A few techniques to achieve invisibility in waveguides

Lecture 2: Invisible perturbations of the reference geometry

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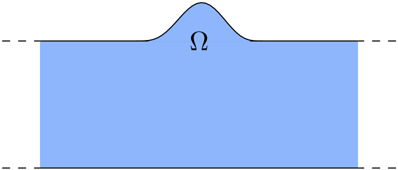
Waveguide problem

► Scattering in time-harmonic regime of a plane wave in the acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



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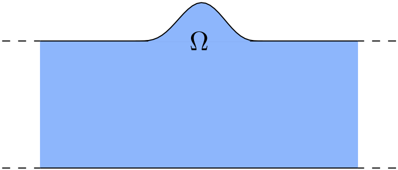

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- For this problem, the modes are

Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2},\ n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2},\ n \geq N.$

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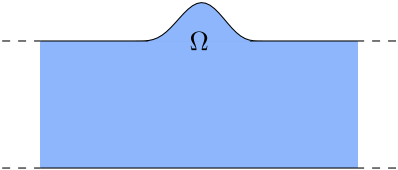

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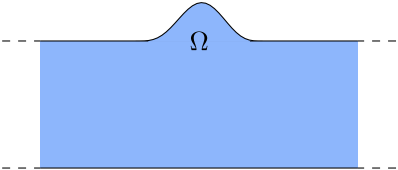

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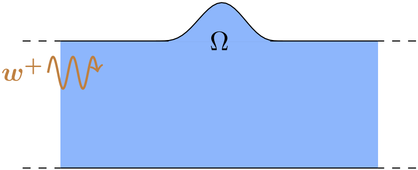

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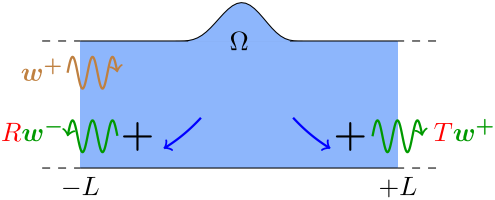

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- We have

$$u = \left| \begin{array}{ll} w_+ + R\,w_- + \dots & \text{for } x \leq -L \\ T\,w_+ + \dots & \text{for } x \geq +L \end{array} \right.$$

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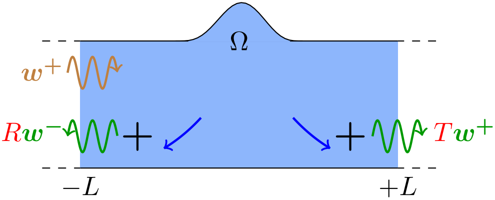

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Definition: R, $T \in \mathbb{C}$ are the reflection and transmission coefficients.

Invisibility

- At infinity, one measures only R and/or T (other terms are too small).
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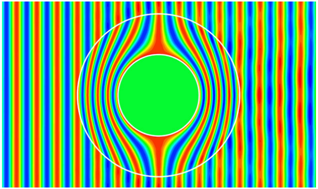

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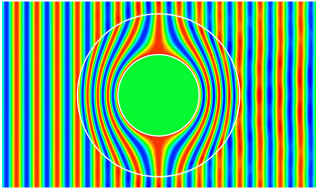
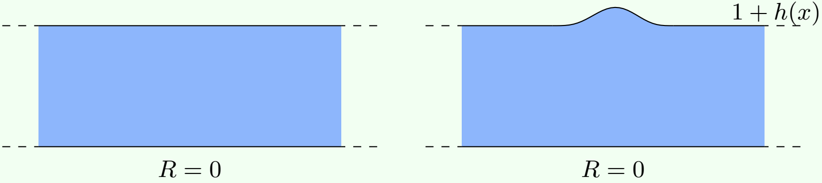


GOAL

We explain how to use perturbative techniques to construct geometries such that $R = 0$ or $T = 1$.

General picture

► Perturbative technique: we construct small non reflecting defects using variants of the implicit functions theorem.



Outline of lecture 2

1. A few notions of asymptotic analysis

2. Invisible smooth perturbations of the reference geometry

3. Non smooth invisible perturbations of the reference geometry

Perturbation of the Poisson's problem

- We study a first simple example with a perturbation in the equation. For Ω a bounded Lipschitz domain and $f \in \mathrm{L}^2(\Omega)$, consider the problem

$$(\mathscr{P}_\varepsilon) \quad \left| \begin{array}{rcll} -\Delta u_\varepsilon + \varepsilon u_\varepsilon & = & f & \text{in } \Omega \\ u_\varepsilon & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

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General procedure:

Step I: we propose an expansion (ansatz) and identify the terms of this expansion.

Step II: we prove error estimates.

Step I - ansatz

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$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

where the terms u_0, u_1, u_2, … have to be determined.

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- Inserting the expansion in $(\mathscr{P}_\varepsilon)$, letting ε tends to zero and identifying the powers in ε, we get

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  $\rightarrow$ This defines the expansion.

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- The proof of error estimates generally relies on two points:
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From the Poincaré inequality

$$\|\varphi\|_{\mathrm{L}^2(\Omega)} \leq C_P \, \|\varphi\|_{\mathrm{H}^1_0(\Omega)} := \|\nabla\varphi\|_{\mathrm{L}^2(\Omega)}, \quad \forall \varphi \in \mathrm{H}^1_0(\Omega),$$

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"The solution of $(\mathscr{P}_\varepsilon)$ is controlled uniformly (C_P is independent of ε, f) by the source term."

Step II - error estimate 2/2

2) Consistency results. Set $\hat{u}_\varepsilon := \sum_{n=0}^{N} \varepsilon^n u_n \in \mathrm{H}^1_0(\Omega)$.

Inserting the **error** $u_\varepsilon - \hat{u}_\varepsilon$ in $(\mathscr{P}_\varepsilon)$, we obtain the **discrepancy**

$$(-\Delta + \varepsilon)(u_\varepsilon - \hat{u}_\varepsilon) = f - \Big(-\sum_{n=0}^{N} \varepsilon^n \Delta u_n + \sum_{n=1}^{N+1} \varepsilon^n u_{n-1}\Big) = -\varepsilon^{N+1} u_N.$$

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Using this consistency result in the stability estimate $(*)$, we find

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{\mathrm{H}^1_0(\Omega)} \leq C_P\, \varepsilon^{N+1} \|u_N\|_{\mathrm{L}^2(\Omega)}.$$

Noting that $\|u_N\|_{\mathrm{L}^2(\Omega)} \leq C_P \|u_N\|_{\mathrm{H}^1_0(\Omega)} \leq C_P^3 \|u_{N-1}\|_{\mathrm{H}^1_0(\Omega)}$, finally we get:

PROPOSITION: We have the error estimate

$$\|u_\varepsilon - \hat{u}_\varepsilon\|_{\mathrm{H}^1_0(\Omega)} \leq C_P^{2N+2}\, \varepsilon^{N+1} \|f\|_{\mathrm{L}^2(\Omega)}.$$

Smooth perturbation of the domain

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry

> Locally $\partial\Omega_\varepsilon$ coincides with the graph of $x \mapsto \varepsilon h(x)$,
> where $h \in \mathscr{C}_0^\infty(-1;1)$ is a given profile function.

- We consider the Laplace problem in the perturbed domain

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What is the dependence of u_ε with respect to ε?

$\rightarrow$ This question has been extensively studied in shape optimization.

A first formal approach

- Let $\mathcal{O}$ be a fixed neighbourhood of the perturbation. To simplify, we assume that $f \in \mathrm{L}^2(\Omega_\varepsilon)$ is zero in $\mathcal{O}$. In Ω_0, we consider the ansatz

$$u_\varepsilon = u_0 + \varepsilon u_1 + \dots$$

where the terms u_0, u_1 have to be determined.

- Observing that at the limit $\varepsilon \to 0$, Ω_ε converges to Ω_0, we get

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- For the boundary conditions, for $(x, y) \in I$, we can write

$$\begin{aligned} 0 = u_\varepsilon(x, \varepsilon h(x)) &= u_\varepsilon(x, 0) + \varepsilon h(x) \partial_y u_\varepsilon(x, 0) + \dots \\ &= u_0(x, 0) + \varepsilon u_1(x, 0) + \varepsilon h(x) \partial_y u_0(x, 0) + \dots . \end{aligned}$$

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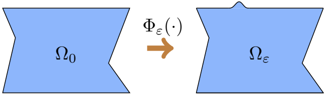
→ Let us see how to justify this formal calculus.

To establish error estimates, we consider a change of variables to work in a fixed geometry.

- For all $\varepsilon \in [0; \varepsilon_0]$, there is a smooth diffeomorphism

$$\begin{array}{rcl} \Phi_\varepsilon : \quad \Omega_0 & \to & \Omega_\varepsilon \\ \mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) & \mapsto & x = \Phi_\varepsilon(\mathrm{x}) = \mathrm{x} + \varepsilon\phi(\mathrm{x}). \end{array}$$



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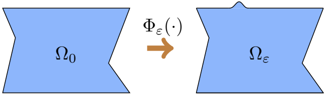

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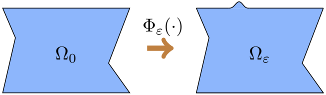

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- We can take ϕ supported in $\mathcal{O}$, of the form

$$\phi(\mathrm{x})=(\phi_1(\mathrm{x}),\phi_2(\mathrm{x}))=(0,h(\mathrm{x}_1)\rho(\mathrm{x}_2))$$

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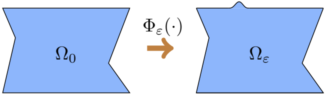

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where ρ is smooth, compactly supported and equal to one in a vicinity of 0.

- Observe that we have $\Phi_\varepsilon|_{\Omega_0 \setminus \overline{\mathcal{O}}} = \mathrm{Id}$.

Error estimates

- Set $U_\varepsilon = u_\varepsilon \circ \Phi_\varepsilon$, $V = v \circ \Phi_\varepsilon$, $F = f \circ \Phi_\varepsilon$. We have

$$\int_{\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)} f v \, dx$$

$$\Leftrightarrow \boxed{\int_{\Omega_0} (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} \nabla U_\varepsilon \cdot (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} \nabla V \, J_{\Phi_\varepsilon} \, \mathrm{dx} = \int_{\Omega_0} F V \, J_{\Phi_\varepsilon} \, \mathrm{dx}.}$$

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Here

$$D\phi = \begin{pmatrix} \partial_{\mathrm{x}_1} \phi_1 & \partial_{\mathrm{x}_2} \phi_1 \\ \partial_{\mathrm{x}_1} \phi_2 & \partial_{\mathrm{x}_2} \phi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \rho \partial_{\mathrm{x}_1} h & h \partial_{\mathrm{x}_2} \rho \end{pmatrix}$$

$$J_{\Phi_\varepsilon} = \det(\mathrm{Id} + \varepsilon D\phi) = 1 + \varepsilon h \partial_{\mathrm{x}_2} \rho.$$

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- Thus we obtain the problem

Find $U_\varepsilon \in \mathrm{H}^1_0(\Omega_0)$ such that

$$-\mathrm{div}(\sigma_\varepsilon \nabla U_\varepsilon) = F J_{\Phi_\varepsilon} \text{ in } \Omega_0$$

with

$$\sigma_\varepsilon := J_{\Phi_\varepsilon} (\mathrm{Id} + \varepsilon (D\phi))^{-1} (\mathrm{Id} + \varepsilon (D\phi)^\top)^{-1} = \mathrm{Id} + \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \dots$$

$$F J_{\Phi_\varepsilon} = F + \varepsilon h \partial_{\mathrm{x}_2} \rho F.$$

Error estimates 3/3

Now the geometry is fixed and we have a pertubation in the equation.

- Considering the expansion

$$U_\varepsilon = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots,$$

we can prove the following error estimate with C independent of $\varepsilon \in (0; \varepsilon_0]$

$$\|U_\varepsilon - \sum_{n=0}^{N} \varepsilon^n U_n\|_{\mathrm{H}_0^1(\Omega_0)} \leq C\,\varepsilon^{N+1} \|f\|_{\mathrm{L}^2(\Omega_0)}.$$

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- Using that

$$\left| \begin{array}{l} U_0 \circ \Phi_\varepsilon^{-1} + \varepsilon U_1 \circ \Phi_\varepsilon^{-1} = U_0 + \varepsilon\,(U_1 - \nabla U_0 \cdot \phi) + \dots \\ U_0 = u_0, \qquad U_1 - \nabla U_0 \cdot \phi = U_1 - h\rho \partial_{\mathrm{x}_2} U_0 = u_1, \end{array} \right.$$

finally we obtain $\|u_\varepsilon - (u_0 + \varepsilon u_1)\|_{\mathrm{H}^1(\Omega_0 \setminus \mathcal{O})} \leq C\,\varepsilon^2 \|f\|_{\mathrm{L}^2(\Omega_0)}.$

Comments

- This is only to give a flavour. Much more refined results exist in the literature concerning shape optimization.

M. Pierre and A. Henrot. Shape Variation and Optimization. A Geometrical Analysis. EMS, 2018.

M.C. Delfour and J.P. Zolésio. Shapes and geometries: metrics, analysis, differential calculus, and optimization. Society for Industrial and Applied Mathematics, 2011.

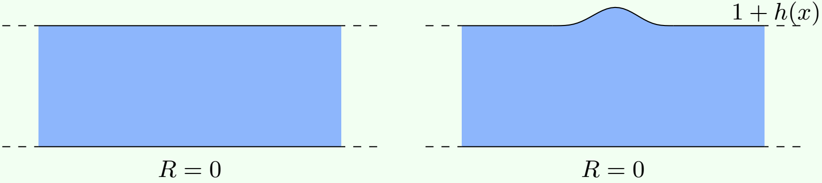
- In particular:

- For this Dirichlet problem, smoothness assumptions of the geometry can be considerably relaxed and result exist when Ω_0 is only measurable.

- Higher order terms can be computed but then smoothness on f is required.

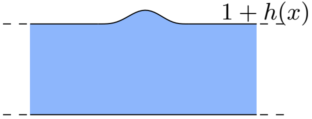
General picture

- Perturbative technique: we construct small non reflecting defects using variants of the implicit functions theorem.



Sketch of the method

- For $h \in \mathscr{C}_0^\infty(\mathbb{R})$, denote $R(h) \in \mathbb{C}$ the reflection coef. in the geometry:



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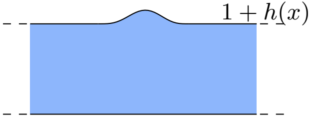

> Note that $R(0) = 0$
> (no obstacle leads to null measurements).

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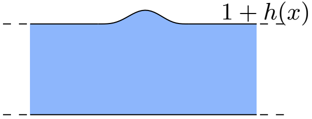

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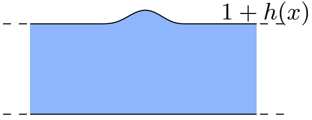

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- We look for h of the form $h = \varepsilon\mu$ with $\varepsilon > 0$ small and μ to determine.

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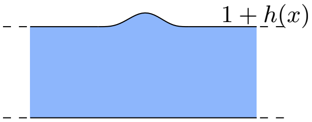

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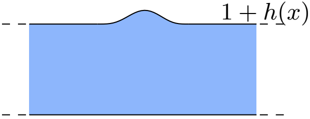

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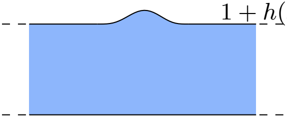

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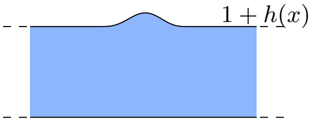

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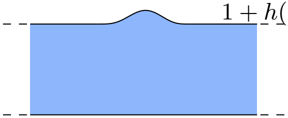

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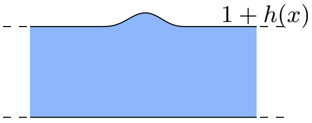

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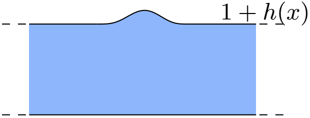

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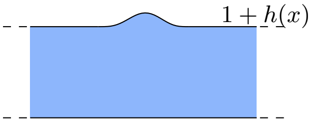

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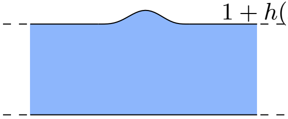

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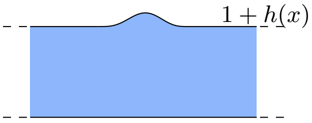

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G^ε is a contraction $\Rightarrow$ the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.

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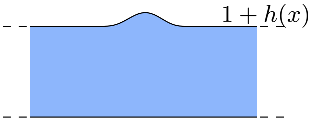

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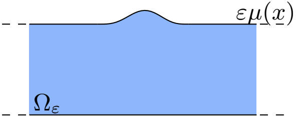
$$0 = R(\varepsilon\mu) \quad \Leftrightarrow \quad \vec{\tau} = G^\varepsilon(\vec{\tau}) \quad \text{where} \left| \begin{array}{l} \vec{\tau} = (\tau_1, \tau_2)^\top \\ G^\varepsilon(\vec{\tau}) = -\varepsilon(\Re e\, \tilde{R}(\varepsilon\mu), \Im m\, \tilde{R}(\varepsilon\mu))^\top. \end{array} \right.$$

G^ε is a contraction $\Rightarrow$ the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.
Set $h^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $R(h^{\mathrm{sol}}) = 0$ (non reflecting perturbation).

Calculus of the differential 1/2



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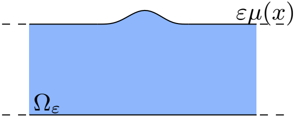

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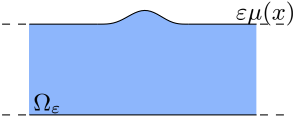

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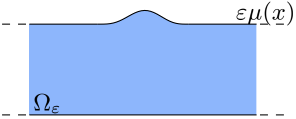

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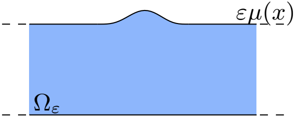

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$\Rightarrow$ Thus we can create geometries Ω_ε where $R_\varepsilon = 0$ for the Dirichlet pb.

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- The invisible perturbation coincides with the graph of the function

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$\Rightarrow$ There exist an infinite number of non reflecting geometries.

- We can show that $|\tau_1^{\text{sol}}| + |\tau_2^{\text{sol}}| = O(\varepsilon)$. Therefore we can choose the principal form of the non reflecting perturbation.

Numerical implementation 1/2

- We can solve the fixed point equation

$$\vec{\tau} = G^{\varepsilon}(\vec{\tau})$$

using an iterative procedure.

- Pick $\varepsilon > 0$, choose μ_0, μ_1, μ_2 once for all. Set $\vec{\tau}^{\,0} = (0,0)$ and for $p \in \mathbb{N}$

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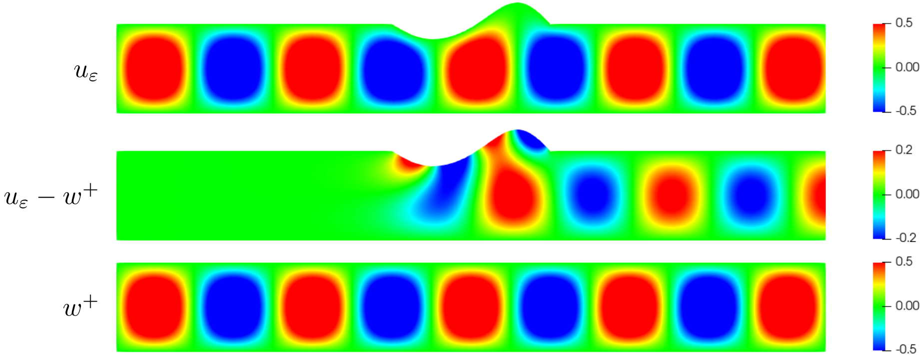
✉ We have to mesh a new domain Ω_p^{ε} at each step $p \geq 0$.

Numerical implementation 2/2

- An example of non reflecting perturbation obtained after 24 iterations ($\varepsilon = 0.2$).



- The algorithm converges though ε is not that small!

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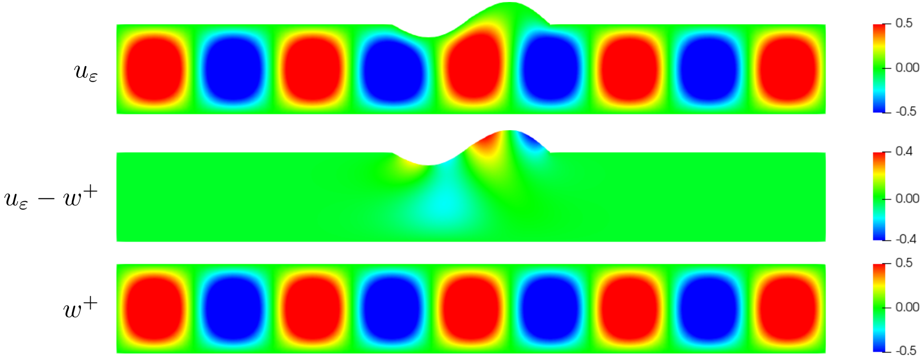
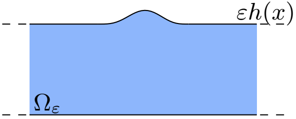


- The scattered field is exponentially decaying both at $\pm\infty$ and this time there is no phase shift for the transmitted field.

Calculus of the differential 1/2



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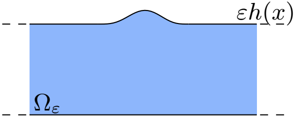

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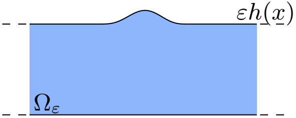

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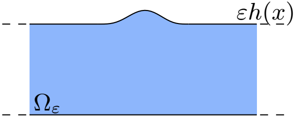

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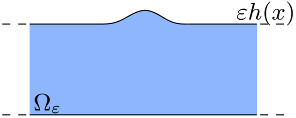
Calculus of the differential 1/2



$$(\mathscr{P}_\varepsilon) \left| \begin{array}{rcll} \Delta u_\varepsilon + k^2 u_\varepsilon & = & 0 & \text{in } \Omega_\varepsilon \\ \partial_{n_\varepsilon} u_\varepsilon & = & 0 & \text{on } \partial\Omega_\varepsilon \\ & u_\varepsilon - w^+ & \text{is outgoing} & \end{array} \right.$$

- We need to compute $dR(0)(\mu)$ that is the term R_1 in the expansion

$$R(\varepsilon h) = R_0 + \varepsilon R_1 + \dots.$$

- Inserting the expansion $u_\varepsilon = u_0 + \varepsilon u_1 + \dots$ in $(\mathscr{P}_\varepsilon)$, we find

$$\left| \begin{array}{rcll} \Delta u_0 + k^2 u_0 & = & 0 & \text{in } \Omega_0 \\ \partial_y u_0 & = & 0 & \text{on } \partial\Omega_0 \\ & u_0 - w^+ & \text{is outgoing} & \end{array} \right. \qquad \left| \begin{array}{rcll} \Delta u_1 + k^2 u_1 & = & 0 & \text{in } \Omega_0 \\ & & & \text{on } \partial\Omega_0 \\ & u_1 & \text{is outgoing.} & \end{array} \right.$$

On the top wall, we have

$$\left| \begin{array}{l} n_\varepsilon = \dfrac{1}{\sqrt{1+\varepsilon^2 (h'(x))^2}} \begin{pmatrix} -\varepsilon h'(x) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} -h'(x) \\ 0 \end{pmatrix} + \dots \\ \nabla u_\varepsilon(x, \varepsilon h(x)) = \nabla u_\varepsilon(x, 0) + \varepsilon h(x) \begin{pmatrix} \partial^2_{xy} u_\varepsilon(x,0) \\ \partial^2_{yy} u_\varepsilon(x,0) \end{pmatrix} + \dots \end{array} \right.$$

so that we get $0 = n_\varepsilon \cdot \nabla u_\varepsilon(x, \varepsilon h(x)) = \partial_y u_0 +$

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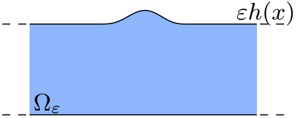

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We use that $u_0 = w^+$
$\Rightarrow \partial^2_{yy} u_0 = 0$

so that we get $0 = n_\varepsilon \cdot \nabla u_\varepsilon(x, \varepsilon h(x)) = \partial_y u_0 + \varepsilon\, (\partial_y u_1 - \varepsilon h'(x)\partial_x u_0) + \dots.$

Calculus of the differential 2/2

- We have $u_0 = w_+$ and u_1 is uniquely defined.
- Set $\Sigma_{\pm L} = \{\pm L\} \times (-1;0)$ for L large enough. From the known formula

$$2ikR(\varepsilon\mu) = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \qquad \text{where } \partial_n = \pm\partial_x \text{ at } x = \pm L,$$

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Integrating by parts, finally we get the final result:

PROPOSITION:

$$dR(0)(\mu) = -\frac{1}{2}\int_{-L}^{L} \partial_x h(x)(w^+(x,0))^2\, dx = -\frac{1}{2}\int_{-L}^{L} \partial_x h(x) e^{2ikx}\, dx.$$

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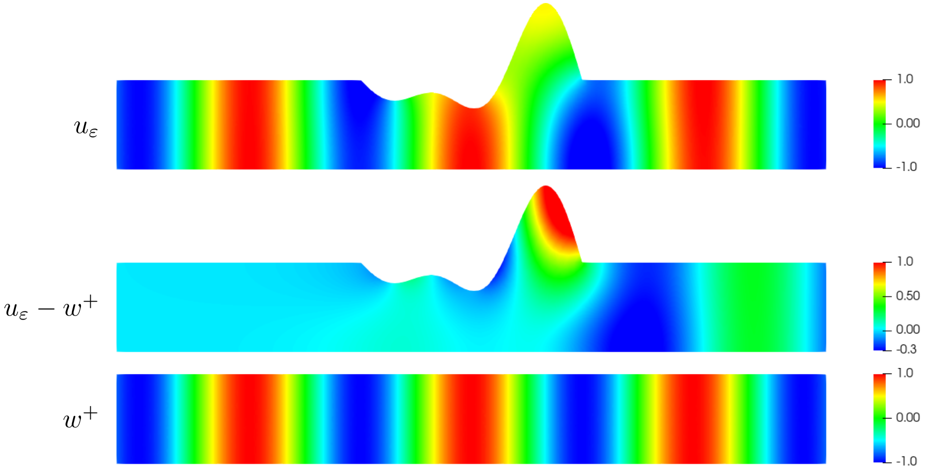
Thus we can construct geometries Ω_ε where $R_\varepsilon = 0$ for the Neumann pb.

Numerics for the Neumann problem 1/2

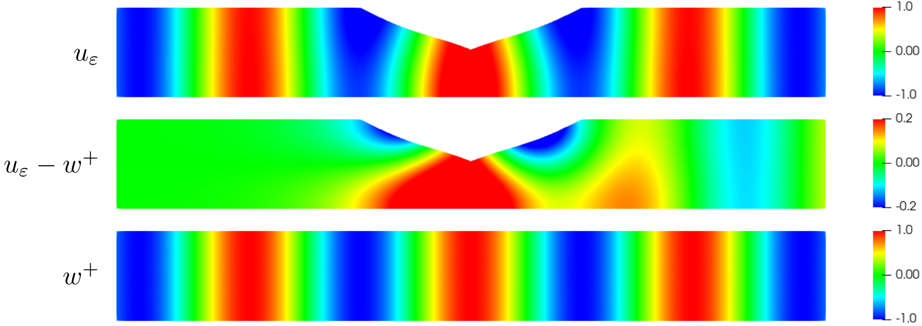
- An example of non reflecting perturbation obtained after 15 iterations ($\varepsilon = 0.4$).



- Again, the algorithm converges though ε is not that small.

Numerics for the Neumann problem 2/2

- Other example non reflecting perturbation.



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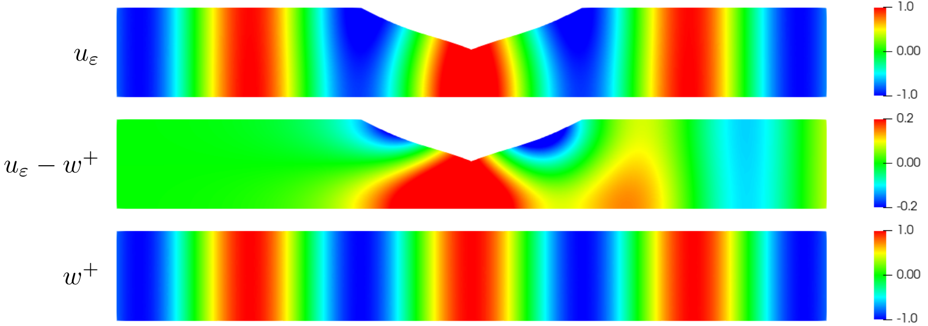
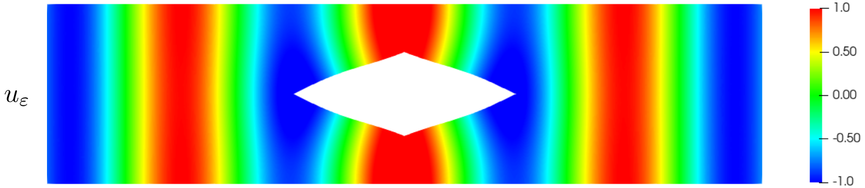


- The defect lies below the line $y = 1$. Symmetrisation:



$T = 1$ for the Neumann problem?

PROPOSITION: We have

$$\begin{aligned} dT(0)(\mu) &= \frac{1}{2ik}\int_{-L}^{L} \partial_x \mu \partial_x w^+ w^- \, dx \\ &= \frac{1}{2}\int_{-L}^{L} \partial_x \mu \, dx = 0. \end{aligned}$$

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$dT(0)$ is null $\Rightarrow$ the approach fails to impose $T = 1$.

An example of singularly perturbed problem

- For $a > 0$, $a \neq 1$, consider the 1D problem

$$(\mathscr{P}_\varepsilon) \quad \left| \begin{array}{l} \varepsilon u_\varepsilon''(x) + u_\varepsilon'(x) - a = 0 \text{ in } \Omega := (0;1) \\ u_\varepsilon(0) = 0, \qquad u_\varepsilon(1) = 1. \end{array} \right.$$

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But since $\|u_\varepsilon(x) - \hat{u}_0(x)\|_{\mathrm{L}^\infty(\overline{\Omega})} = |1-a|$, (u_ε) does not cv to $\hat{u}_0$ in $\mathrm{H}^1(\Omega)$.

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The expansion $(*)$ does not provide a good representation of u_ε.

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$$u_\varepsilon(x) = ax + (1-a)\,\frac{1-e^{-x/\varepsilon}}{1-e^{-1/\varepsilon}} \qquad\qquad \hat{u}_0(x) = ax + (1-a).$$

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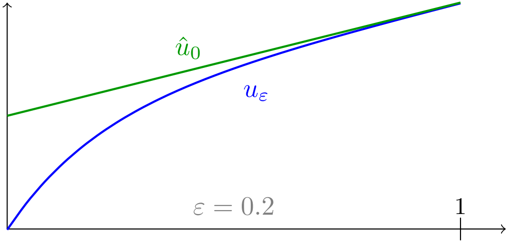

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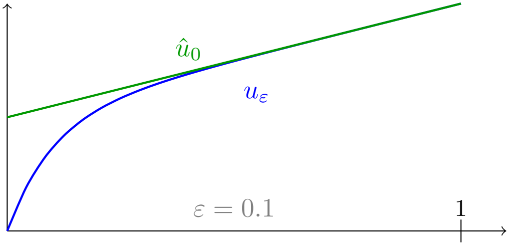

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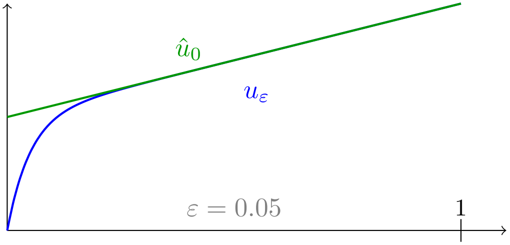

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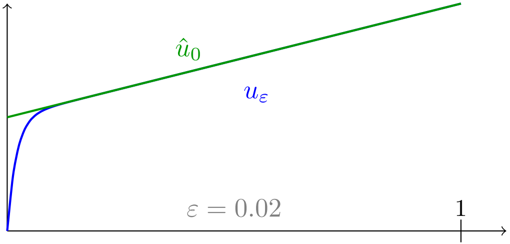

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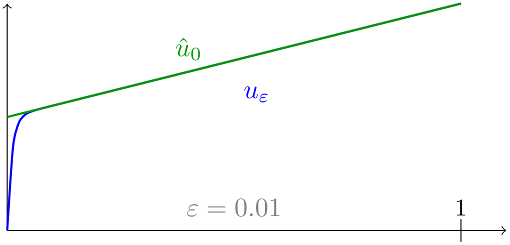

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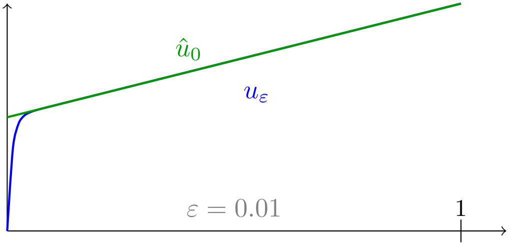

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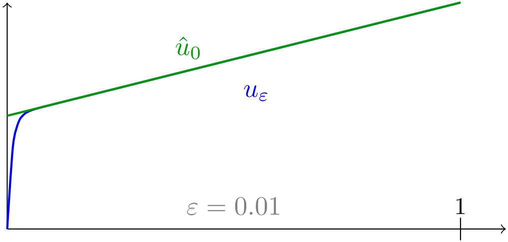

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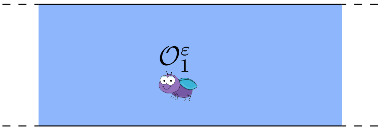
- To approximate correctly u_ε near the origin, we will have to incorporate terms which depend on the rapid variable x/ε.

One small obstacle

Can one hide a small Dirichlet obstacle centered at M_1 ?

- Set $\mathcal{O}_1^\varepsilon := M_1 + \varepsilon\mathcal{O}$ where $M_1 \in \mathbb{R} \times \omega$ and $\mathcal{O}$ is a bounded Lipschitz domain. We consider the problem



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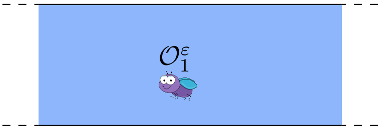

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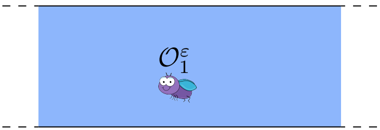

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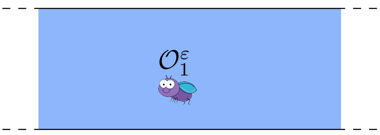

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Non zero terms! $(\mathrm{cap}(\mathcal{O}) > 0)$

$\Rightarrow$ One single small obstacle cannot be non reflecting.

Derivation of the asymptotic of u_ε 1/3

- To simplify, we remove the index $_1$ of the obstacle. Consider the ansatz

$$u_\varepsilon = u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x} - M)) + \varepsilon\Big(u_1 + \zeta(x)\, v_1(\varepsilon^{-1}(\mathrm{x} - M))\Big) + \dots$$

where $\zeta \in \mathscr{C}_0^\infty(\Omega_0)$ is equal to one in a neighbourhood of M.

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$$u_\varepsilon = u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x} - M)) + \varepsilon\Big(u_1 + \zeta(x)\, v_1(\varepsilon^{-1}(\mathrm{x} - M))\Big) + \dots$$

where $\zeta \in \mathscr{C}_0^\infty(\Omega_0)$ is equal to one in a neighbourhood of M.

- Inserting this expansion in $(\mathscr{P}_\varepsilon)$, first we find

$$\left|\begin{array}{rcl} \Delta u_0 + k^2 u_0 &=& 0 \quad \text{in } \Omega_0 = \mathbb{R}\times\omega \\ u_0 &=& 0 \quad \text{on } \partial\Omega_0 \\ u_0 - w^+ && \text{is outgoing.} \end{array}\right.$$

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- v_0 serves to impose Dirichlet BC on $\partial\mathcal{O}^\varepsilon$ at order ε^0. For $x \in \partial\mathcal{O}^\varepsilon$,

$$u_0(x) = u_0(M) + (\mathrm{x} - M)\cdot\nabla u_0(M) + \dots \quad \text{(note that } \mathrm{x} - M \text{ is of order } \varepsilon\text{).}$$

Therefore we impose $v_0 = -u_0(M)$ on $\partial\mathcal{O}$.

Derivation of the asymptotic of u_ε 2/3

- Introduce the fast variable $\xi = \varepsilon^{-1}(\mathrm{x} - M)$. In a vicinity of M, we have

$$(\Delta_x + k^2\mathrm{Id})\left(v_0(\varepsilon^{-1}(\mathrm{x}-M)) + \varepsilon\, v_1(\varepsilon^{-1}(\mathrm{x}-M)) + \dots\right)$$

$$= \varepsilon^{-2}\,\boxed{\Delta_\xi v_0(\xi)} + \varepsilon^{-1}\,\Delta_\xi v_1(\xi) + \dots\ .$$

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$$v_0(\xi) = -u_0(M)\,W(\xi).$$

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- As $|\xi| \to +\infty$, we have

$$W(\xi) = \frac{\mathrm{cap}(\mathcal{O})}{|\xi|} + \vec{q} \cdot \nabla\Phi(\xi) + O(|\xi|^{-3}),$$

where $\Phi := \xi \mapsto -1/(4\pi|\xi|)$ is the Green function of the Laplacian in $\mathbb{R}^3$, $\mathrm{cap}(\mathcal{O}) > 0$, $\vec{q} \in \mathbb{R}^3$.

Derivation of the asymptotic of u_ε

- Now, we turn to the terms of order ε in the expansion of u^ε

$$u_\varepsilon = u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x}-M)) + \varepsilon\Big(u_1 + \zeta(x)\, v_1(\varepsilon^{-1}(\mathrm{x}-M))\Big) + \dots .$$

- By inserting $u_0 + \zeta(x)\, v_0(\varepsilon^{-1}(\mathrm{x}-M))$ into $(\mathscr{P}_\varepsilon)$ and replacing v_0 by its main contribution at infinity, we find that u_1 must solve

$$\left|\begin{array}{rcll} -\Delta u_1 - k^2 u_1 & = & -\left([\Delta_x,\zeta] + k^2\zeta\mathrm{Id}\right)\left(w^+(M)\,\dfrac{\mathrm{cap}(\mathcal{O})}{|\mathrm{x}-M|}\right) & \text{in } \Omega_0 \\ u_1 & = & 0 & \text{on } \partial\Omega_0. \end{array}\right.$$

where $[\Delta_x,\zeta]\varphi := \Delta_x(\zeta\varphi) - \zeta\Delta_x\varphi = 2\nabla\varphi\cdot\nabla\zeta + \varphi\Delta\zeta$ (commutator).

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$\rightarrow$ This uniquely defines u_1.

Asymptotic of the scattering coefficients

- We consider the ansatz

$$R_\varepsilon = R_0 + \varepsilon R_1 + \dots \qquad T_\varepsilon = T_0 + \varepsilon T_1 + \dots.$$

- Set $\Sigma_{\pm L} = \{\pm L\} \times \omega$ for L large enough. From the known formula

$$2ikR_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^+ - u_\varepsilon \partial_n w^+ d\sigma, \qquad 2ikT_\varepsilon = \int_{\Sigma_{\pm L}} \partial_n u_\varepsilon w^- - u_\varepsilon \partial_n w^- d\sigma,$$

where $\partial_n = \pm\partial_x$ at $x = \pm L$,

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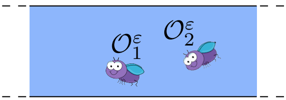
Integrating by parts, finally we get the final result:

PROPOSITION: We have

$$\begin{aligned} R_\varepsilon &= 0 + \varepsilon \, (4i\pi \operatorname{cap}(\mathcal{O}) w^+(M_1)^2) + O(\varepsilon^2) \\ T_\varepsilon &= 1 + \varepsilon \, (4i\pi \operatorname{cap}(\mathcal{O}) |w^+(M_1)|^2) + O(\varepsilon^2). \end{aligned}$$

Several small obstacles



- One small obstacle cannot be non reflecting. Let us try with **TWO**, located at M_1, M_2.

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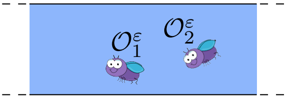

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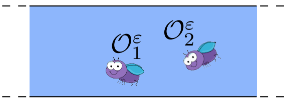

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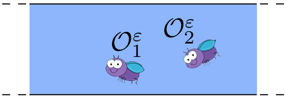

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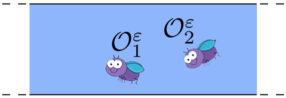

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→ Hard part is to justify the asymptotics for the fixed point problem.

→ We cannot impose $T_\varepsilon = 1$ with this strategy.

→ When there are more propagating waves, we need more obstacles.

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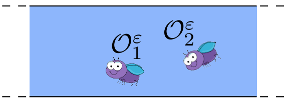

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Acting as a team, obstacles can become invisible!

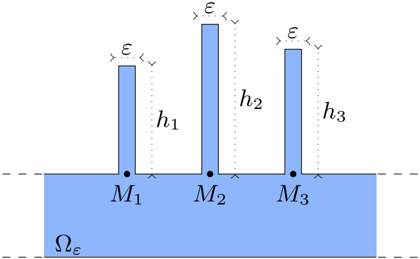
1. A few notions of asymptotic analysis

2. Invisible smooth perturbations of the reference geometry

3. Non smooth invisible perturbations of the reference geometry
   - An example of singularly perturbed problem
   - Invisible clouds of small obstacles
   - Perfect invisibility for the Neumann problem

$T = 1$ for the Neumann problem

- We study the same problem in the geometry Ω_ε



Singular perturbation of the geometry!

- We obtain

$$R_\varepsilon = 0 + \varepsilon\left(ik\sum_{n=1}^{3}(w^+(M_n))^2\tan(kh_n)\right) + O(\varepsilon^2)$$

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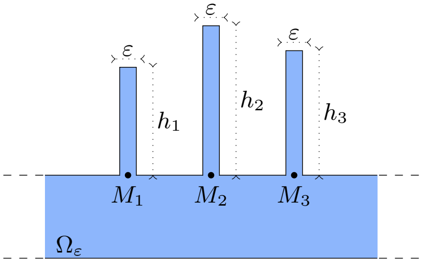

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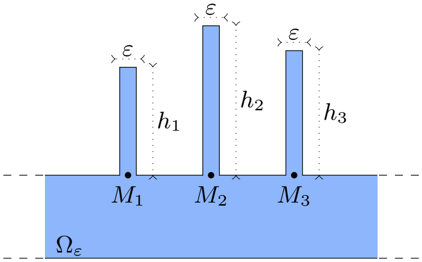

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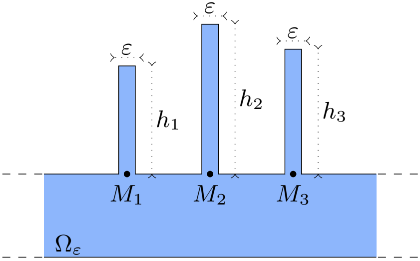

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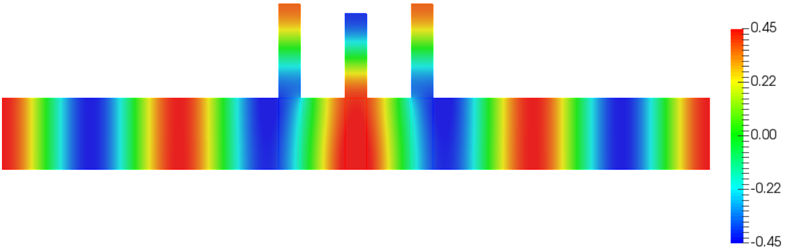
3) Energy conservation + $[T_\varepsilon = 1 + O(\varepsilon)]$ $\Rightarrow$ $T_\varepsilon = 1$.

Numerical results

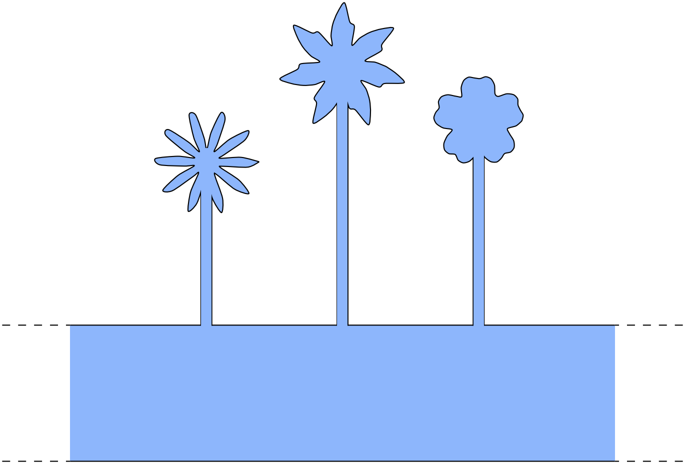
- Perturbed waveguide ($\Re e\,(u_\varepsilon(x,y)e^{-i\omega t})$)



- Reference waveguide ($\Re e\,(u_i(x,y)e^{-i\omega t})$)

Comments

- We could also have hidden gardens of flowers!



Conclusion of lecture 2

What we did

1) Perturbation in the PDE. Recall the standard scheme

Step I: ansatz and identification of the terms of the ansatz;

Step II: error estimates (stability estimate + consistency result).

2) Smooth perturbation of the geometry. Use a change of variable to show error estimates in a fixed geometry.

3) Construction of smooth and non smooth invisible defects in waveguides.

✉ Use the first term in the asymptotic whose dependence wrt the perturbation is explicit and linear to cancel the whole expansion by solving a fixed point problem.

Next lecture

♠ We will explain how to use resonant phenomena to construct large invisible defects.

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- Part III

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