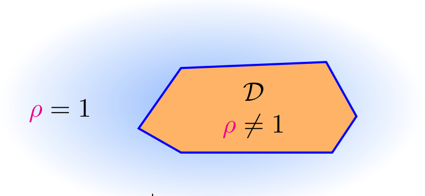
Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.



$$\left|\begin{array}{l} \text{Find } u \text{ such that} \\ \begin{array}{rcll} -\Delta u & = & k^2 \rho\, u & \text{in } \mathbb{R}^2, \\ u & = & u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \end{array} \\ \displaystyle\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}} \right) = 0. \end{array}\right. \tag{1}$$

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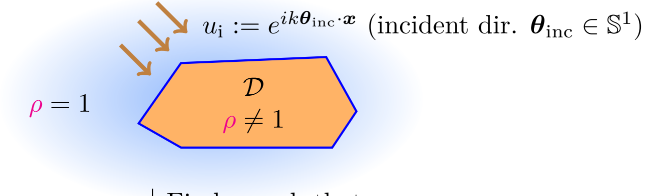

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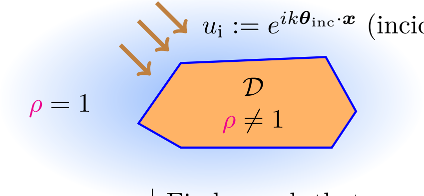

$u_{\mathrm{i}} := e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}}$ (incident dir. $\boldsymbol{\theta}_{\mathrm{inc}} \in \mathbb{S}^1$)

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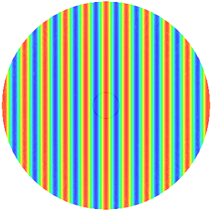
DEFINITION:
- u_{i} = incident field (data)
- u = total field (uniquely defined by (1))
- u_{s} = scattered field (uniquely defined by (1)).

Far field pattern

$\Re e\, u_{\mathrm{i}}$



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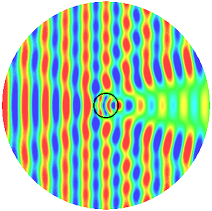

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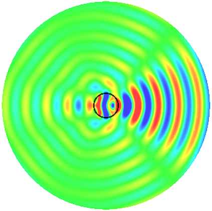

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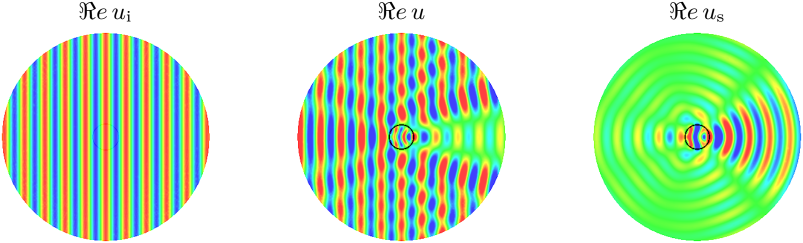

- The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\mathrm{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \Big(\, u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{sca}}, \boldsymbol{\theta}_{\mathrm{inc}}) \, + O(1/r) \Big).$$

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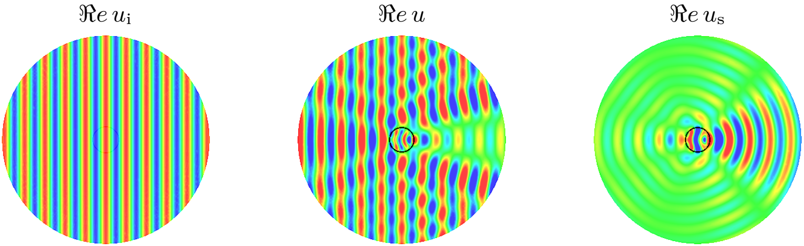

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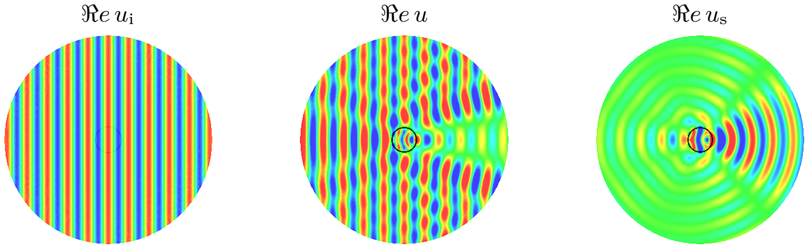

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At infinity, one measures the far field pattern (other terms are too small).

Setting

- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_{\mathrm{s}}^{\infty}(\cdot,\cdot)$ on a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.

  – In literature, most of the techniques require a continuum of data.

(Nachman, 1988, Sylvester & Uhlmann, 1987, Bukhgeim, 2008, Imanuvilov & Yamamoto, 2012)

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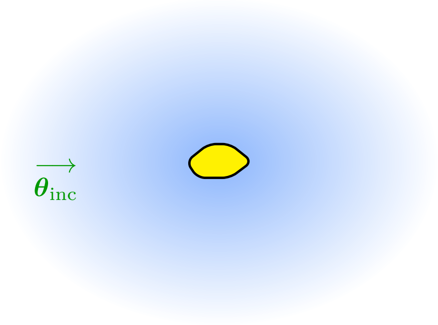
- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



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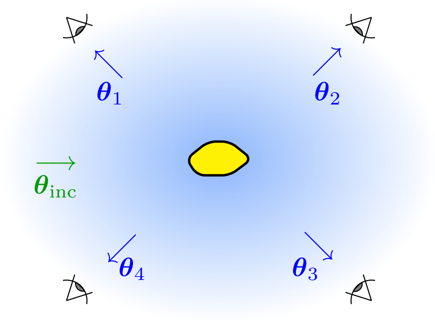

$\rightarrow$ We measure $u_s^\infty(\boldsymbol{\theta}_1), \ldots, u_s^\infty(\boldsymbol{\theta}_N)$.

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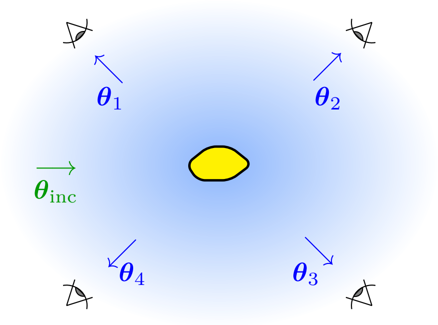

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We explain how to construct inclusions such that

$$u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0.$$

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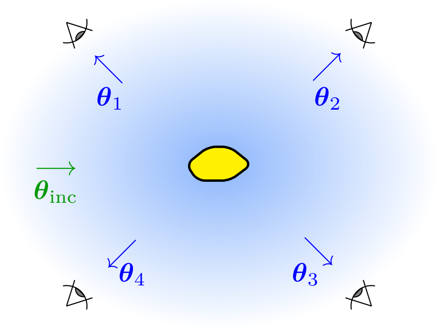

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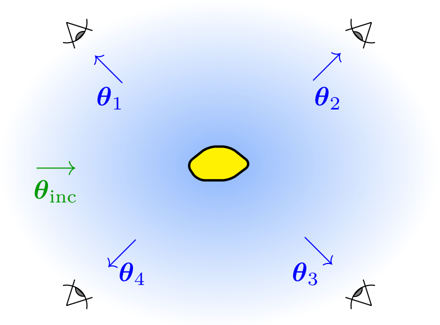

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- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_N$ (given).

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution to the problem

$$\left|\begin{array}{l} \text{Find } u = u_{\text{s}} + e^{ik\boldsymbol{\theta}_{\text{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2 \rho\, u \quad \text{ in } \mathbb{R}^2, \\ \qquad\qquad u_{\text{s}} \text{ is outgoing} \end{array}\right.$$

satisfies $u_{\text{s}}^{\infty}(\boldsymbol{\theta}_1) = \dots = u_{\text{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

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Sketch of the method

- We will work as in the proof of the implicit functions theorem.

  The idea was used in `Nazarov 11` to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
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- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

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- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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where $\vec{\tau} = (\tau_1, \dots, \tau_{2N})^\top$

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$$\exists \mu_0, \mu_1, \ldots, \mu_{2N} \in \mathrm{L}^\infty(\mathcal{D}) \text{ s.t. } \left| \begin{array}{l} dF(0)(\mu_0) = 0 \\ [dF(0)(\mu_1), \ldots, dF(0)(\mu_{2N})] = \mathrm{Id}_{2N}. \end{array} \right.$$

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$$0 = F(\varepsilon\mu) \qquad \Leftrightarrow \qquad \boxed{\vec{\tau} = G^\varepsilon(\vec{\tau})}$$

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Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$

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If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$. Set $\sigma^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (invisible inclusion).

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

$$F(\sigma) = (\Re e\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Re e\, u_s^\infty(\boldsymbol{\theta}_N), \Im m\, u_s^\infty(\boldsymbol{\theta}_1), \ldots, \Im m\, u_s^\infty(\boldsymbol{\theta}_N)).$$

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- We obtain the expansion (Born approx.), for small ε

$$u_{\mathrm{s}}^{\varepsilon\,\infty}(\boldsymbol{\theta}_n) = 0 + \varepsilon\, c\, k^2 \int_{\mathcal{D}} \mu\, e^{ik(\boldsymbol{\theta}_{\mathrm{inc}} - \boldsymbol{\theta}_n)\cdot\boldsymbol{x}}\, d\boldsymbol{x} \; + O(\varepsilon^2).$$

Calculus of $dF(0)$ 1/2

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$$\begin{aligned} dF(0)(\mu) = \Big(&\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\rm inc} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\rm inc} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \\ &\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\rm inc} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\rm inc} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}\Big) \end{aligned}$$

Calculus of $dF(0)$ 2/2

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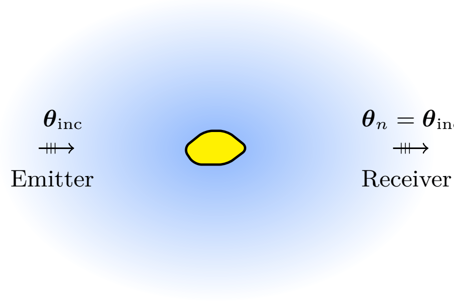
- Clearly, we need to avoid the configuration $\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n = 0$.



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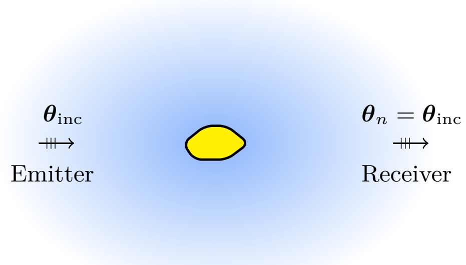

- And one can prove that if $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$, $n = 1, \dots, N$, the answer is yes.

Main result

PROPOSITION: Assume that $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\mathrm{inc}}$ for $n = 1, \ldots, N$. For $\boldsymbol{\varepsilon}$ **small enough**, define $\rho^{\,\mathrm{sol}} = 1 + \varepsilon\mu^{\,\mathrm{sol}}$ with

$$\mu^{\,\mathrm{sol}} = \mu_0 + \sum_{n=1}^{2N} \tau_n^{\,\mathrm{sol}}\,\mu_n.$$

Then the solution of the scattering problem

$$\left|\begin{array}{l} \text{Find } u^\varepsilon = u_{\mathrm{s}}^\varepsilon + e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}} \text{ such that} \\ -\Delta u \quad = \quad k^2\rho^{\,\mathrm{sol}}\,u \quad \text{ in } \mathbb{R}^2, \\ \qquad\qquad u_{\mathrm{s}} \text{ is outgoing} \end{array}\right.$$

satisfies $u_{\mathrm{s}}^\infty(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^\infty(\boldsymbol{\theta}_N) = 0$.

COMMENTS:

$\rightarrow$ We need $\boldsymbol{\varepsilon}$ to be **small enough** to prove that G^ε is a contraction.

$\rightarrow$ We have $\mu^{\,\mathrm{sol}} \not\equiv 0$ (non trivial inclusion). To see it, compute $dF(0)(\mu^{\,\mathrm{sol}})$.

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→ We have $\tau^{\text{sol}} = O(\varepsilon) \Rightarrow \mu^{\text{sol}} \approx \mu_0$. We control the main form of the defect.

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Existence of invisible inclusions may appear not so surprising since there are $2N$ measurements and $\rho \in \mathrm{L}^\infty(\mathcal{D})$.

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Existence of invisible inclusions may appear not so surprising since there are $2N$ measurements and $\rho \in \mathrm{L}^\infty(\mathcal{D})$. *Let us see the case* $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$...

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1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments
2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
   - Can one hide a perturbation of the wall?

The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

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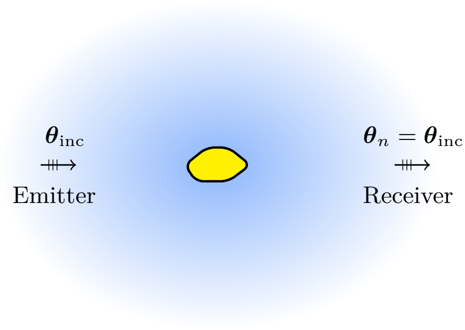
  What if $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$?



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- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- `Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014`
- `Corners and edges always scatter, J. Elschner, G. Hu, 2015`

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

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1. Invisibility in free space
    - The general scheme
    - The forbidden case
    - Numerical experiments
2. Invisibility for waveguide problems
    - Construction of invisible penetrable defects
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Data and algorithm

- We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \dots, 0)^\top$ then define

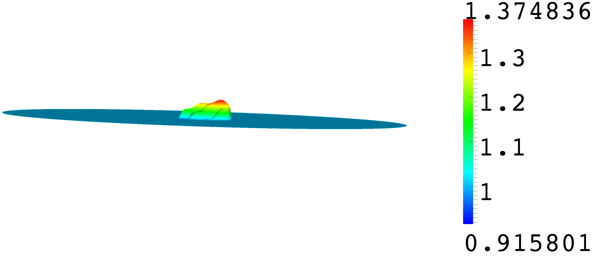
$$\boxed{\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n}).}$$

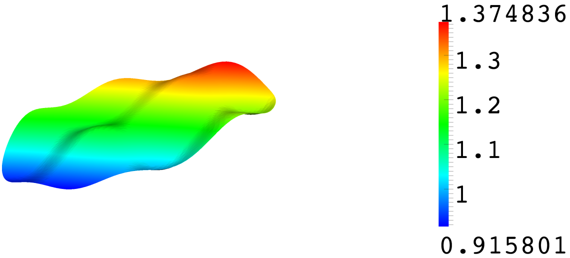
- At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

- For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{ll} \boldsymbol{\theta}_{\mathrm{inc}} = (\cos(\psi_{\mathrm{inc}}), \sin(\psi_{\mathrm{inc}})), & \psi_{\mathrm{inc}} = 0^\circ \\ \boldsymbol{\theta}_1 = (\cos(\psi_1), \sin(\psi_1)), & \psi_1 = 90^\circ \\ \boldsymbol{\theta}_2 = (\cos(\psi_2), \sin(\psi_2)), & \psi_2 = 180^\circ \\ \boldsymbol{\theta}_3 = (\cos(\psi_3), \sin(\psi_3)), & \psi_3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

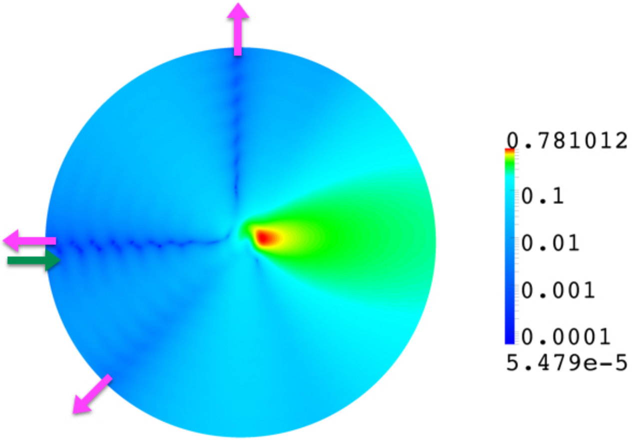


Figure: $|u_s|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles 90°, 180° and 225°. The domain is equal to B_8.

Results: far field pattern

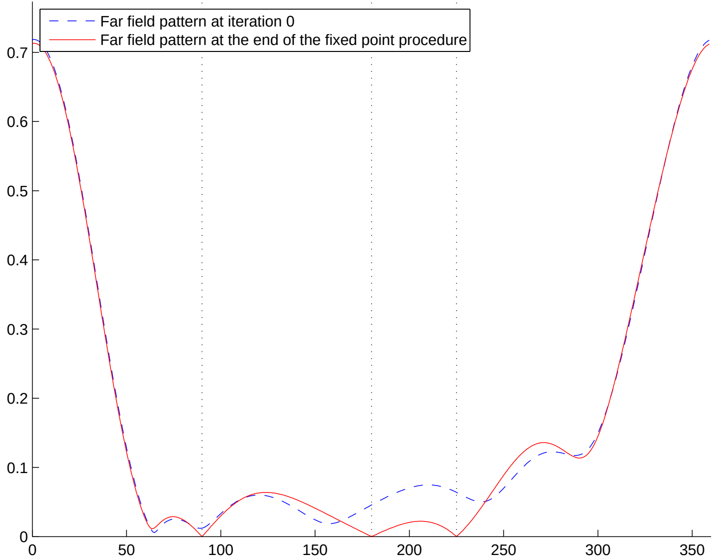


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

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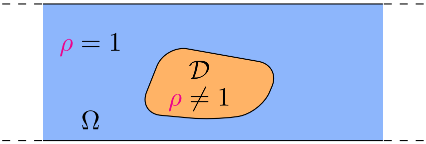
Waveguide problem

► Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\Omega := \{(x, y) \in \mathbb{R} \times (0; 1)\}$.



Find $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ s. t.

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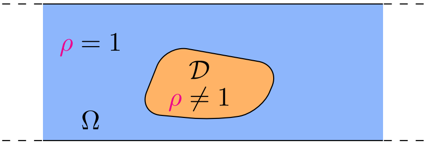

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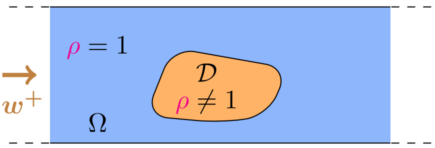

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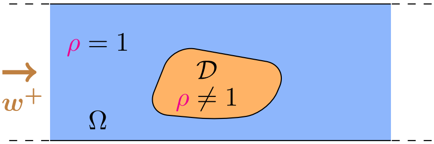

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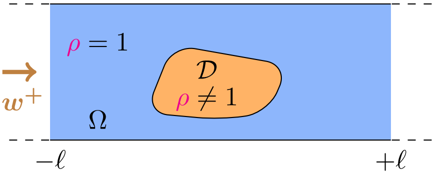

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(*$\chi^{\pm}$ are smooth cut-off functions s.t. $\chi^{\pm} = 1$ for $\pm x \geq 2\ell$, $\chi^{\pm} = 0$ for $\pm x \leq \ell$*)

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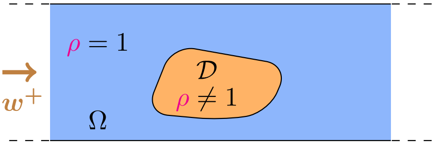

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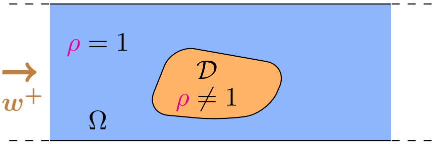

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► Conservation of energy implies $|s^-|^2 + |1 + s^+|^2 = 1$.

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- u_s is outgoing means that there are some $s^{\pm} \in \mathbb{C}$ such that

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Definition: Inclusion is said $\left|\begin{array}{l}\text{non reflective if } s^- = 0\\ \text{perfectly invisible if } s^+ = 0.\end{array}\right.$

- Due to conservation of energy $|s^-|^2 + |1 + s^+|^2 = 1$,

$$\boxed{s^+ = 0 \quad \Rightarrow \quad s^- = 0} \qquad \text{(and } u_s \text{ is expo. decay. at } \pm\infty\text{).}$$

The converse is wrong ($s^- = 0 \not\Rightarrow s^+ = 0$).

Penetrable inclusion

- Set $F(\sigma) = (\Re e\,\dfrac{s^-}{ik^2}, \Im m\,\dfrac{s^-}{ik^2}, \Re e\,\dfrac{s^+}{ik^2}, \Im m\,\dfrac{s^+}{ik^2})$ with $\sigma = \rho - 1$.

  Again, we wish to find $\sigma \not\equiv 0$ such that $F(\sigma) = 0$.

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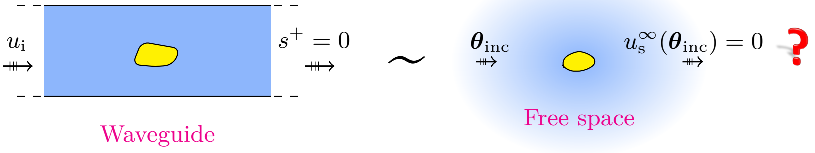

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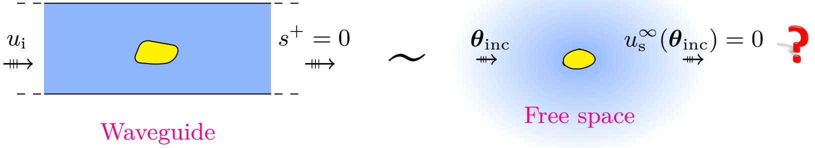

Impose $\left| \begin{array}{l} s^- = 0 \\ \Im m\, s^+ = 0 \end{array} \right.$. *Then,* $\left| \begin{array}{l} |s^-|^2 + |1 + s^+|^2 = 1 \\ s^+ = O(\varepsilon) \end{array} \right.$ $\Rightarrow$ $s^+ = 0$.

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> Can we increase the perturbation to obtain larger defects ?

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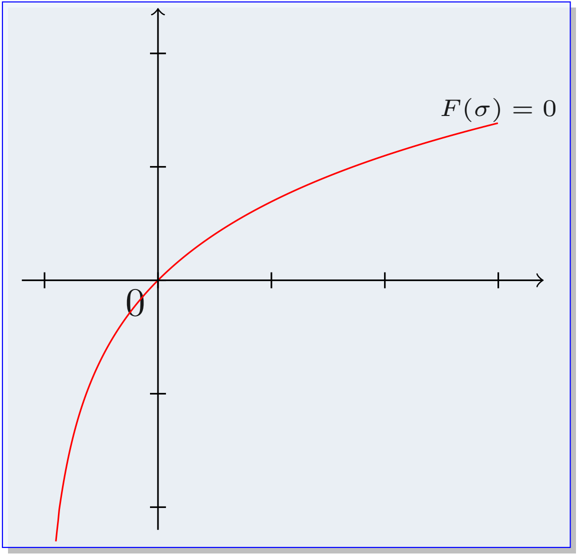

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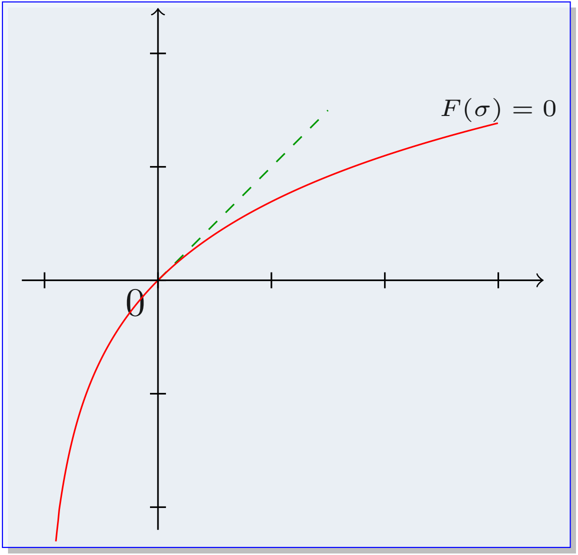

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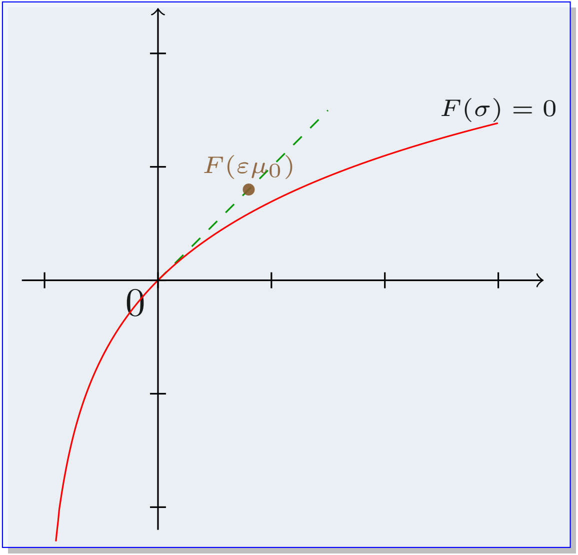

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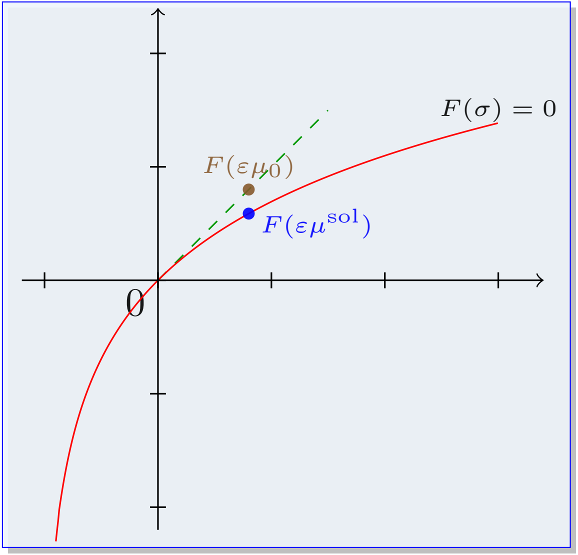

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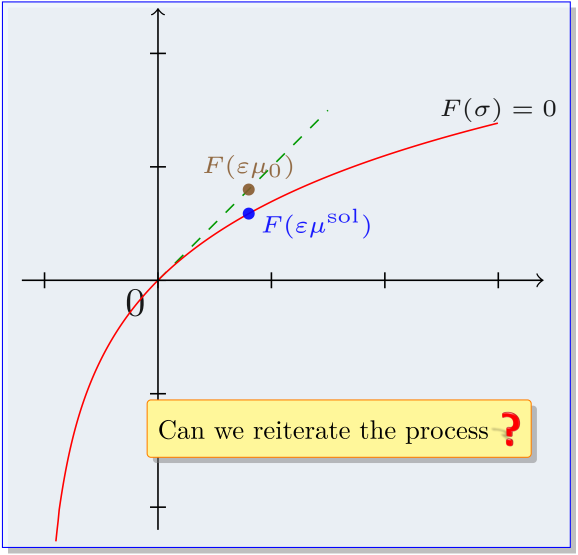

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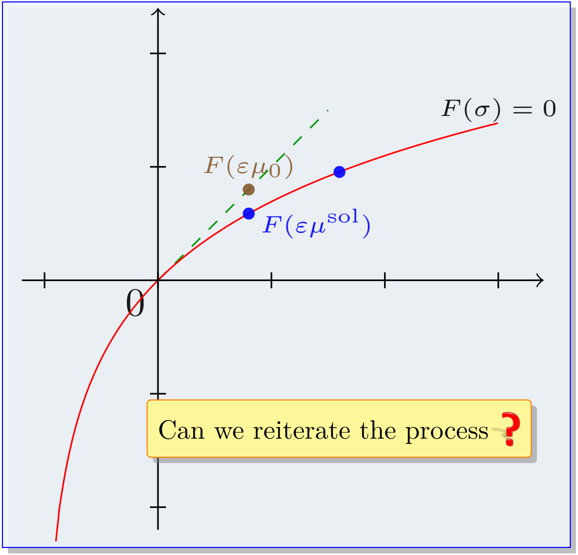
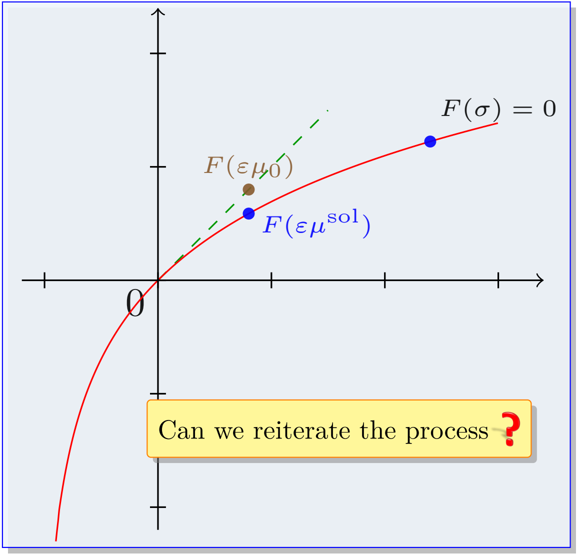


Can we reiterate the process ?

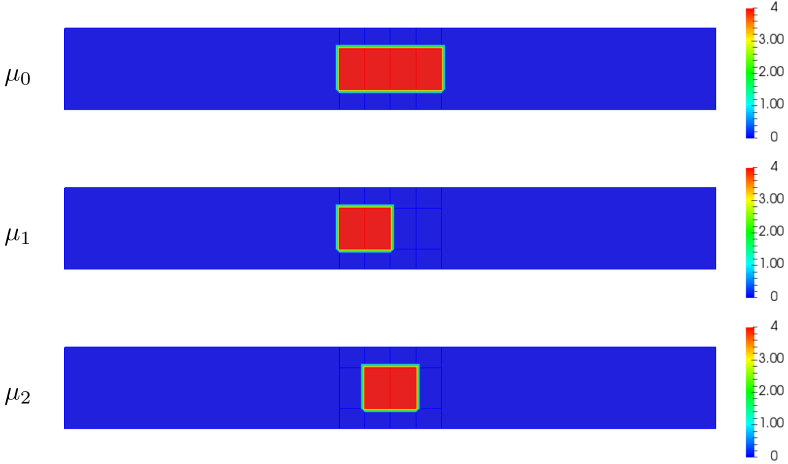
Can we increase the perturbation?

- Schematic view of the process to construct larger invisible defects:



Numerical results to impose $s^- = 0$

- We set $k = 0.8\,\pi$ and $\mathcal{D} = (-\pi/k; \pi/k) \times (1/4; 3/4)$.
- We replace "Find $\boldsymbol{\sigma} \in \mathbf{L}^\infty(\mathcal{D})$ such that $F(\sigma) = 0_{\mathbb{R}^2}$"
  by "Find $\boldsymbol{\sigma} \in \mathbf{span}(\boldsymbol{\mu_0}, \boldsymbol{\mu_1}, \boldsymbol{\mu_2})$ such that $F(\sigma) = 0_{\mathbb{R}^2}$".



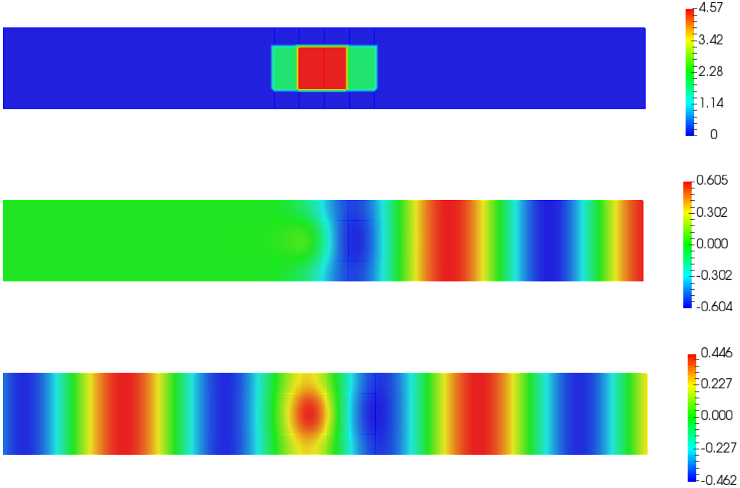
Numerical results to impose $s^{-} = 0$

- After 250 steps of iterations, we obtain

$\sigma = \rho - 1$

$\Re e\, u_{\mathrm{s}}$

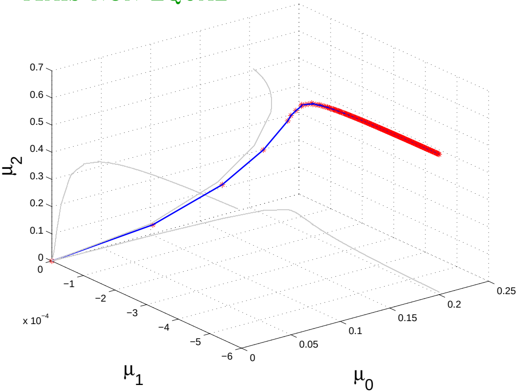
$\Re e\, u$



Numerical results to impose $s^- = 0$

- Each $*$ on the curves represents one $\sigma \in \text{span}(\mu_0, \mu_1, \mu_2)$ s.t. $F(\sigma) = 0_{\mathbb{R}^2}$.

Axis non equal



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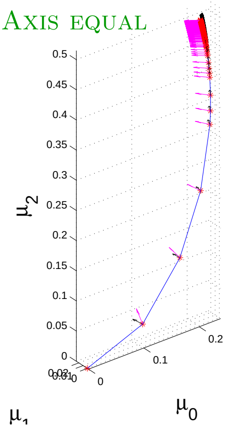

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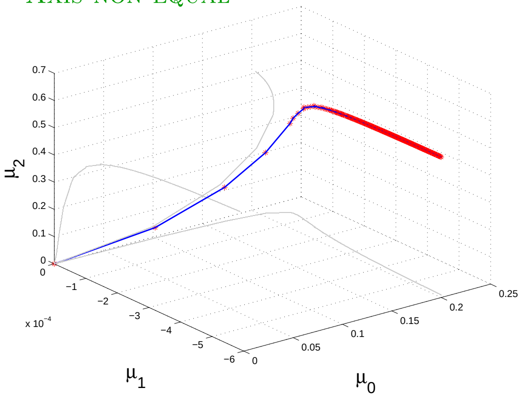

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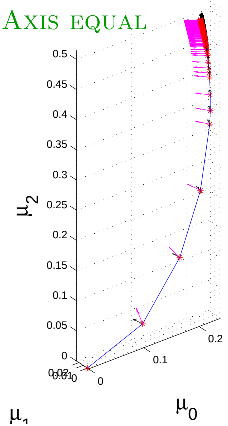
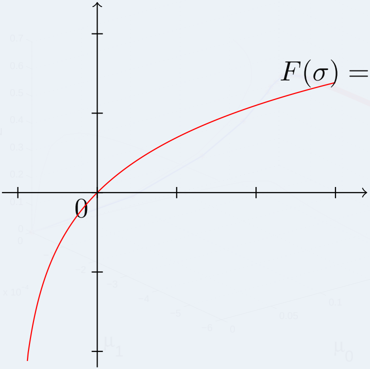


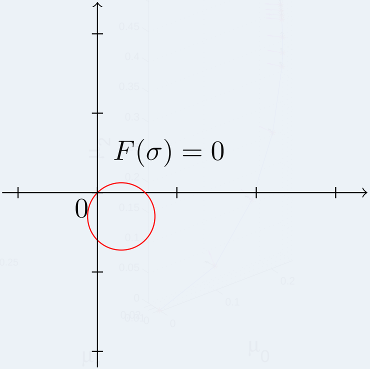
→ First results are encouraging. Still some questions: with more elements in the basis $(\mu_0, \ldots, \mu_N)$, at each step, how to choose the new directions?

Numerical results to impose $s^- = 0$

Depending on the directions, we may have



or



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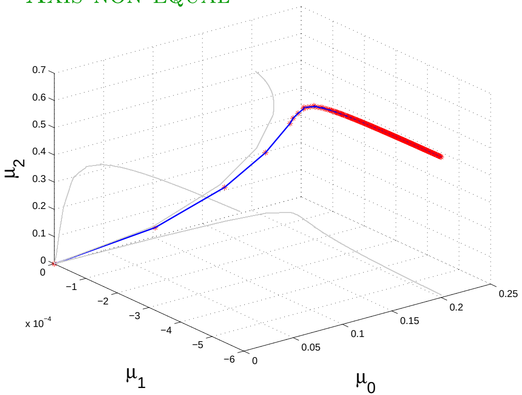

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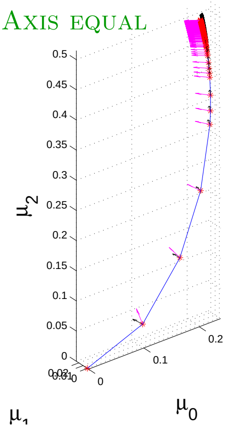

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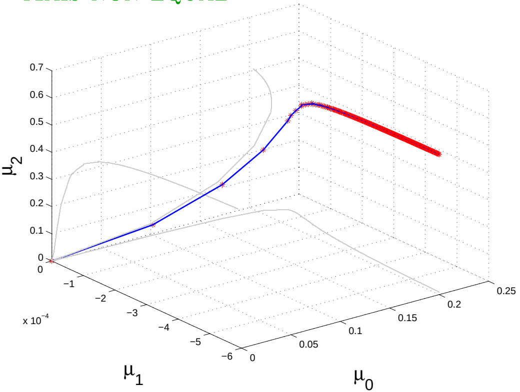

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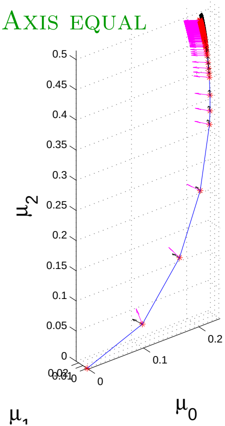


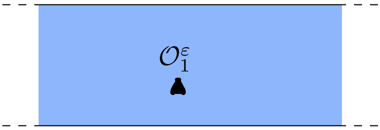
→ First results are encouraging. Still some questions: with more elements in the basis $(\mu_0, \dots, \mu_N)$, at each step, how to choose the new directions?

→ We are not able to prove that $ds^-(\sigma) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{C}$ is onto for $\sigma \not\equiv 0$.

1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments
2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
   - Can one hide a perturbation of the wall?

Small Dirichlet obstacle

- Can one hide a small Dirichlet obstacle centered at M_1?



Find $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ s. t.

$$\begin{array}{rcll} -\Delta u & = & k^2 u & \text{in } \Omega^\varepsilon := \Omega \setminus \overline{\mathcal{O}_1^\varepsilon}, \\ u & = & 0 & \text{on } \partial\Omega^\varepsilon, \\ & & & u_{\mathrm{s}} \text{ is outgoing.} \end{array}$$

- Again, u_{s} is outgoing means that there are some $s^\pm \in \mathbb{C}$ such that

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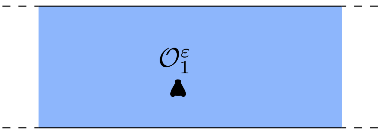

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Due to Dirichlet B.C., $w^\pm$ are not the same as previously (but this not important).

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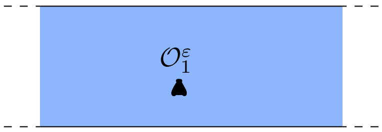

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- In 3D, we obtain

$$\begin{array}{rcl} s^- & = & 0 + \varepsilon\ \left(4i\pi\,\mathrm{cap}(\mathcal{O}) w^+(M_1)^2\right) + O(\varepsilon^2) \\ s^+ & = & 0 + \varepsilon\ \left(4i\pi\,\mathrm{cap}(\mathcal{O}) |w^+(M_1)|^2\right) + O(\varepsilon^2). \end{array}$$

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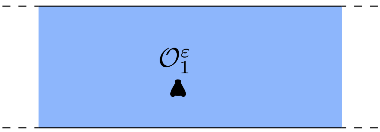

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Non zero terms!
$(\mathrm{cap}(\mathcal{O}) > 0)$

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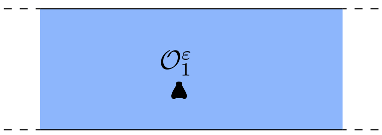

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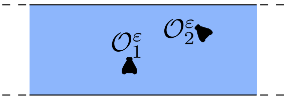
Non zero terms!
($\mathrm{cap}(\mathcal{O}) > 0$)

$\Rightarrow$ One single small obstacle cannot even be non reflective.

Small Dirichlet obstacles



- Let us try with two small Dirichlet obstacles centered at M_1, M_2.

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$$\begin{aligned} s^- &= 0 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} w^+(M_n)^2) + O(\varepsilon^2) \\ s^+ &= 0 + \varepsilon\,(4i\pi\,\mathrm{cap}(\mathcal{O})\sum_{n=1}^{2} |w^+(M_n)|^2) + O(\varepsilon^2). \end{aligned}$$

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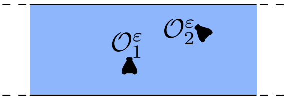

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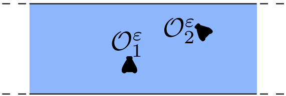

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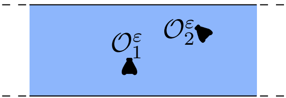

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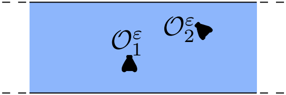

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Comments:

→ Hard part is to justify the asymptotics for the fixed point problem.

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→ When there are more propagative waves, we need more obstacles.

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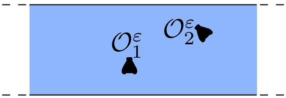

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Acting as a team, flies can become invisible!

1 Invisibility in free space

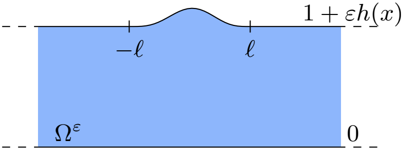
- The general scheme
- The forbidden case
- Numerical experiments

2 Invisibility for waveguide problems

- Construction of invisible penetrable defects
- Can one hide a small Dirichlet obstacle?
- Can one hide a perturbation of the wall?

Can one hide a perturbation of the wall?

- Pick $h \in \mathscr{C}_0^\infty(-\ell;\ell)$, $\ell > 0$. Set $k \in (0;\pi)$, $w^\pm = e^{\pm ikx}/\sqrt{2k}$, $u_\mathrm{i} = w^-$.



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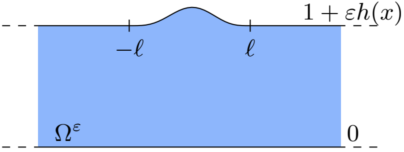

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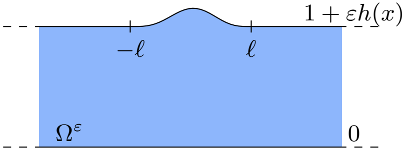

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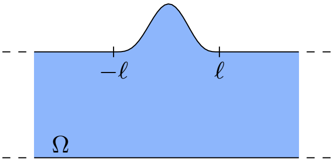
$\Rightarrow$ With this approach, we can impose $s^- = 0$ but not $s^+ = 0$.

Can one hide a perturbation of the wall?

- More generally, for any Neumann waveguide, one can show that $s^+ = 0$ implies
$$\int_\Omega |\nabla u_\mathrm{s}|^2 - k^2 |u_\mathrm{s}|^2 \, d\boldsymbol{x} = 0.$$



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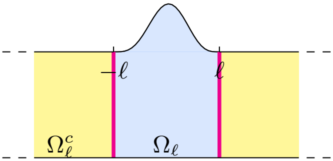

- Decomposing in Fourier series, one finds

$$\int_{\Omega_\ell^c} |\nabla u_s|^2 - k^2 |u_s|^2 \, d\boldsymbol{x} \geq 0.$$

- Note that $s^+ = 0 \Rightarrow u_s \in \mathrm{Y} := \{v \in \mathrm{H}^1(\Omega_\ell) \,|\, \int_{x=\pm\ell} v \, d\sigma = 0\}$. Define

$$\lambda_\dagger := \inf_{v \in \mathrm{Y} \setminus \{0\}} \left(\int_{\Omega_\ell} |\nabla v|^2 \, d\boldsymbol{x} \right) \Big/ \left(\int_{\Omega_\ell} |v|^2 \, d\boldsymbol{x} \right) > 0.$$

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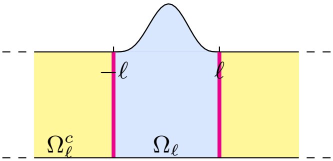

- Decomposing in Fourier series, one finds

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Proposition: For a given shape, $s^+ = 0$ cannot hold for $k^2 \in (0; \lambda_\dagger)$.

Can one hide a perturbation of the wall?

- More generally, for any Neumann wave-guide, one can show that $s^+ = 0$ implies

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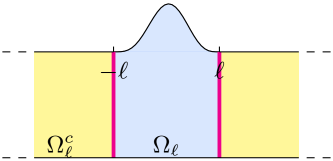

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$\rightarrow$ For a small smooth perturbation of amplitude εh, one finds $|\lambda_\dagger - \pi^2| \leq C \, \varepsilon$.

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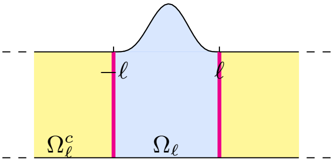

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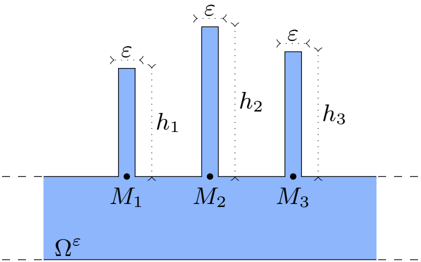
→ To impose invisibility at low frequency, we need to work with special shapes.

Non smooth perturbation of the wall

- We study the same problem in the geometry Ω^ε



- We obtain
$$\begin{aligned} s^- &= 0+\varepsilon\left(ik\sum_{n=1}^{3}(w^+(M_n))^2\tan(kh_n)\right)+O(\varepsilon^2)\\ s^+ &= 0+\varepsilon\left(i/2\sum_{n=1}^{3}\tan(kh_n)\right)+O(\varepsilon^2)\end{aligned}$$

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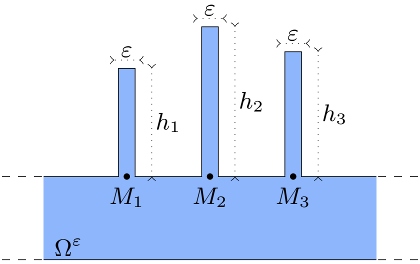

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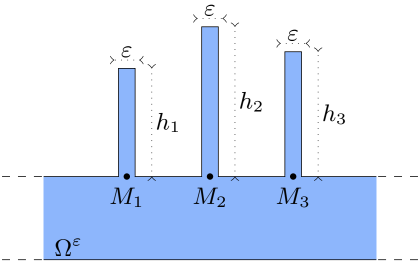

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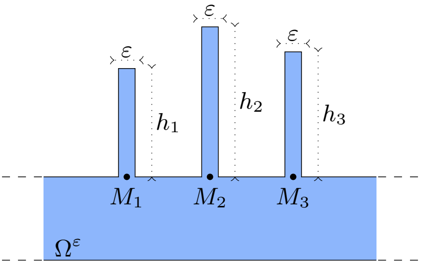

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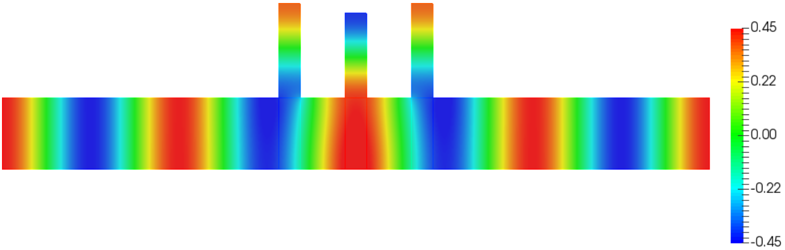
3) Energy conservation + $[s^+ = O(\varepsilon)] \quad \Rightarrow \quad s^+ = 0$.

Numerical results

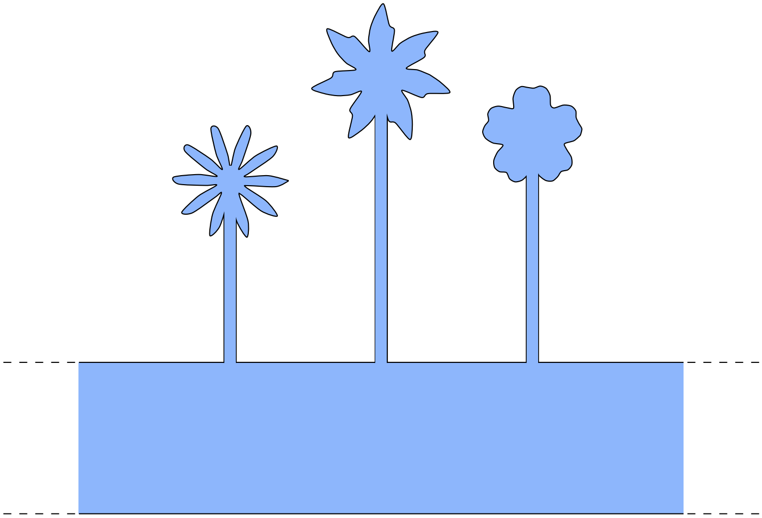
- Perturbed waveguide ($\Re e\,(u(x)e^{-i\omega t})$)



- Reference waveguide ($\Re e\,(u_{\mathrm{i}}(x)e^{-i\omega t})$)

Remark

- We could also have worked with gardens of flowers!



1. Invisibility in free space
   - The general scheme
   - The forbidden case
   - Numerical experiments
2. Invisibility for waveguide problems
   - Construction of invisible penetrable defects
   - Can one hide a small Dirichlet obstacle?
   - Can one hide a perturbation of the wall?

Conclusion

What we did

- ♠ We explained how to construct invisible perturbations of a reference situation in a setting with a finite number of measurements.

Future work

1) We want to continue the analysis of the reiteration process to construct large invisible defects of the reference medium.

2) It would be interesting to consider other models (Maxwell, elasticity, ...) and to investigate cases where the differential is not onto.

3) For a given perturbation, can we study the frequencies (invisible modes) such that invisibility holds?

4) We wish to better understand the link between the invisible modes and the so-called trapped modes in waveguides.

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Invisible mode

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Thank you for your attention!!!