A few techniques to achieve invisibility in waveguides

Lecture 3: Playing with resonances to achieve invisibility

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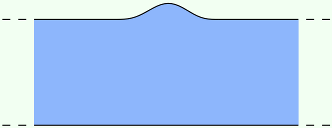
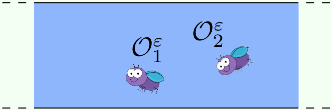
Lecture 2

- We explained how to construct **small** non reflecting or invisible obstacles by working with perturbative techniques.





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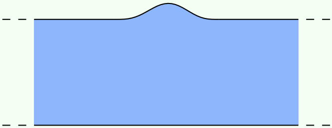
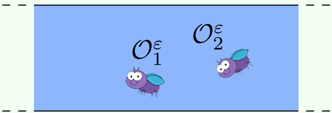
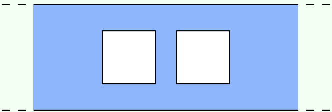




Lecture 3

♠ We wish to obtain non reflection or invisibility for **large** obstacles by working with resonant phenomena.



Outline of lecture 3

1. Construction of non reflecting obstacles using Fano resonance

2. Cloaking of given obstacles in acoustics using resonant ligaments

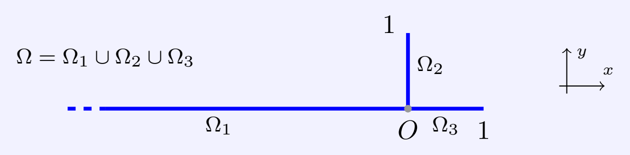
1 Construction of non reflecting obstacles using Fano resonance
- A 1D toy problem
- The Fano resonance in 2D waveguides
- Non reflection and complete reflection
- Numerical experiments

2 Cloaking of given obstacles in acoustics using resonant ligaments

A 1D toy problem

- Fano resonance phenomenon appears in many fields in physics. First, we illustrate it for a simple 1D problem.

A 1D toy problem



- Consider the scattering problem

$$\varphi'' + k^2\varphi = 0 \text{ in } \Omega, \quad \left| \begin{array}{l} \varphi_1 = \varphi_2 = \varphi_3 \text{ at } O \\ \varphi_1' = \varphi_2' + \varphi_3' \text{ at } O \\ \varphi_2' = \varphi_3' = 0 \text{ on } \partial\Omega \end{array} \right. \quad \text{with } \underbrace{\varphi_1 = e^{ikx} + R\, e^{-ikx}}_{\text{radiation condition}},\ R \in \mathbb{C}.$$

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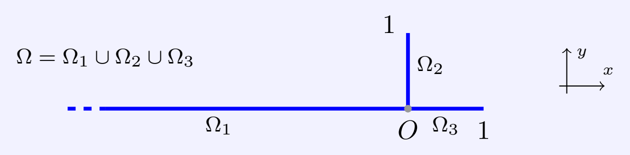

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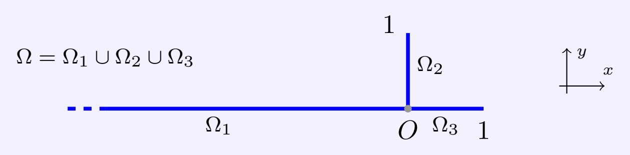

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- Well-posedness $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.

- Uniqueness $\Leftrightarrow$ $k \notin (2\mathbb{N}+1)\pi/2$. Existence for all $k \in \mathbb{R}$ ($F \in \ker {}^t\mathbb{M}^\perp$)

$$R = \frac{\cos(k) + 2i\sin(k)}{\cos(k) - 2i\sin(k)}.$$

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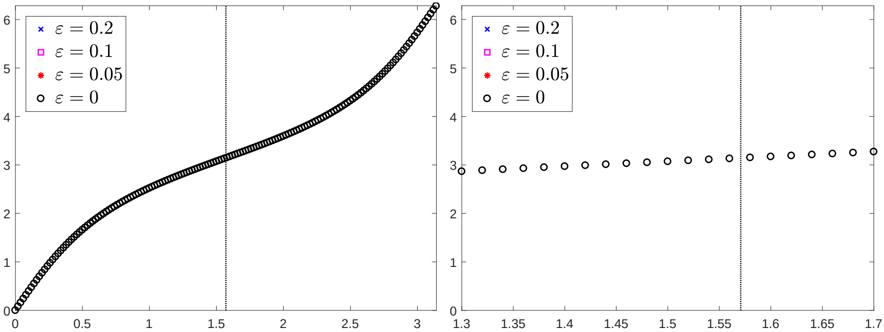

Figure: $k \mapsto \theta^\varepsilon(k)$ for several ε (non uniqueness for $\varepsilon = 0$, $k = \pi/2$).

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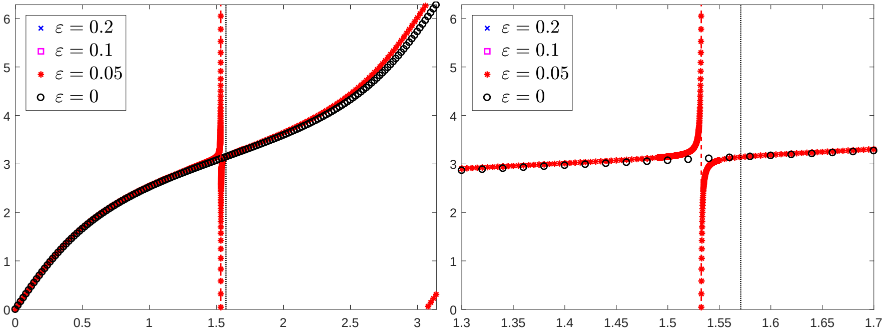

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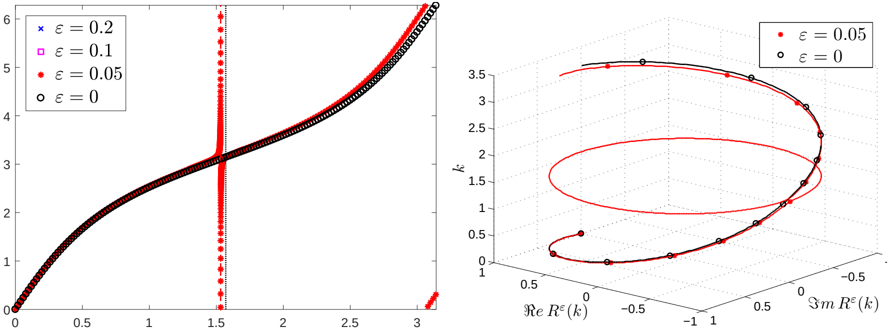

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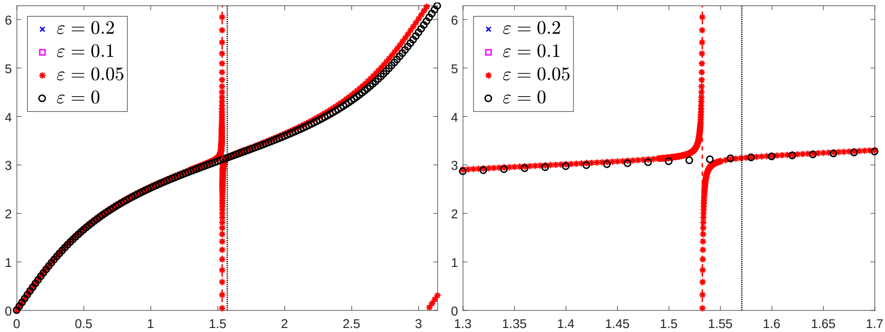

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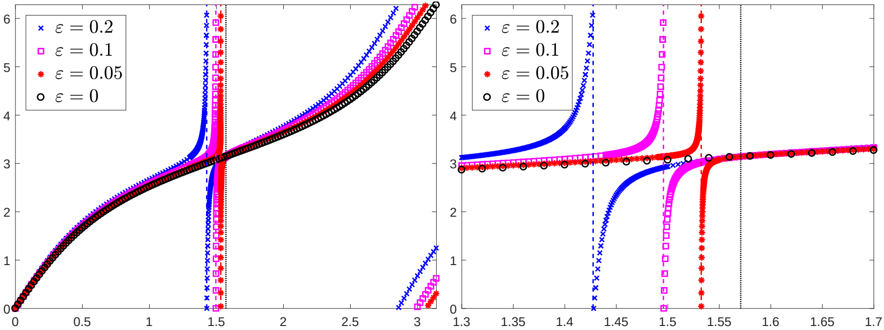

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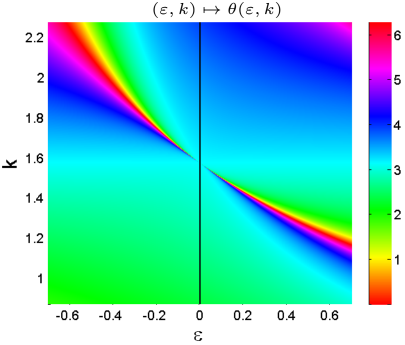
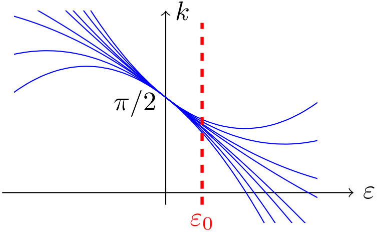
- Set $R(\varepsilon, k) = e^{i\theta(\varepsilon,k)}$ (functions of two variables).





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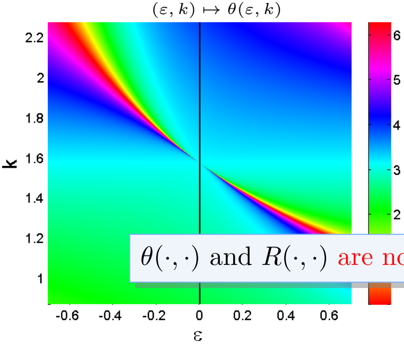

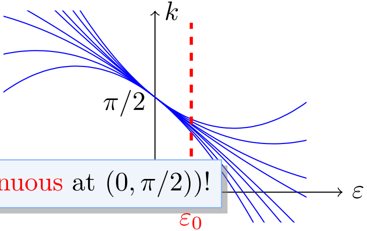

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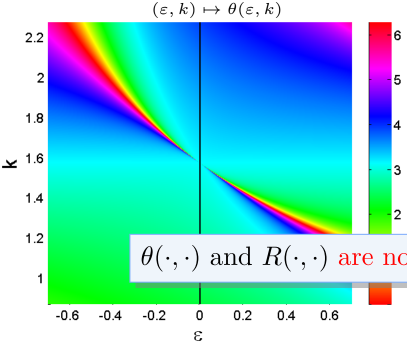

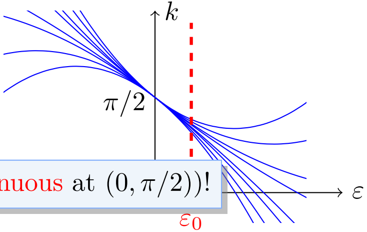

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Next steps

1) Prove a similar Fano resonance phenomenon for a 2D waveguide.
2) Use it to provide examples of non reflection and complete reflection.

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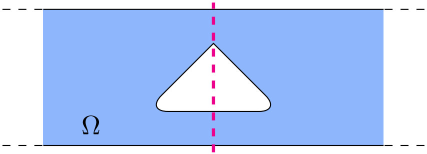
Setting

- Scattering in time-harmonic regime in a symmetric (to simplify) acoustic waveguide Ω coinciding with $\{(x, y) \in \mathbb{R} \times (0; 1)\}$ outside a compact region.



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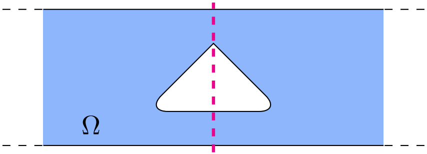

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- We assume that trapped modes exist for $\lambda = \lambda^0 \in (0;\pi^2)$:

$$u_{\mathrm{tr}} \in \mathrm{H}^1(\Omega) \setminus \{0\} \text{ satisfies } (*) \text{ for } \lambda = \lambda^0 \text{ (non uniqueness).}$$

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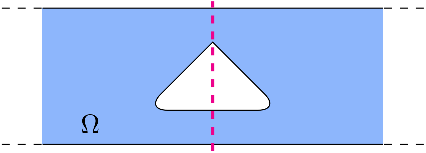

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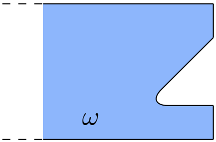
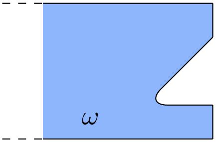
- Due to symmetry, $u_{\rm tr}$ is also a trapped mode for the half waveguide pb.



$$\left| \begin{array}{rcll} \Delta v + \lambda v & = & 0 & \text{in } \omega, \\ \partial_n v & = & 0 & \text{on } \partial\omega \cap \partial\Omega, \\ \mathrm{ABC}(v) = v / \partial_n v & = & 0 & \text{on } \partial\omega \setminus \partial\Omega. \end{array} \right.$$

(depends on the sym.)

Scattering problem in the half waveguide



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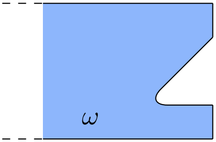

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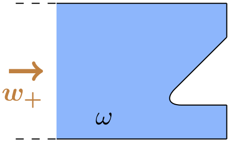

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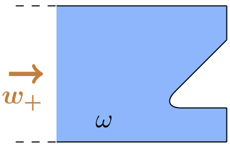

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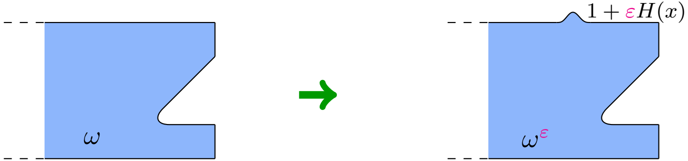
- R is uniquely defined (even for $\lambda = \lambda^0$) and $|R| = 1$ (cons. of energy).

Small perturbation of the geometry

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry



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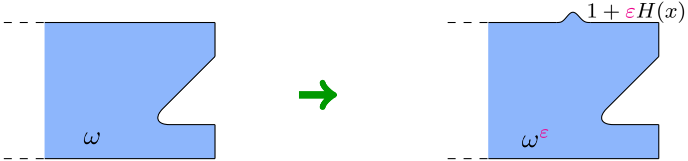

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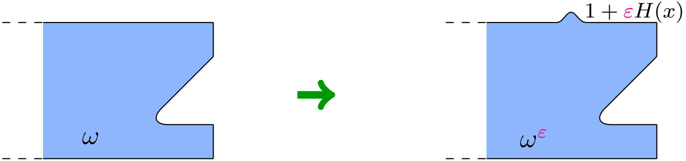

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We can study the behaviour of $(\varepsilon, \lambda) \mapsto R(\varepsilon, \lambda)$ in a neighbourhood of $(0, \lambda^0)$ where trapped modes exist.

$\rightarrow$ *One proves that R is* ***not continuous*** *at $(0, \lambda^0)$ (one approach: work with the augmented scattering matrix which is* ***continuous*** *at $(0, \lambda^0)$).*

The Fano resonance

PROPOSITION: There is $\lambda'_p > 0$ such that

$$\left| \begin{array}{ll} \lim_{\varepsilon \to 0} R(\varepsilon, \lambda^0 + \varepsilon \lambda') = R, & \text{for } \lambda' \neq \lambda'_p \\ \lim_{\varepsilon \to 0} R(\varepsilon, \lambda^0 + \varepsilon \lambda'_p + \varepsilon^2 \mu) = R + \dfrac{a}{ib\mu - c}, & \mu \in \mathbb{R}. \end{array} \right.$$

Here a, b, c are some constants that one can characterize.

$\rightarrow$ When $\mu \in \mathbb{R}$, the quantity $R + \dfrac{a}{ib\mu - c}$ runs on the whole unit circle.

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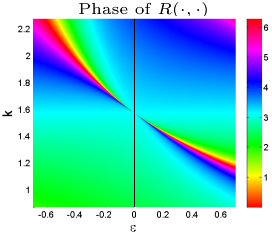
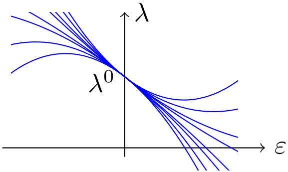
→ When $\mu \in \mathbb{R}$, the quantity $R + \dfrac{a}{ib\mu - c}$ runs on the whole unit circle.

→ We find the same picture as in 1D: $R(\cdot,\cdot)$ is not continuous at $(0, \lambda^0)$.





The Fano resonance

PROPOSITION: There is $\lambda'_p > 0$ such that

$$\left| \begin{array}{ll} \lim\limits_{\varepsilon\to 0} R(\varepsilon, \lambda^0 + \varepsilon\lambda') = R, & \text{for } \lambda' \neq \lambda'_p \\ \lim\limits_{\varepsilon\to 0} R(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) = R + \dfrac{a}{ib\mu - c}, & \mu \in \mathbb{R}. \end{array} \right.$$

Here a, b, c are some constants that one can characterize.

→ When $\mu \in \mathbb{R}$, the quantity $R + \dfrac{a}{ib\mu - c}$ runs on the whole unit circle.

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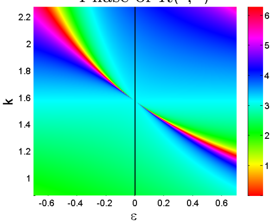

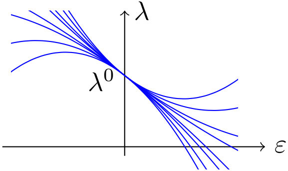

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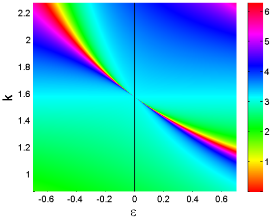
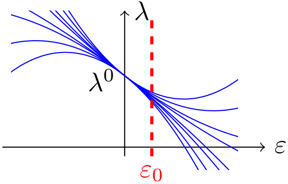
Phase of $R(\cdot,\cdot)$





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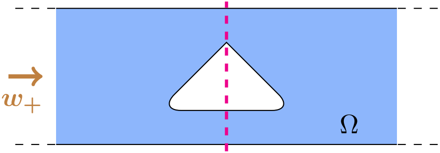
Exploiting symmetry

- We come back to the problem in the total waveguide Ω



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- $(*)$ admits the solution

$$v = \left|\begin{array}{rll} e^{ikx} + R\,e^{-ikx} + \tilde{v}, & x < 0 & \text{(reflection)}\\ T\,e^{-ikx} + \tilde{v}, & x > 0 & \text{(transmission)}\end{array}\right.$$

with R, $T \in \mathbb{C}$ and $\tilde{v} \in \mathrm{H}^1(\Omega)$. We have $|R|^2 + |T|^2 = 1$.

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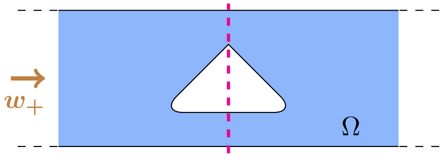

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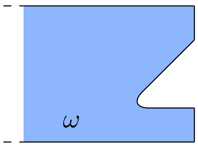
- Introduce the two half-waveguide problems



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$$\left|\begin{array}{rll} \Delta U + \lambda U = 0 & \text{in } \omega \\ \partial_n U = 0 & \text{on } \partial\omega \setminus \partial\Omega \\ U = 0 & \text{on } \partial\omega \cap \partial\Omega. \end{array}\right.$$

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- Half-waveguide problems admit the solutions

$$u = e^{ikx} + R_N\, e^{-ikx} + \tilde{u}, \qquad \text{with } \tilde{u} \in \mathrm{H}^1(\omega)$$
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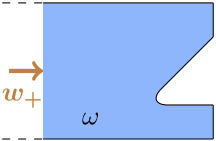

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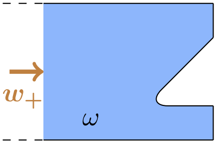

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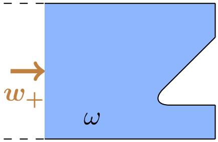

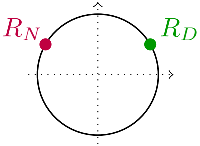

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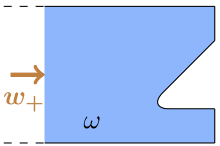

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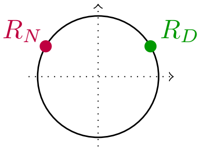

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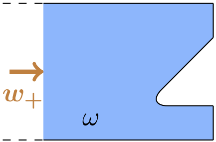

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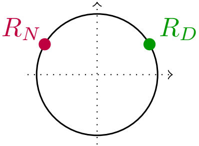

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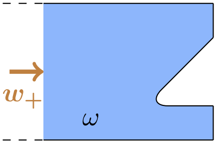

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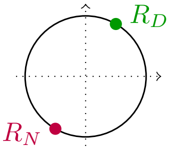

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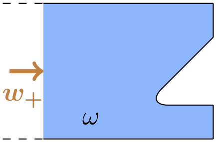

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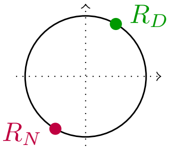

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Non reflection $R = 0$
$\Leftrightarrow \quad R_N = -R_D$

Perfect reflection $T = 0$
$\Leftrightarrow \quad R_N = R_D$

Non reflection and perfect reflection

$$R = \frac{R_N + R_D}{2} \qquad\qquad T = \frac{R_N - R_D}{2}$$

- To set ideas, we assume that u_{tr} is symmetric w.r.t. (Oy).
  $\Rightarrow u_{\mathrm{tr}}$ is a trapped mode for the pb with Neumann B.Cs.

i) No trapped modes for the Dirichlet pb at $\lambda = \lambda^0$. This implies

$$|R_D(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) - R_D(0, \lambda^0)| \le C\,\varepsilon, \qquad \forall \varepsilon \in (0; \varepsilon_0],\ \mu \in [-c\varepsilon^{-1}; c\varepsilon].$$

ii) $\mu \mapsto R_N(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$ rushes on the unit circle for $\mu \in [-c\varepsilon^{-1}; c\varepsilon]$.

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PROPOSITION:

$\exists \lambda_\varepsilon$, with $\lambda_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $R(\varepsilon, \lambda_\varepsilon) = 0$ (non reflection).

$\exists \tilde{\lambda}_\varepsilon$, with $\tilde{\lambda}_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $T(\varepsilon, \tilde{\lambda}_\varepsilon) = 0$ (perfect reflection).

1. Construction of non reflecting obstacles using Fano resonance
   - A 1D toy problem
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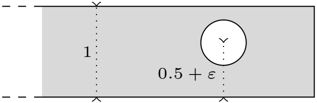
The Fano resonance

- Numerics using FE methods (`Freefem++`) with DtN maps or PMLs.
- Left: domain ω^ε. Right: u_{tr} (trapped mode) for $\varepsilon = 0$.



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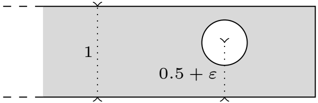


- Since $|R^\varepsilon| = 1$ (conservation of energy), $\exists \theta^\varepsilon \in]-\pi;\pi]$ s.t. $R^\varepsilon = e^{i\theta^\varepsilon}$.

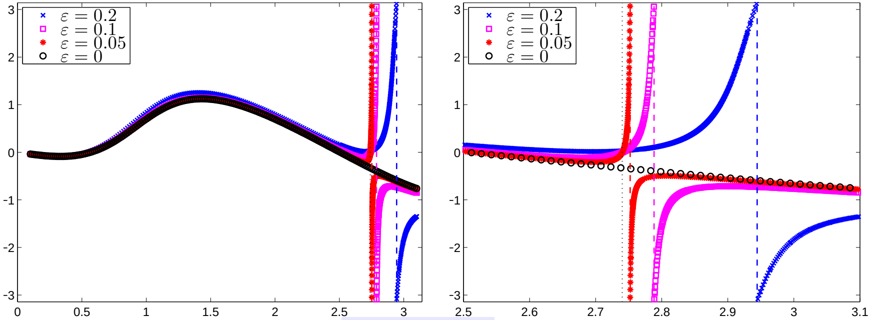
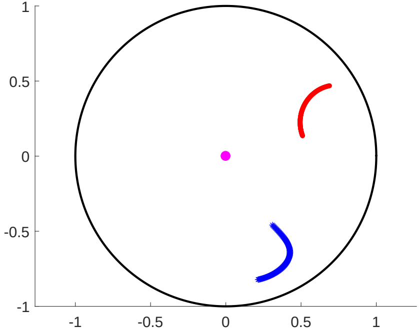


Figure: $k \mapsto \theta^\varepsilon(k)$ for several ε (non uniqueness for $\varepsilon = 0$, $k = 2.7403$).

Non reflection/perfect reflection

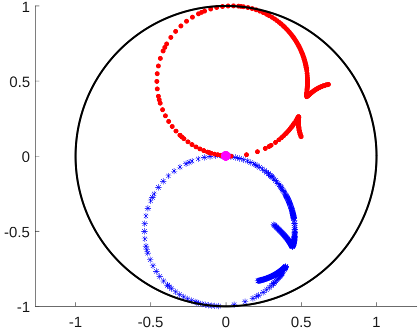
- **Scattering coefficients** for $k \in (2.5; 3.1)$.

No shift $(\boldsymbol{\varepsilon} = \mathbf{0})$



$k \mapsto R(0, k)$ $k \mapsto T(0, k)$

Small shift $(\boldsymbol{\varepsilon} > \mathbf{0})$

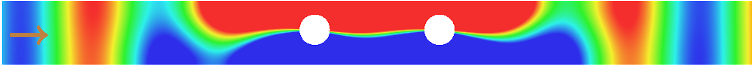


$k \mapsto R(0.05, k)$ $k \mapsto T(0.05, k)$

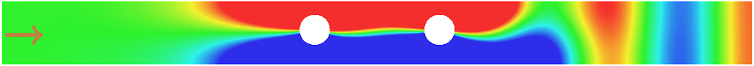
Non reflection/perfect reflection

- Example of setting where $R(\varepsilon, \lambda^\varepsilon) = 0$ (non reflection).

$\Re e\, v$



$\Re e\, (v - v_i)$



- Example of setting where $T(\varepsilon, \lambda^\varepsilon) = 0$ (perfect reflection).

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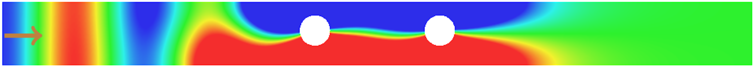


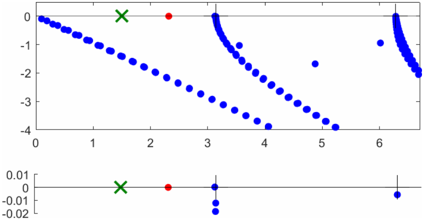
Frequency behaviour

No shift $(\varepsilon = \mathbf{0})$ | Small shift $(\varepsilon > \mathbf{0})$

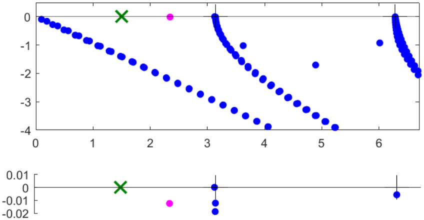
- $k \mapsto \Re e\, v(k)$

- **Complex spectrum** computed with PMLs (we zoom at the real axis).

• Trapped mode



• Complex resonance



Comments

What we did

- We illustrated the Fano resonance phenomenon in a 2D waveguide.

  If trapped modes exist for $(\varepsilon, \lambda) = (0, \lambda^0)$, *then for* $\varepsilon > 0$ *small,* $\lambda \mapsto R(\varepsilon, \lambda)$ *has a quick variation at* λ^0. `Symmetry is not needed.`

- We use it to show examples of non reflection and perfect reflection.
  `Symmetry is essential.`

- The phenomenon appears with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dimension.

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- ♠ We use it to show examples of non reflection and perfect reflection. `Symmetry is essential.`

- ♠ The phenomenon appears with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dimension.

Other directions

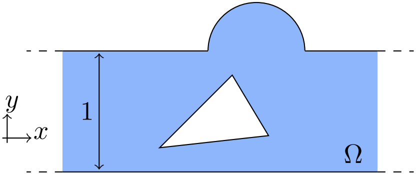
1) Without symmetry, one can show that T still passes through zero.
2) Is there non reflection/perfect reflection for $k > \pi$ (monomode regime was essential in the mechanism)?
3) What happens if λ^0 is not a simple eigenvalue?

1 Construction of non reflecting obstacles using Fano resonance

2 Cloaking of given obstacles in acoustics using resonant ligaments

Setting

- We consider the propagation of waves in a 2D acoustic waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).



$$(\mathscr{P}) \left| \begin{array}{rcll} \Delta u + k^2 u & = & 0 & \text{in } \Omega, \\ \partial_n u & = & 0 & \text{on } \partial\Omega \end{array} \right.$$

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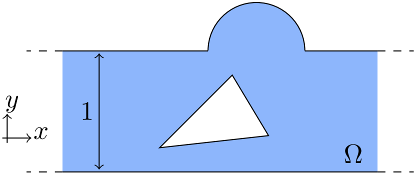

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- The scattering of these waves leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

$$u_+ = \left|\begin{array}{l} e^{ikx} + R_+\, e^{-ikx} + \dots \\ T\ \ e^{+ikx} + \dots \end{array}\right. \qquad u_- = \left|\begin{array}{l} T\ \ e^{-ikx} + \dots \\ e^{-ikx} + R_-\, e^{+ikx} + \dots \end{array}\right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

$R_\pm$, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

Goal

We wish to slightly perturb the walls of the guide to obtain $R_{\pm} = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ cloaking at "infinity".

Goal

We wish to slightly perturb the walls of the guide to obtain $R_{\pm} = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ cloaking at "infinity".

Difficulty: the scattering coefficients have a not explicit and not linear dependence wrt the geometry.

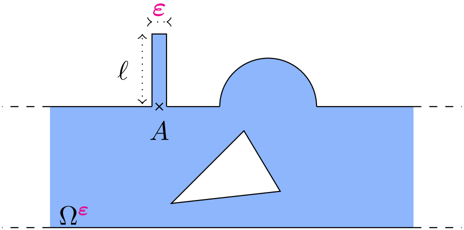
We wish to cloak big obstacles and not only small perturbations.

1. Construction of non reflecting obstacles using Fano resonance

2. Cloaking of given obstacles in acoustics using resonant ligaments
   - Asymptotic analysis in presence of thin resonators
   - Almost zero reflection
   - Cloaking

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^\varepsilon)\left|\begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u = 0 & \text{on } \partial\Omega^\varepsilon \end{array}\right.$$

- In this geometry, we have the scattering solutions

$$u_+^\varepsilon = \left|\begin{array}{r} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ T^\varepsilon\, e^{+ikx} + \dots \end{array}\right. \qquad u_-^\varepsilon = \left|\begin{array}{r} T^\varepsilon\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^\varepsilon\, e^{+ikx} + \dots \end{array}\right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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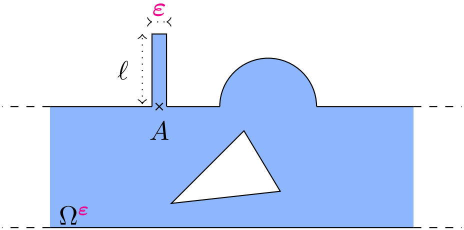

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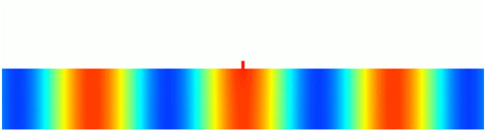
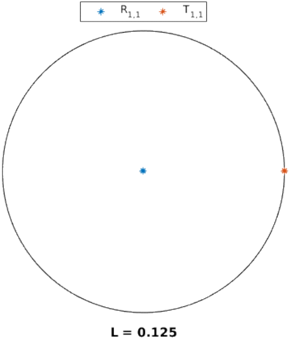
In general, the thin ligament has only a weak influence on the scattering coefficients: $R_\pm^\varepsilon \approx R_\pm$, $T^\varepsilon \approx T$. But not always ...

Numerical experiment

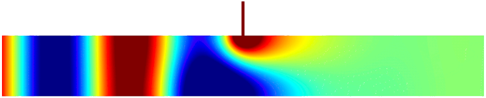
- We vary the length of the ligament:

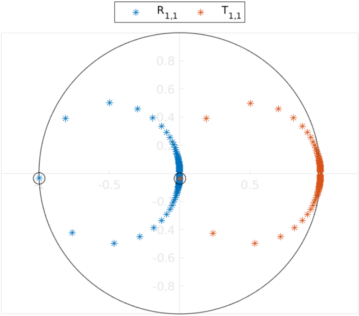




Numerical experiment

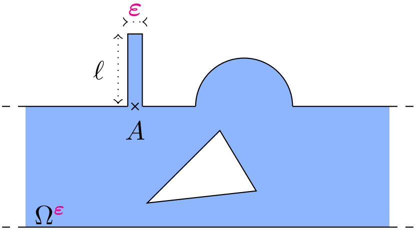
► For one particular length of the ligament, we get a standing mode (zero transmission):





Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u_+^ε, R_+^ε, T^ε as $\varepsilon \to 0$.



$$(\mathscr{P}^\varepsilon) \left| \begin{array}{rl} \Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{array} \right.$$

$$u_+^\varepsilon = \left| \begin{array}{r} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ T^\varepsilon\, e^{+ikx} + \dots \end{array} \right.$$

► To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin & Zhang 17, 18, Brandao, Holley, Schnitzer 20, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u_+^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u_+^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\text{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

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The features of $(\mathscr{P}_{1\mathrm{D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u^\varepsilon_+(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u^\varepsilon_+(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{\text{1D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

The features of $(\mathscr{P}_{\text{1D}})$ play a key role in the physical phenomena and in the asymptotic analysis.

- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_{\text{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_{\text{1D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

$$u^{\varepsilon}_{\pm}(x,y) = u_{\pm} + o(1) \qquad \text{in } \Omega,$$

$$u^{\varepsilon}_{\pm}(x,y) = u_{\pm}(A)\, v_0(y) + o(1) \qquad \text{in the resonator},$$

$$R^{\varepsilon}_{\pm} = R_{\pm} + o(1), \qquad T^{\varepsilon} = T + o(1).$$

Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

→ **Not interesting for our purpose** because we want $\left| \begin{array}{l} R_\pm^\varepsilon = 0 + \dots \\ T^\varepsilon = 1 + \dots \end{array} \right.$

Asymptotic analysis – Resonant case

- Now assume that $\ell = \ell_{\mathrm{res}}$. Then we find $v^{-1}(y) = a\sin(k(y-1))$ for some a to determine.

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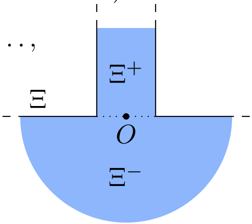
$$(\Delta_{\mathrm{x}} + k^2)u_{+}^{\varepsilon}(\varepsilon^{-1}(\mathrm{x} - A)) = \varepsilon^{-2}\Delta_{\xi}u^{\varepsilon}(\xi) + \dots,$$

when $\varepsilon \to 0$, we are led to study the problem

$$(\star)\ \left|\ \begin{array}{rl} -\Delta_{\xi}Y = 0 & \text{in } \Xi \\ \partial_{\nu}Y = 0 & \text{on } \partial\Xi. \end{array}\right.$$



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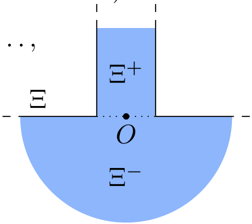

- Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

$$Y^1(\xi) = \begin{cases} \xi_y + C_\Xi + O(e^{-\pi\xi_y}) & \text{as } \xi_y \to +\infty, \quad \xi \in \Xi^+ \\ \dfrac{1}{\pi}\ln\dfrac{1}{|\xi|} + O\left(\dfrac{1}{|\xi|}\right) & \text{as } |\xi| \to +\infty, \quad \xi \in \Xi^-. \end{cases}$$

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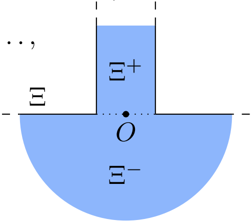

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$$u_+^{\varepsilon}(x) = C^A\, Y^1(\xi) + c^A + \dots \qquad (c^A,\ C^A \text{ constants to determine}).$$

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Asymptotic analysis – Resonant case

- In the ansatz $u^{\varepsilon}_{+} = u^0 + \dots$ in Ω, we deduce that we must take

$$u_0 = u_+ + ak\gamma$$

where γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega. \end{array} \right.$

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- Matching the constant behaviour in the resonator, we obtain

$$v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi).$$

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- This is a Fredholm problem with a non zero kernel. A solution exists iff the **compatibility condition** is satisfied. This sets

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}$$

and **ends the calculus of the first terms**.

Asymptotic analysis – Resonant case

- Finally for $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

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This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- Similarly for $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, by modifying only the last step with the compatibility relation, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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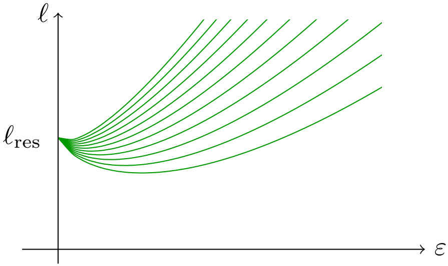
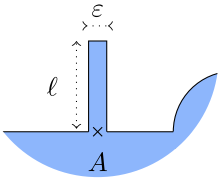
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \ell_{\text{res}} + \varepsilon(\eta - \pi^{-1}|\ln \varepsilon|)), \ \varepsilon > 0\} \subset \mathbb{R}^2.$$





According to η, the limit of the scattering coefficients along the path as $\varepsilon \to 0^+$ is different.

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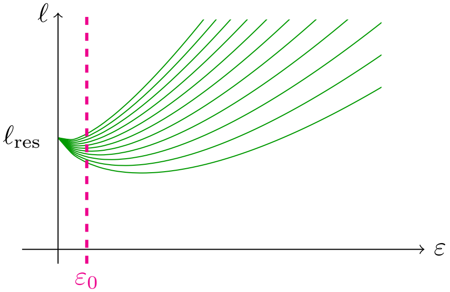

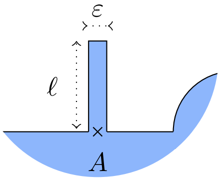

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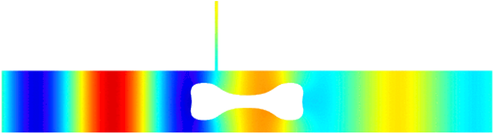
- For a fixed small ε_0, the scattering coefficients have a rapid variation for ℓ varying in a neighbourhood of the resonance length.

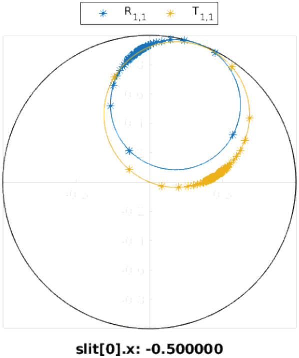
1. Construction of non reflecting obstacles using Fano resonance

2. Cloaking of given obstacles in acoustics using resonant ligaments
   - Asymptotic analysis in presence of thin resonators
   - Almost zero reflection
   - Cloaking

Almost zero reflection

From this expansion, we find that asymptotically, when the length of the resonator is perturbed **around $\boldsymbol{\ell}_{\mathbf{res}}$**, R_+^ε, T^ε run on **circles** whose **features depend on the choice for A**.





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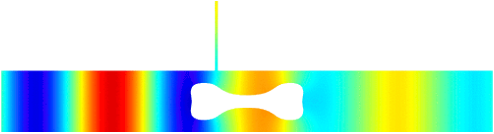

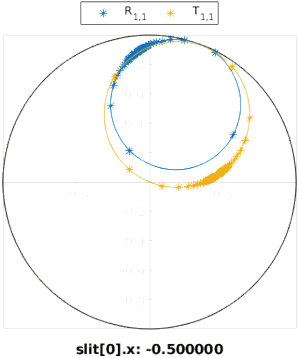

- Using the expansions of $u_\pm(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**.

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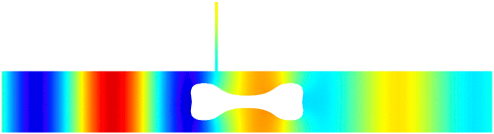

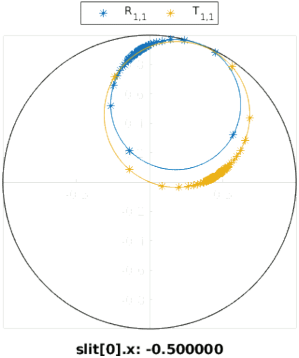

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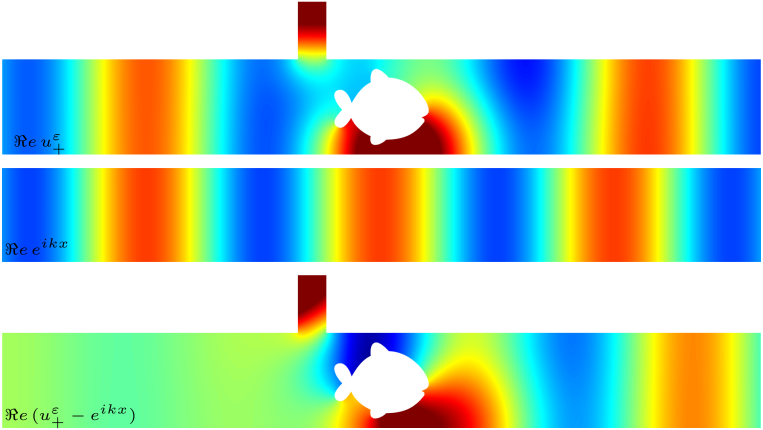
Proposition: There are **positions of the resonator A** such that the circle $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_+^\varepsilon = 0 + o(1)$.

Almost zero reflection

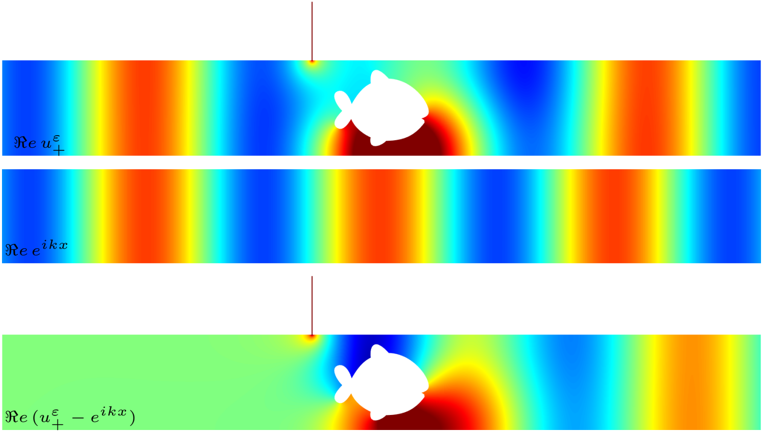
- Example of situation where we have almost zero reflection ($\varepsilon = \mathbf{0.3}$).



$\rightarrow$ Simulations realized with the `Freefem++` library.

Almost zero reflection

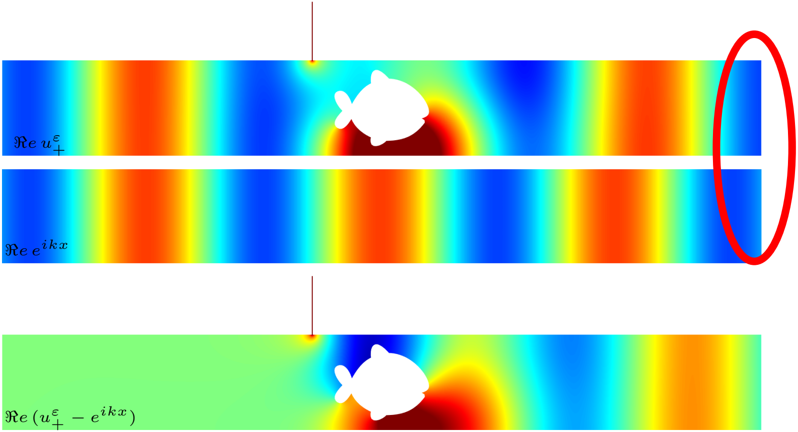
- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.01}$).



→ Simulations realized with the `Freefem++` library.

Almost zero reflection

- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.01}$).



→ Simulations realized with the `Freefem++` library.

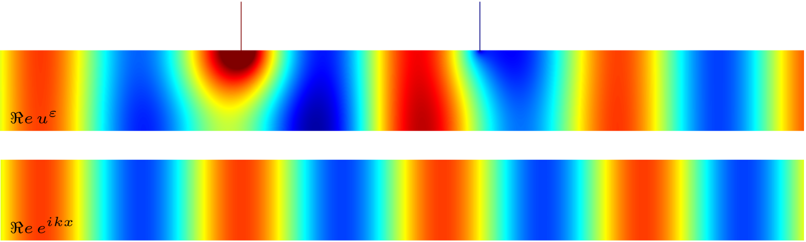
> To cloak the object, it remains to compensate the phase shift!

1 Construction of non reflecting obstacles using Fano resonance

2 Cloaking of given obstacles in acoustics using resonant ligaments
- Asymptotic analysis in presence of thin resonators
- Almost zero reflection
- Cloaking

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

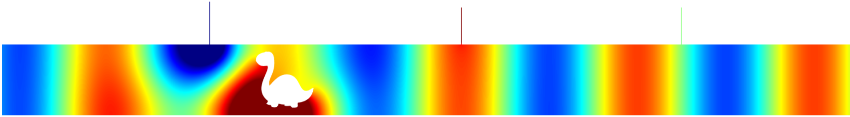


- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

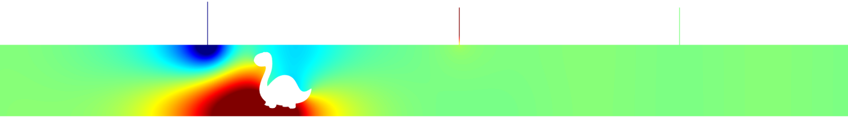
Cloaking with three resonators

- Gathering the two previous results, we can cloak any object with three resonators.

$\Re e\, u_+$



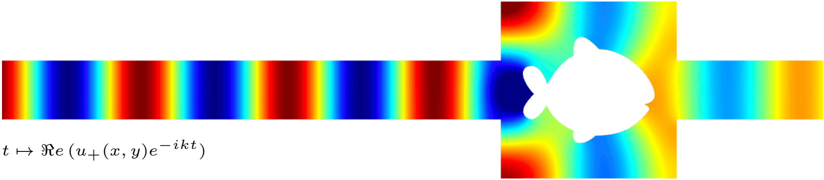
$\Re e\, u_+^\varepsilon$



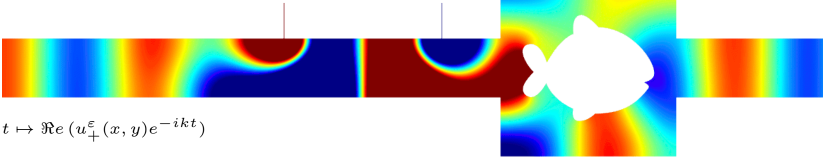
$\Re e\, (u_+^\varepsilon - e^{ikx})$

Cloaking with two resonators

▶ Working a bit more, one can show that two resonators are enough to cloak any object.



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$

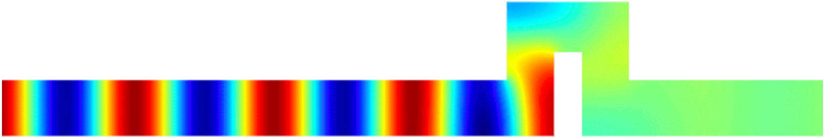


$t \mapsto \Re e\,(u_+^{\varepsilon}(x,y)e^{-ikt})$

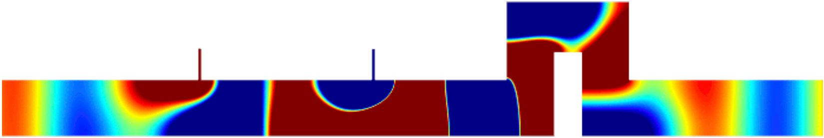
$t \mapsto \Re e\,(e^{ik(x-t)})$

Cloaking with two resonators

- Another example



$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$



$t \mapsto \Re e\,(u_+^\varepsilon(x,y)e^{-ikt})$

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Recap of the cloaking strategy

What we did

♠ We explained how to approximately cloak any object in monomode regime using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

Recap of the cloaking strategy

What we did

♠ We explained how to approximately cloak any object in monomode regime using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

Possible extensions and open questions

1) We can similarly hide penetrable obstacles or work in 3D.
2) We can do cloaking at a finite number of wavenumbers (thin structures are resonant at one wavenumber otherwise act at order ε).
3) With Dirichlet BCs, other ideas must be found.
4) Can we realize exact cloaking ($T = 1$ exactly)? This question is also related to robustness of the device.

Bibliography

- Part I

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S.P. Shipman, H. Tu. Total resonant transmission and reflection by periodic structures, SIAP, vol. 72, 1:216-239, 2012.

- Part II

L. Chesnel, J. Heleine and S.A. Nazarov. Acoustic passive cloaking using thin outer resonators. submitted, ZAMP, vol. 73, 98, 2022.