A few techniques to achieve invisibility in waveguides

Lecture 1: Rudiments of waveguide theory

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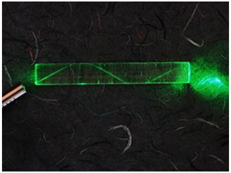
Introduction

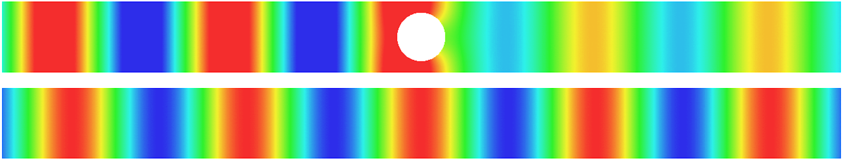
- Waveguides appear in many fields of physics: acoustics, water waves, electromagnetics, classical mechanics, quantum mechanics, ...



- One can think to musical instruments, loudspeakers, optical fibers, conductive metal pipes, ...

General setting

- We will be interested in the scattering of incident waves:



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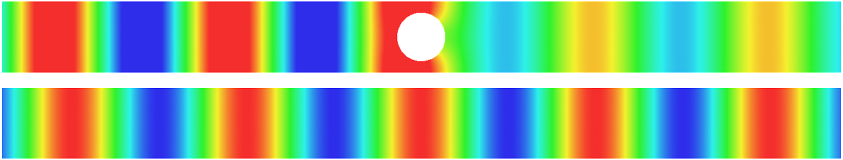

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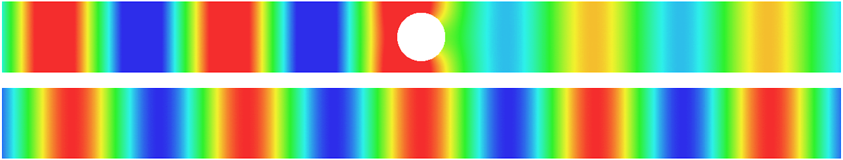

- We wish to study questions of invisibility:

How to identify situations (geometry, frequency, ...)
where waves go through like if there were no defect

- One can wish to have good energy transmission through the structure.
- One can wish to hide objects.

Goals of the mini course

1) To explain how to model propagation of scalar waves in waveguides in time-harmonic regime.

2) To present different tools of applied mathematics to identify situations of invisibility:
   - Asymptotic analysis;
   - Spectral theory for self-adjoint and non self-adjoint problems;
   - Finite element methods.

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Structure of the mini course

Lecture 1. Rudiments of waveguide theory.

Lecture 2. Invisible perturbations of the reference geometry.

→ *Construction of small amplitude invisible obstacles.*

Lecture 3. Playing with resonances to achieve invisibility.

→ *Construction of large amplitude invisible obstacles.*

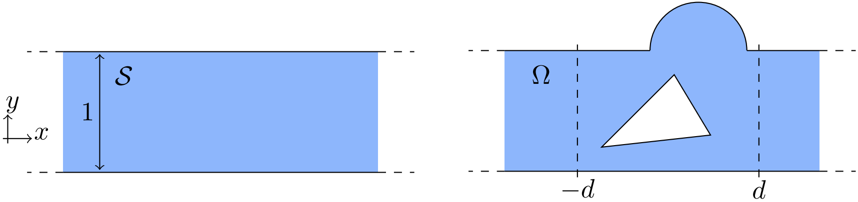
Lecture 4. A spectral problem characterizing zero reflection.

→ *Given an obstacle, find frequencies such that one has zero reflection.*

Lecture 1: Rudiments of waveguide theory

Setting

- Set $I := (0;1)$, $\mathcal{S} := \mathbb{R} \times I$. Let $\Omega \subset \mathbb{R}^2$ be a waveguide which coincides with $\mathcal{S}$ outside of a compact region.



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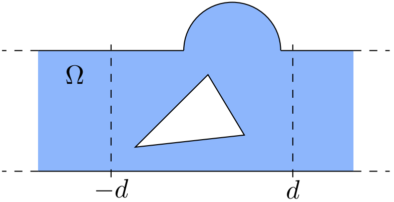

- Consider the wave equation, for $t \geq 0$,

$$\left| \begin{array}{rcll} \dfrac{1}{c^2}\dfrac{\partial^2 U}{\partial t^2} - \Delta U & = & F & \text{in } \Omega \\ U & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

with some initial conditions. Assume that the celerity $c > 0$ is constant. This problem appears for example in electromagnetics.

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- For time-harmonic F, *i.e.* of the form

$$F(x,y,t) = f(x,y)e^{-i\omega t},$$

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Goal of the lecture

To understand the features of $(\mathscr{P}_D)$ according to the values of $k > 0$.

Dirichlet problem for $k < \pi$ 1/2

- Assume that $f \in \mathrm{L}^2(\Omega)$. The variational formulation of $(\mathscr{P}_D)$ writes

$$\left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that} \\ a(u,v) = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega), \end{array} \right.$$

with $\mathrm{H}^1_0(\Omega) := \{w \in \mathrm{H}^1(\Omega) \,|\, w = 0 \text{ on } \partial\Omega\}$ and

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- Since $a(\cdot,\cdot)$ is continuous, with Riesz we can define the linear bounded operator $A(k) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ such that

$$(A(k)u, v)_{\mathrm{H}^1(\Omega)} = a(u,v), \qquad \forall u, v \in \mathrm{H}^1_0(\Omega).$$

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THEOREM: Pick $k \in (0;\pi)$. The operator $A(k)$ decomposes as

$$A(k) = B + K$$

where $B : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is an isomorphism, $K : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is compact.

Dirichlet problem for $k < \pi$ 2/2

- We deduce that $A(k)$ satisfies the Fredholm alternative:

- Either $A(k)$ is injective and then is it an isomorphism;

- Or $A(k)$ has a kernel of finite dim. $\text{span}(u_1, \ldots, u_P)$ and then the equation

$$A(k)u = F$$

has a solution (defined up to $\text{span}(u_1, \ldots, u_P)$) if and only if F satisfies the compatibility conditions

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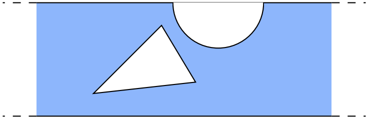
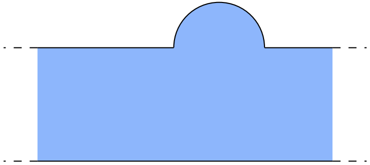


$A(k)$ isom. for all $k \in (0;\pi)$



$A(k)$ not always isom. for $k \in (0;\pi)$

Computation of modes

- Modes are defined as the solutions with separate variables

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$$\alpha''(x)\varphi(y) + \alpha(x)\varphi''(y) + k^2\alpha(x)\varphi(y) = 0$$

which gives

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- We deduce that

$$\lambda_n = n\pi, \qquad \varphi_n(y) = \sqrt{2}\sin(n\pi y), \qquad n \in \mathbb{N}^* := \{1, 2, \dots\}.$$

When $k \notin \mathbb{N}\pi$, the modes coincide with the family $\{w_n^\pm\}_{n\in\mathbb{N}^*}$ where

$$w_n^\pm(x,y) = e^{\pm i\beta_n x}\varphi_n(y), \qquad \beta_n := \sqrt{k^2 - n^2\pi^2}.$$

Comments

- $\sqrt{\cdot}$ is chosen such that $\Im m \sqrt{\cdot} \geq 0$.
- Pick $k \in (N\pi; (N+1)\pi)$ for some $N \in \mathbb{N}$. Two families of modes:

⋆ There are N propagating modes

$$w_n^{\pm}(x,y) = e^{\pm i\sqrt{k^2-n^2\pi^2}x}\varphi_n(y), \qquad n = 1, ..., N.$$

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$$w_n^{\pm}(x,y) = e^{\mp\sqrt{n^2\pi^2-k^2}x}\varphi_n(y), \qquad n = N+1, N+2, ...,$$

which are expon. decaying as $x \to \pm\infty$ and expon. growing as $x \to \mp\infty$.

Ill-posedness in $\mathrm{H}^1_0(\Omega)$

DEFINITION: Let $T : \mathrm{X} \to \mathrm{Y}$ be a continuous linear map between two Hilbert spaces. T is said to be Fredholm iff

i) $\dim(\ker T) < +\infty$ and $\operatorname{range} T$ is closed;

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Message: loss of Fredholmness in $\mathrm{H}^1_0(\Omega)$ is due to the existence of propagating modes.

Problem with dissipation 1/4

- To model dissipation, for $\eta > 0$, work on the problem

$$(\mathscr{P}_\eta) \quad \left| \begin{array}{rcll} -\Delta u_\eta - (k^2 + ik\eta)u_\eta & = & f & \text{in } \Omega \\ u_\eta & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

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- With the convention $U_\eta(x,y,t) = u_\eta(x,y)e^{-i\omega t}$, this originates from

$$\left| \begin{array}{rcll} \dfrac{\partial^2 U_\eta}{\partial t^2} + \eta \dfrac{\partial U_\eta}{\partial t} - \dfrac{1}{c^2}\Delta U_\eta & = & F & \text{in } \Omega,\ t > 0, \\ U_\eta & = & 0 & \text{on } \partial\Omega,\ t > 0. \end{array} \right. \qquad (1)$$

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- Assume that $F \equiv 0$. Multiplying (1) by $\partial_t \overline{U}$ and integrating in Ω gives

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega \left|\frac{\partial U_\eta}{\partial t}\right|^2 + \frac{1}{c^2}|\nabla U_\eta|^2\,dxdy = -\eta\int_\Omega \left|\frac{\partial U_\eta}{\partial t}\right|^2 dxdy.$$

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Conclusion: the energy

$$E(t) = \frac{1}{2}\int_\Omega \left|\frac{\partial U_\eta}{\partial t}\right|^2 + \frac{1}{c^2}|\nabla U_\eta|^2\,dxdy$$

indeed decreases, due to the term $\eta\partial_t U_\eta$, when $\eta > 0$.

Problem with dissipation 2/4

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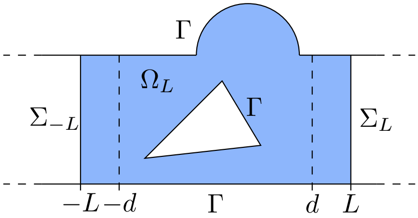
- To take the limit $\eta \to 0$ (limiting absorption principle), let us rewrite the problem in a bounded domain. For $L > d$, set

$$\Omega_L := \{(x, y) \in \Omega \,|\, |x| < L\}, \qquad \Sigma_{\pm L} := \{\pm L\} \times I, \qquad \Gamma := \partial\Omega \cap \partial\Omega_L.$$



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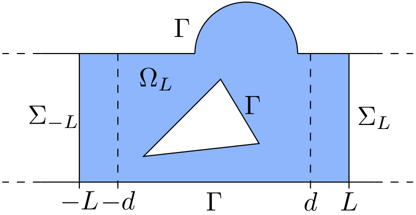

We must impose *ad hoc* transparent conditions on the artificial boundaries $\Sigma_\pm$ that do not create spurious reflections.

Problem with dissipation 3/4

- Set $\mathcal{S}_\pm := \pm(L;+\infty)\times I$ and define the Dirichlet-to-Neumann operators

$$\begin{array}{rccc}\Lambda^\eta_\pm : & \mathrm{H}^{1/2}_{00}(\Sigma_{\pm L}) & \rightarrow & \mathrm{H}^{-1/2}(\Sigma_{\pm L}) \\ & \varphi & \mapsto & \dfrac{\partial v_\varphi}{\partial \nu},\end{array}$$

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$$\left|\begin{array}{rcll}\Delta v_\varphi + (k^2+ik\eta)v_\varphi & = & 0 & \text{in } \mathcal{S}_\pm \\ v_\varphi & = & 0 & \text{on } \partial\Omega\cap\partial\mathcal{S}_\pm \\ v_\varphi & = & \varphi & \text{on } \Sigma_{\pm L}.\end{array}\right.$$

PROPOSITION: $u_\eta \in \mathrm{H}^1_0(\Omega)$ solves $(\mathscr{P}_\eta)$ iff $u_\eta|_{\Omega_L}$ satisfies

$$(\mathscr{P}^L_\eta)\quad\left|\begin{array}{lcll}\multicolumn{4}{l}{\text{Find } u_\eta \in \mathrm{H}^1_0(\Omega_L;\Gamma) \text{ such that}} \\ -\Delta u_\eta - (k^2+ik\eta)u_\eta & = & f & \text{in } \Omega_L \\ \dfrac{\partial u_\eta}{\partial \nu} & = & \Lambda^\eta_\pm(u_\eta) & \text{on } \Sigma_{\pm L}\end{array}\right.$$

where $\partial_\nu = \pm\partial_x$ at $x=\pm L$.

Problem with dissipation 4/4

- For the $\Lambda^\eta_\pm$, we have the explicit representation

$$\begin{aligned}\Lambda^\eta_\pm(\varphi) &= \sum_{n=1}^{+\infty} i\beta_n^\eta\,(\varphi,\varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\,e^{\pm i\beta_n^\eta(x\mp L)}\,\varphi_n(y)|_{\Sigma_{\pm L}}\\ &= \sum_{n=1}^{+\infty} i\beta_n^\eta\,(\varphi,\varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})}\,\varphi_n(y)\end{aligned}$$

where $\beta_n^\eta := \sqrt{k^2+i\eta-n^2\pi^2}$ (the solution decomposes on the exponentially decaying modes).

Problem without dissipation 1/2

- Taking the limit $\eta \to 0$, we define the operators

$$\Lambda_{\pm}(\varphi) = \sum_{n=1}^{+\infty} i\beta_n \, (\varphi, \varphi_n)_{\mathrm{L}^2(\Sigma_{\pm L})} \, \varphi_n(y)$$

and consider the problem

$$(\mathscr{P}^L) \quad \left| \begin{array}{lcll} \text{Find } u \in \mathrm{H}^1_0(\Omega_L; \Gamma) \text{ such that} & & & \\ -\Delta u - k^2 u & = & f & \text{in } \Omega_L \\ \dfrac{\partial u}{\partial \nu} & = & \Lambda_{\pm}(u) & \text{on } \Sigma_{\pm L}. \end{array} \right.$$

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- Its variational formulation writes

$$\left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega_L; \Gamma) \text{ such that} \\ a^{\mathrm{out}}(u, v) = \ell(v), \qquad \forall v \in \mathrm{H}^1_0(\Omega_L; \Gamma), \end{array} \right.$$

with

$$a^{\mathrm{out}}(u, v) = \int_{\Omega_L} \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} \, dx dy - \langle \Lambda_+(u), v \rangle_{\Sigma_L} - \langle \Lambda_-(u), v \rangle_{\Sigma_{-L}}.$$

Problem without dissipation 2/2

- Since $a^{\text{out}}(\cdot,\cdot)$ is continuous, with Riesz we can define the linear bounded operator $A^{\text{out}}(k) : \mathrm{H}_0^1(\Omega_L;\Gamma) \to \mathrm{H}_0^1(\Omega_L;\Gamma)$ such that

$$(A^{\text{out}}(k)u, v)_{\mathrm{H}^1(\Omega)} = a^{\text{out}}(u,v), \qquad \forall u, v \in \mathrm{H}_0^1(\Omega).$$

Theorem: For $k \in (N\pi;(N+1)\pi)$, $N \in \mathbb{N}^*$, $A^{\text{out}}(k)$ decomposes as

$$A^{\text{out}}(k) = B + K$$

where $B : \mathrm{H}_0^1(\Omega_L;\Gamma) \to \mathrm{H}_0^1(\Omega_L;\Gamma)$ is an isomorphism and $K : \mathrm{H}_0^1(\Omega_L;\Gamma) \to \mathrm{H}_0^1(\Omega_L;\Gamma)$ is compact.

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THEOREM: Fix $k \in (\pi;+\infty) \setminus \{\mathbb{N}\pi\}$ and assume that $A^{\text{out}}(k)$ is injective. There is $C > 0$ (independent of η), η_0 such that we have

$$\|u_\eta - u\|_{\mathrm{H}^1(\Omega_L)} \leq C\eta \, \|f\|_{\mathrm{L}^2(\Omega_L)}, \qquad \forall \eta \in (0;\eta_0].$$

Scattering problems 1/2

- In sequel, we will not work with source terms f but instead consider the scattering of incident waves.
- To simplify, assume that $k \in (\pi; 2\pi)$ so that only the waves

$$w_{\pm}(x, y) = e^{\pm i\beta_1 x}\varphi_1(y) = e^{\pm i\sqrt{k^2-\pi^2}x}\varphi_1(y).$$

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- The scattering of the rightgoing wave w_+ coming from the left branch of the waveguide leads us to consider the problem

$$(\mathscr{P}_+) \quad \left| \begin{array}{l} \text{Find } u_+ \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \text{ such that } u_+ - w_+ \text{ is outgoing and} \\ \begin{array}{rcll} \Delta u_+ + k^2 u_+ & = & 0 & \text{in } \Omega \\ u_+ & = & 0 & \text{on } \partial\Omega. \end{array} \end{array} \right.$$

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The sentence "$u_+ - w_+$ is outgoing" means that we impose

$$u_+ - w_+ = \left\{ \begin{array}{ll} \displaystyle\sum_{n=1}^{+\infty} \alpha_n^+ e^{i\beta_n x}\,\varphi_n(y) & \text{for } x > L \\ \displaystyle\sum_{n=1}^{+\infty} \alpha_n^- e^{-i\beta_n x}\,\varphi_n(y) & \text{for } x < -L \end{array} \right., \text{ for some } \alpha_n^{\pm} \in \mathbb{C}.$$

Scattering problems 2/2

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PROPOSITION: We have $T_+ = T_- =: T$ and the identities

$$|R_\pm|^2 + |T|^2 = 1 \qquad \text{(conservation of energy)}.$$

Numerical approximation

Computers don't like infinite!

- Ω is unbounded. Let us work in Ω_L on the formulation

Find $u_+ \in \mathrm{H}^1_0(\Omega_L;\Gamma)$ such that for all $v \in \mathrm{H}^1_0(\Omega_L;\Gamma)$,

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> Find $u^h_+ \in \mathrm{V}_h$ such that for all $v^h \in \mathrm{V}_h$,
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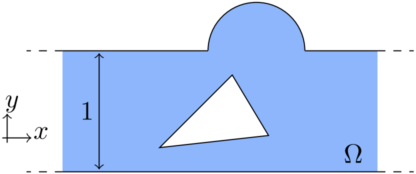
For h small enough, L large enough (one has exponential convergence with respect to L), u^h_+ yields a good approximation of u_+.

Neumann problem

► In acoustics (also relevant in optics, microwaves, water-waves theory,...), one considers the problem



$$(\mathscr{P}) \left| \begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

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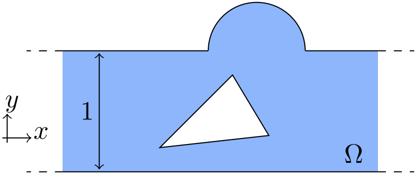

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- For this problem with $k \in (N\pi; (N+1)\pi)$, the modes are

$$\begin{array}{l|l} \text{Propagating} & w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!] \\ \text{Evanescent} & w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N. \end{array}$$

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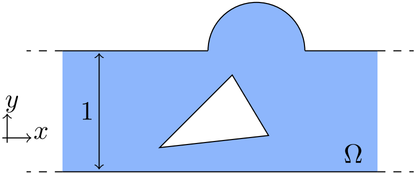

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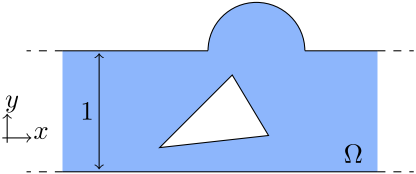

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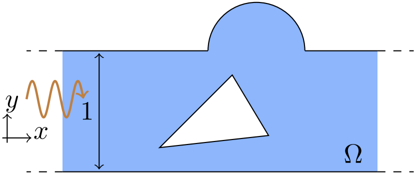

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- The scattering of the wave e^{ikx} leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

$$u = \left| \begin{array}{lr} e^{ikx} + R\, e^{-ikx} + \dots & x \to -\infty \\ \qquad\quad T\, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

R, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

Conclusion of lecture 1

What we did

We studied waveguides problems in time-harmonic regime.

- Dirichlet BCs with $k < \pi$: $\mathrm{H}_0^1(\Omega)$ ok because no propagating modes.

- Dirichlet BCs with $k > \pi$: $\mathrm{H}_0^1(\Omega)$ not ok because propagating modes
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Next lecture

♠ We will study questions of invisibility. How to create defects which provide the same scattering coefficients as in the reference strip?