A few techniques to achieve invisibility in waveguides

Lecture 4: A spectral problem characterizing zero reflection

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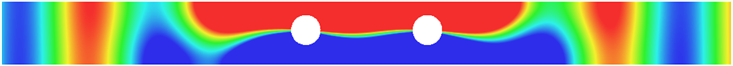
Idefix team, EDF/Ensta/Inria, France

Toulouse, 27/06/2025

Lecture 3

- We explained how to exploit the Fano resonance phenomenon together with symmetry to obtain zero reflection for **large** obstacles.



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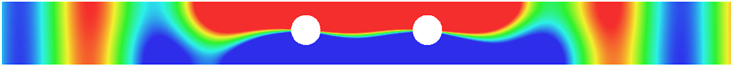

Lecture 4 : **Two distinct goals**

1. A simple example of large invisible defect in acoustics

ASYMPTOTIC ANALYSIS:

k is given, we construct simple examples of Ω such that $\boldsymbol{T = 1}$.

2. A spectral approach to determine non reflecting wavenumbers

SPECTRAL THEORY:

Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

Outline of the talk

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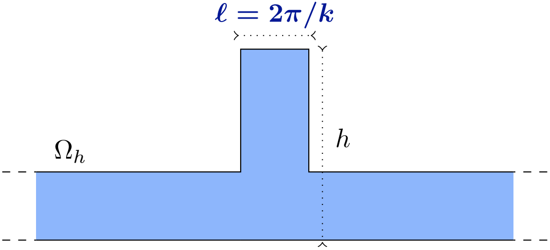

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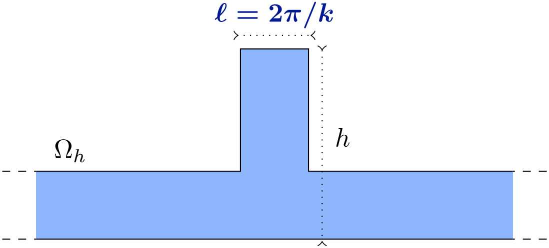

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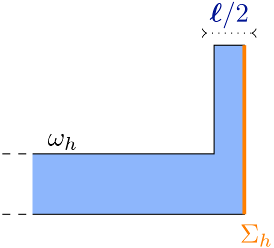

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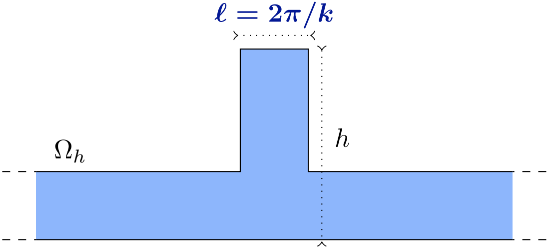

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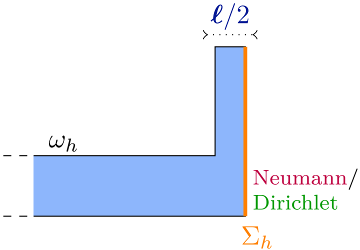

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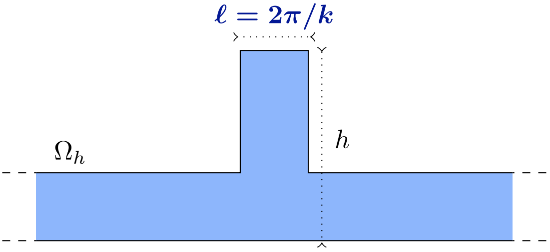

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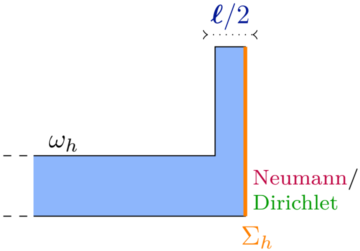

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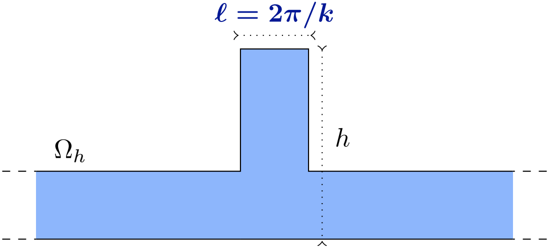

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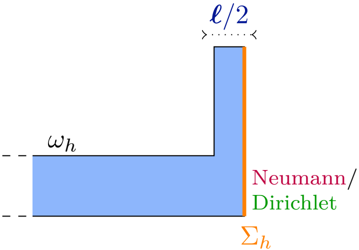

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Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

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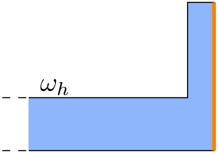

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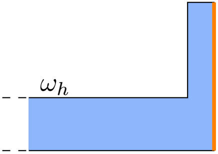

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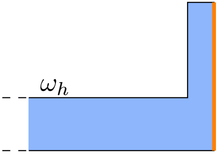

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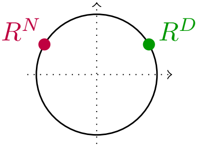

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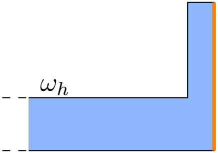

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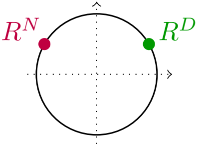

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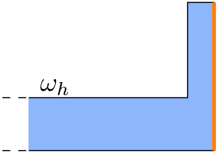

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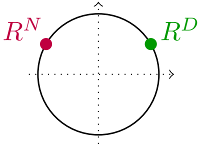

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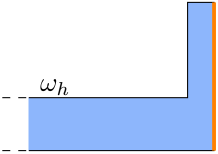

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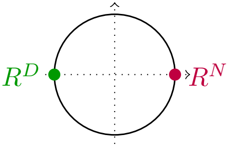

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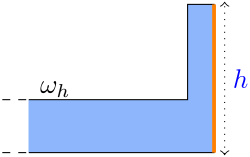

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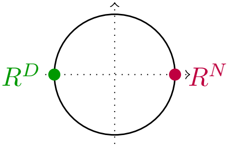

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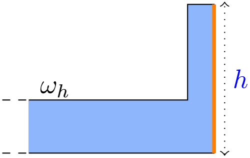

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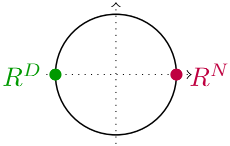

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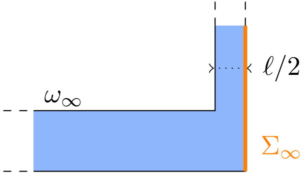
$\rightarrow$ It remains to study the behaviour of $R^D = R^D(h)$ as $h \to +\infty$.

Asymptotics of R^D as $h \to +\infty$

Depends on the nb. of propagating modes in the vertical branch of ω_∞



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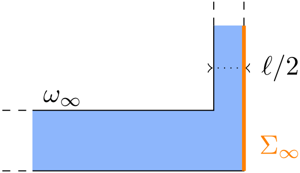

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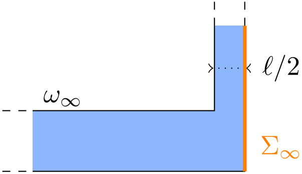

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where $R^D_{\text{asy}}(h)$ runs periodically on the unit circle $\mathscr{C}$.

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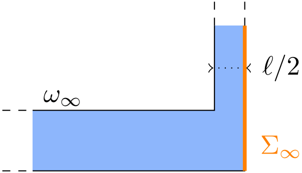

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▶ Additionally one can prove that $h \mapsto R^D(h)$ runs continuously on $\mathscr{C}$.

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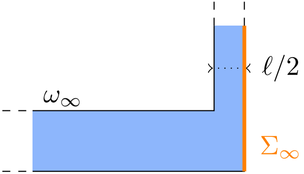

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$\Rightarrow$ There is a sequence (h_n) with $h_n \to +\infty$ such that $R^D(h_n) = -1$.

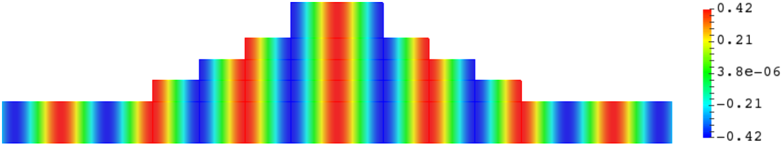
Conclusion

THEOREM: There is an unbounded sequence (h_n) such that for $h = h_n$, we have $T = 1$ (perfect invisibility).

Numerical results

- Works also in the geometry below. When we vary h, the height of the central branch, T runs exactly on the circle $\mathscr{C}(1/2, 1/2)$.
$\rightarrow$ Numerically, we simply sweep in h and extract the h such that $T(h) = 1$.

- Perfectly invisible defect ($t \mapsto \Re e\,(v(x,y)e^{-i\omega t})$)



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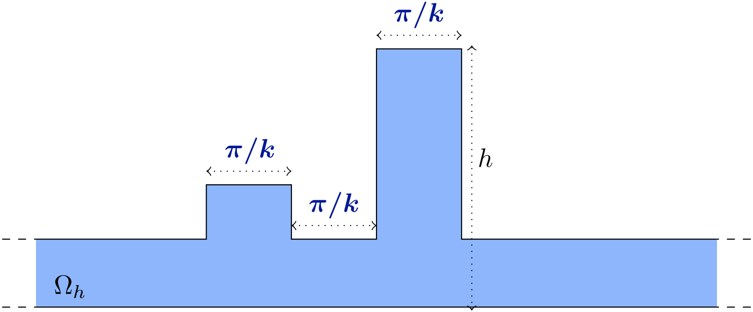
Remark

- Actually Ω does not have to be symmetric and we can work in the following geometry:



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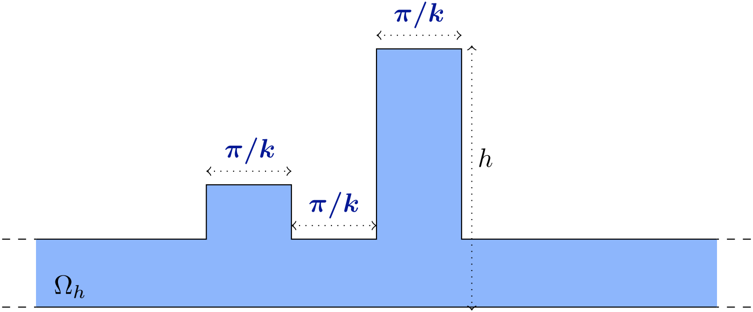


- In this Ω_h, we can show that there holds $R + T = 1$.

- With the identity of energy $|R|^2 + |T|^2 = 1$, this guarantees that T must be on the circle $\mathscr{C}(1/2, 1/2)$.

- Finally, with asy. analysis, we show that T goes through 1 as $h \to +\infty$.

Outline of the talk

1. A simple example of large invisible defect in acoustics

Asymptotic analysis:

k is given, we construct simple examples of Ω such that $T = 1$.

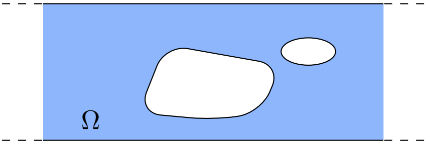
2. A spectral approach to determine non reflecting wavenumbers

Spectral theory:

Ω is given, we explain how to **find non reflecting** k by solving an unusual **spectral problem**.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



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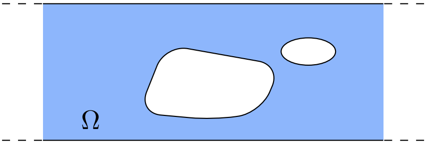

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Propagating: $w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$

Evanescent: $w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y),\ \beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

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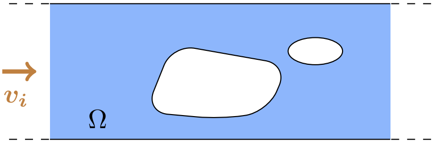

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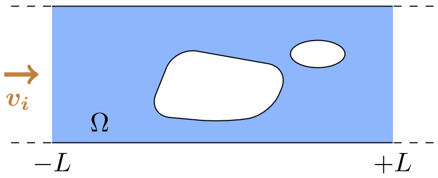
Propagating	$w_n^{\pm}(x,y) = e^{\pm i\beta_n x}\cos(n\pi y)$, $\beta_n = \sqrt{k^2 - n^2\pi^2}$, $n \in [\![0, N-1]\!]$
Evanescent	$w_n^{\pm}(x,y) = e^{\mp \beta_n x}\cos(n\pi y)$, $\beta_n = \sqrt{n^2\pi^2 - k^2}$, $n \geq N$.

- Set $v_i = \sum_{n=0}^{N-1} \alpha_n w_n^+$ for some given $(\alpha_n)_{n=0}^{N-1} \in \mathbb{C}^N$.

Scattering problem

- Consider the scattering problem with $k \in ((N-1)\pi; N\pi)$, $N \in \mathbb{N}^*$



Find $v = v_i + v_s$ s. t.

$$\begin{aligned} \Delta v + k^2 v &= 0 \quad \text{in } \Omega, \\ \partial_n v &= 0 \quad \text{on } \partial\Omega, \\ & v_s \text{ is outgoing.} \end{aligned}$$

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- v_s is outgoing $\Leftrightarrow$ $$v_s = \sum_{n=0}^{+\infty} \gamma_n^{\pm} w_n^{\pm} \quad \text{for } \pm x \geq L, \text{ with } (\gamma_n^{\pm}) \in \mathbb{C}^{\mathbb{N}}.$$

Goal of the section

DEFINITION: v is a non reflecting mode if v_s is expo. decaying for $x \leq -L$
$\Leftrightarrow \quad \gamma_n^- = 0,\ n \in [\![0, N-1]\!] \quad \Leftrightarrow$ energy is completely transmitted.

GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

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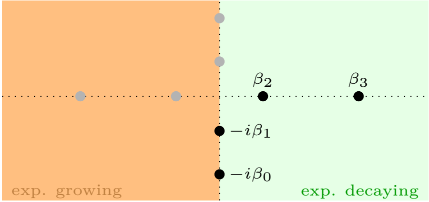
GOAL

For a given geometry, we present a method to find values of k such that there is a non reflecting mode v.

$\rightarrow$ Note that non reflection occurs for **particular** $\boldsymbol{v_i}$ to be computed.

Classical complex scaling to compute v_s 1/2

REMINDER: $v_s = \sum_{n=0}^{N-1} \gamma_n^{\pm} e^{\pm i\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \gamma_n^{\pm} e^{\mp \beta_n x} \cos(n\pi y), \ \pm x \geq L.$



Modal exponents for v_s ($x \leq -L$)

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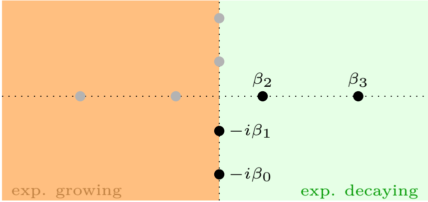

Modal exponents for v_s $(x \leq -L)$

- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{I}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\, e^{i\theta} & \text{for } x \leq -L \\ x & \text{for } |x| < L \\ +L + (x-L)\, e^{i\theta} & \text{for } x \geq L. \end{array} \right.$$

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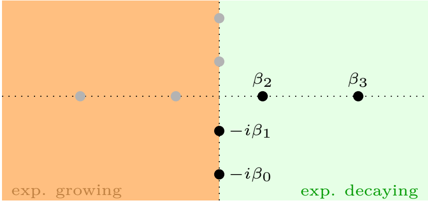

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- Set $v_\theta := v_s \circ (\mathcal{I}_\theta(x), y)$.

1) $v_\theta = v_s$ for $|x| < L$.

2) v_θ is exp. decaying at infinity.

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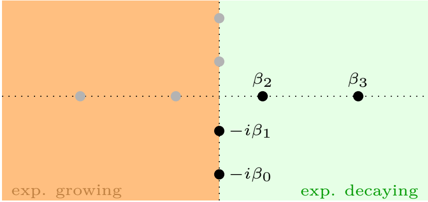

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$$v_\theta = \sum_{n=0}^{N-1} \tilde{\gamma}_n^{\pm} e^{\pm i\tilde{\beta}_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde{\gamma}_n^{\pm} e^{\mp\tilde{\beta}_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde{\beta}_n = \beta_n e^{i\theta}$$

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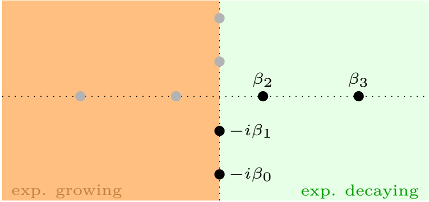

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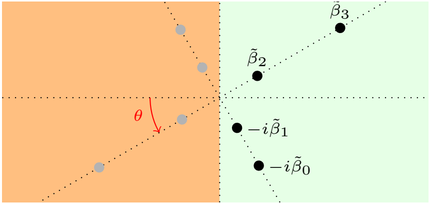

Modal exponents for v_θ $(x \leq -L)$

$$v_\theta = \sum_{n=0}^{N-1} \tilde\gamma_n^{\pm} e^{\pm i\tilde\beta_n x} \cos(n\pi y) + \sum_{n=N}^{+\infty} \tilde\gamma_n^{\pm} e^{\mp\tilde\beta_n x} \cos(n\pi y), \quad \pm x \geq L \quad \tilde\beta_n = \beta_n e^{i\theta}$$

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1) $v_\theta = v_s$ for $|x| < L$.
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Classical complex scaling to compute v_s 2/2

- v_θ solves

$$(*)\left|\begin{array}{rcll} \alpha_\theta \dfrac{\partial}{\partial x}\left(\alpha_\theta \dfrac{\partial v_\theta}{\partial x}\right) + \dfrac{\partial^2 v_\theta}{\partial y^2} + k^2 v_\theta & = & 0 & \text{in } \Omega \\ \partial_n v_\theta & = & -\partial_n v_i & \text{on } \partial\Omega. \end{array}\right.$$

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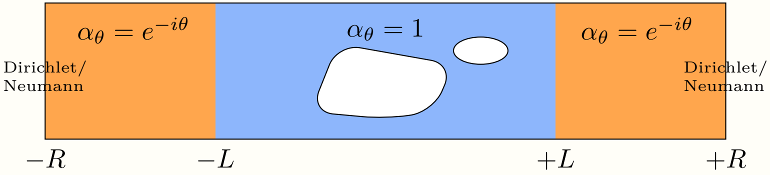
$\alpha_\theta(x) = 1$ for $|x| < L$ $\qquad$ $\alpha_\theta(x) = e^{-i\theta}$ for $|x| \geq L$

- Numerically we solve $(*)$ in the truncated domain



$\Rightarrow$ We obtain a good approximation of v_s for $|x| < L$.

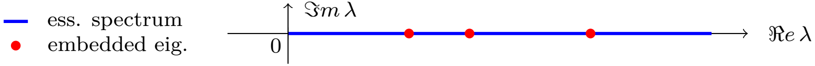
- This is the method of Perfectly Matched Layers (PMLs).

Spectral analysis

- Define the operators A, A_θ of $\mathrm{L}^2(\Omega)$ such that

$$Av = -\Delta v, \qquad A_\theta v = -\Big(\alpha_\theta \frac{\partial}{\partial x}\Big(\alpha_\theta \frac{\partial v}{\partial x}\Big) + \frac{\partial^2 v}{\partial y^2}\Big) \qquad + \partial_n v = 0 \text{ on } \partial\Omega.$$

- A is selfadjoint and positive.
- $\sigma(A) = \sigma_{\mathrm{ess}}(A) = [0; +\infty)$.
- $\sigma(A)$ may contain embedded eigenvalues in the essential spectrum.



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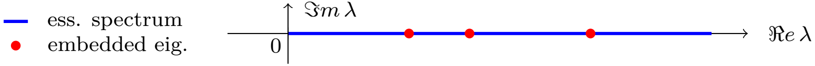

- A_θ is not selfadjoint. $\sigma(A_\theta) \subset \{\rho\, e^{i\gamma},\ \rho \geq 0,\ \gamma \in [-2\theta; 0]\}$.
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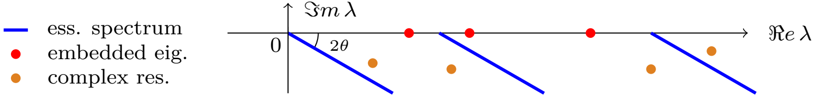
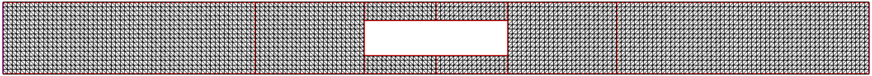


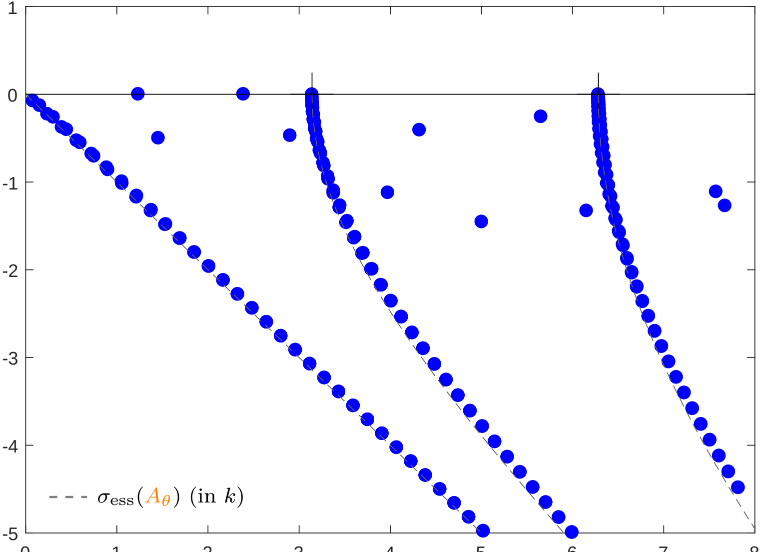
Numerical results

- We work in the geometry



Numerical results

- Discretized spectrum of A_θ in k (not in k^2). We take $\theta = \pi/4$.



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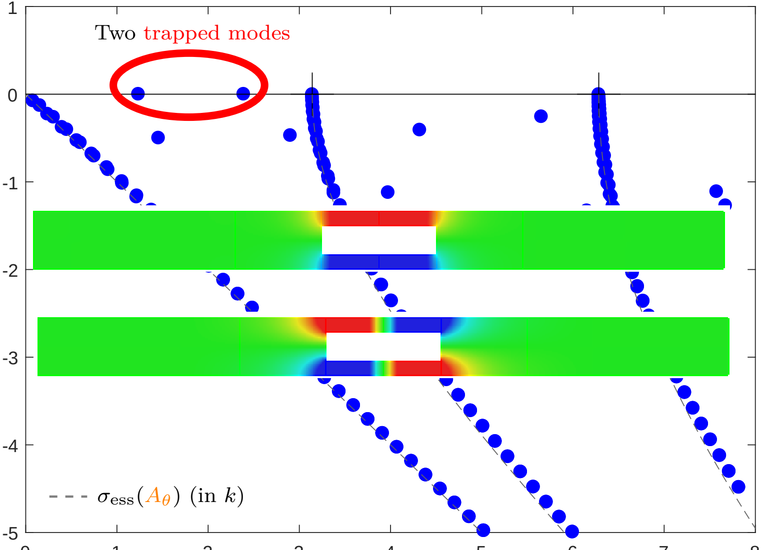

A new complex spectrum for non reflecting v

- Usual complex scaling selects scattered fields which are

outgoing at $-\infty$ and outgoing at $+\infty$.

IMPORTANT REMARK: **general** v decompose as

$$v = v_i + \sum_{n=0}^{N-1} \gamma_n^- \, w_n^- + \sum_{n=N}^{+\infty} \gamma_n^- \, w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ \, w_n^+ \quad x \geq L.$$

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Important remark: **non reflecting** v decompose as

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IMPORTANT REMARK: **non reflecting** v decompose as

$$v = \sum_{n=0}^{N-1} \boldsymbol{\alpha_n w_n^+} + \sum_{n=N}^{+\infty} \gamma_n^- w_n^- \quad x \leq -L, \quad v = \sum_{n=0}^{+\infty} \gamma_n^+ w_n^+ \quad x \geq L.$$

- In other words, **non reflecting** v are

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- In other words, **non reflecting** v are

ingoing at $-\infty$ and outgoing at $+\infty$.

Let us **change the sign** of the complex scaling at $-\infty$!

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- For $\theta \in (0; \pi/2)$, consider the complex change of variables

$$\mathcal{J}_\theta(x) = \left| \begin{array}{ll} -L + (x+L)\ e^{-i\theta} & \text{for } x \le -L \\ x & \text{for } |x| < L \\ +L + (x-L)\ e^{+i\theta} & \text{for } x \ge L. \end{array} \right.$$

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- Set $u_\theta := v \circ (\mathcal{J}_\theta(x), y)$.

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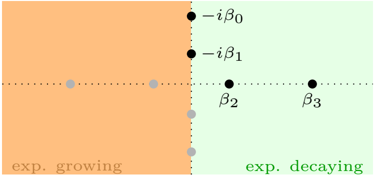
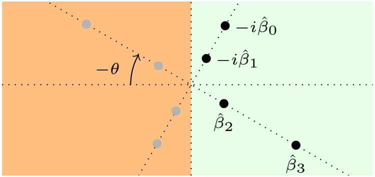


Modal exponents for v $(x \le -L)$



Modal exponents for u_θ $(x \le -L)$

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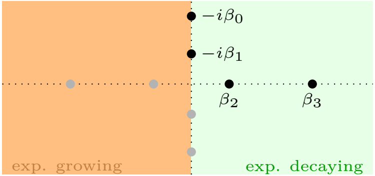

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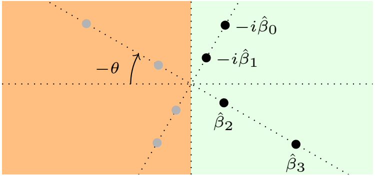

Modal exponents for u_θ $(x \leq -L)$

- u_θ solves

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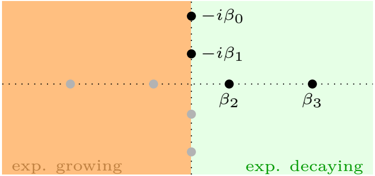

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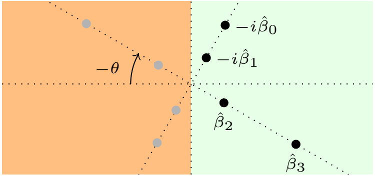

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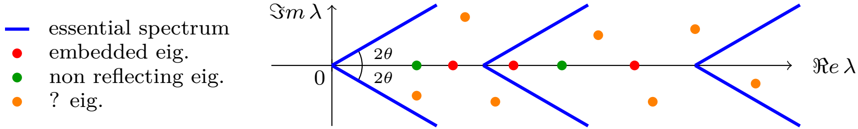
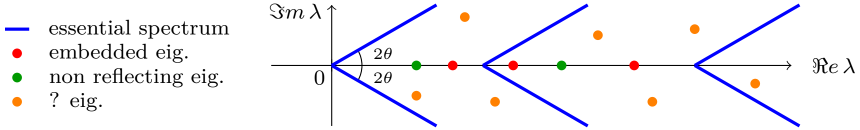
- real eigenvalues of B_θ = real eigenvalues of A+non reflecting k^2.



Remarks



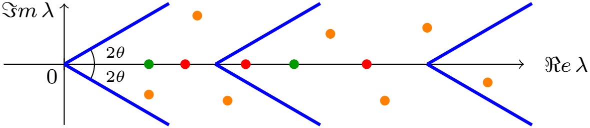
1) ● **? eig.** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

Different from **complex resonances** for which the eigenfunctions are exp. growing both at $\pm\infty$ ...

2) It is not simple to prove that $\sigma(B_\theta) \setminus \sigma_{\text{ess}}(B_\theta)$ is discrete.

Remarks

- essential spectrum
- embedded eig.
- non reflecting eig.
- ? eig.



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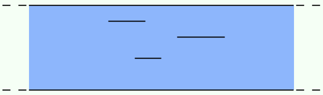
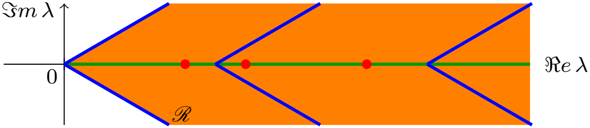


$\rightarrow$ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Remarks

- essential spectrum (blue line)
- embedded eig. (red dot)
- non reflecting eig. (green dot)
- ? eig. (orange dot)



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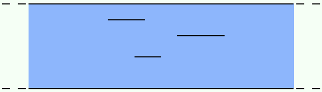
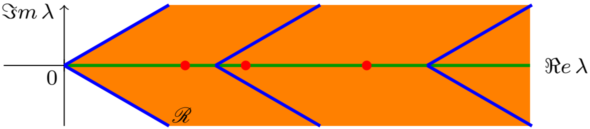


→ **Not true in general!**

$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

Remarks

- essential spectrum (blue line)
- embedded eig. (red dot)
- non reflecting eig. (green dot)
- ? eig. (orange dot)



1) ● **? eig.** correspond to solutions of the Helmholtz equation which are exp. growing at one side of Ω, exp. decaying at the other.

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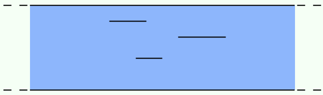

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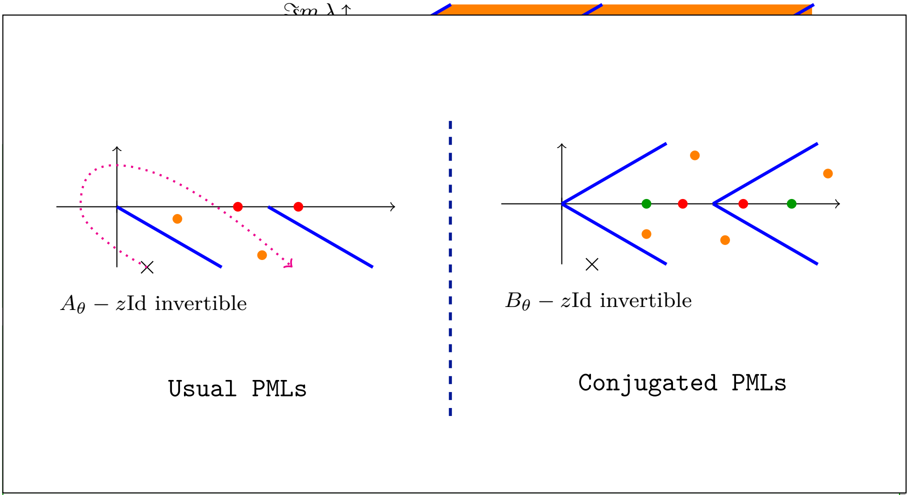
$e^{ikx} \circ \mathcal{J}_\theta$ is an eigenfunction for all $k \in \mathscr{R}$.

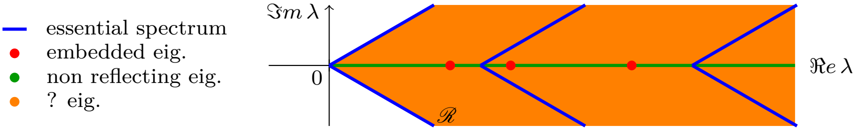
→ $\mathbb{C} \setminus \sigma_{\text{ess}}(B_\theta)$ is not connected ⇒ we cannot apply simply the analytic Fredholm thm.

Remarks



$\rightarrow \mathbb{C} \setminus \sigma_{\mathrm{ess}}(B_\theta)$ is not connected $\Rightarrow$ we cannot apply simply the analytic Fredholm thm.

Remarks



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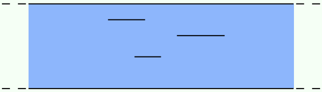

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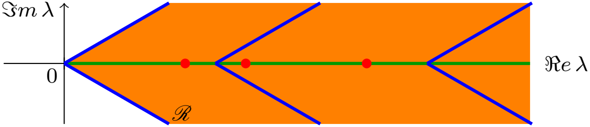

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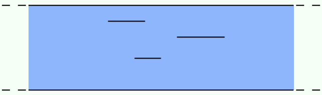

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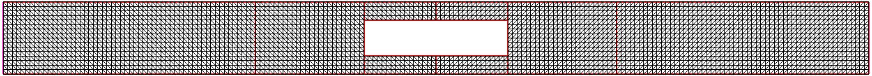
→ A compact perturbation can change drastically the spectrum (B_θ is not selfadjoint). Numerical consequences?

Numerical results

- Again we work in the geometry



- Define the operators $\mathcal{P}$ (Parity), $\mathcal{T}$ (Time reversal) such that

$$\mathcal{P}v(x,y) = v(-x,y) \quad \text{and} \quad \mathcal{T}v(x,y) = \overline{v(x,y)}.$$

Prop.: For symmetric $\Omega = \{(-x,y)\,|\,(x,y)\in\Omega\}$, B_θ is $\mathcal{PT}$ symmetric:

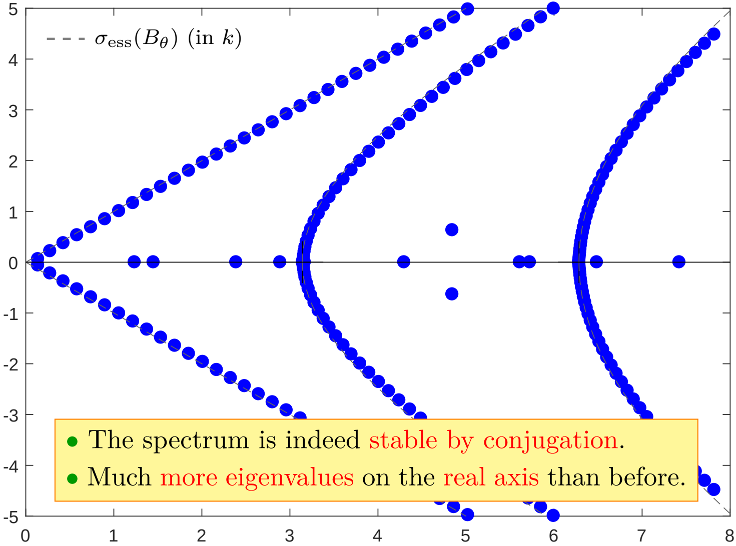
$$\mathcal{PT}B_\theta\mathcal{PT} = B_\theta.$$

As a consequence, $\sigma(B_\theta) = \overline{\sigma(B_\theta)}$.

$\Rightarrow$ If λ is an "isolated" eigenvalue located close to the real axis, then $\lambda \in \mathbb{R}$!

Numerical results

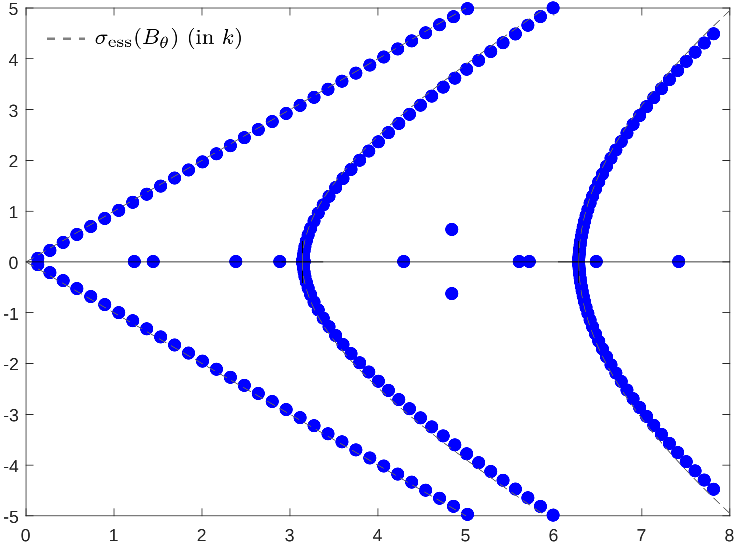
- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



- The spectrum is indeed stable by conjugation.
- Much more eigenvalues on the real axis than before.

Numerical results

- Discretized spectrum in k (not in k^2). We take $\theta = \pi/4$.



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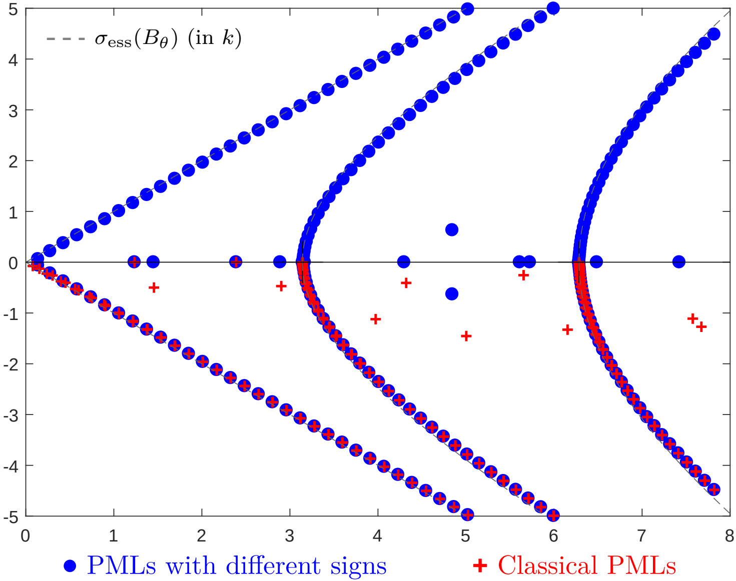
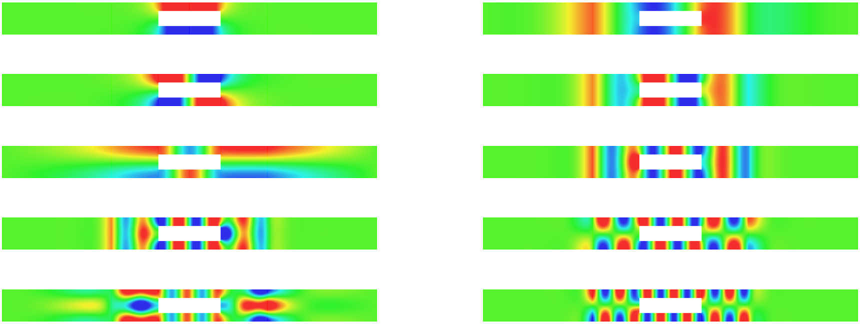


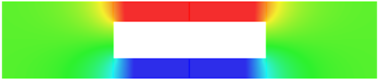
Numerical results

- We display the eigenmodes for the ten first real eigenvalues in the whole computational domain (including PMLs).



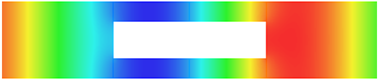
Numerical results

- Let us focus on the eigenmodes such that $0 < k < \pi$.



First trapped mode
$k = 1.2355...$

Second trapped mode
$k = 2.3897...$

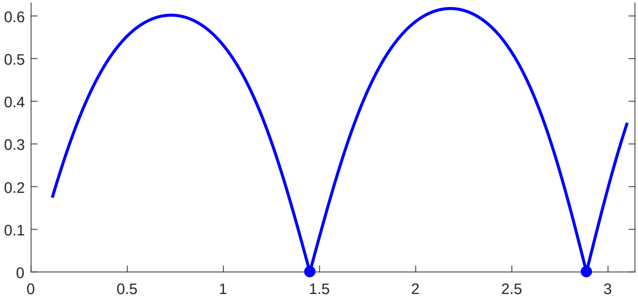


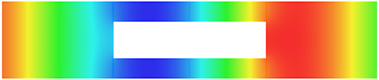
First non reflecting mode
$k = 1.4513...$

Second non reflecting mode
$k = 2.8896...$

Numerical results

- To check our results, we compute $k \mapsto |R(k)|$ for $0 < k < \pi$.





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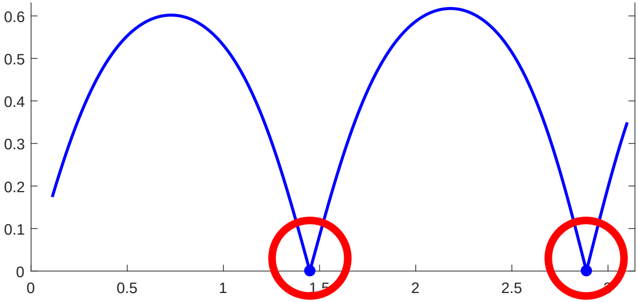
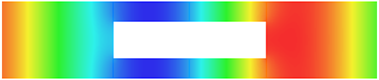




First non reflecting mode

$k = 1.4513...$

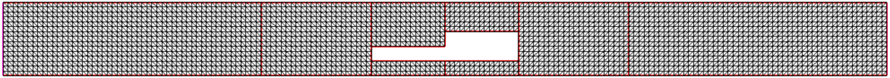
Second non reflecting mode

$k = 2.8896...$

There is perfect agreement!

Numerical results

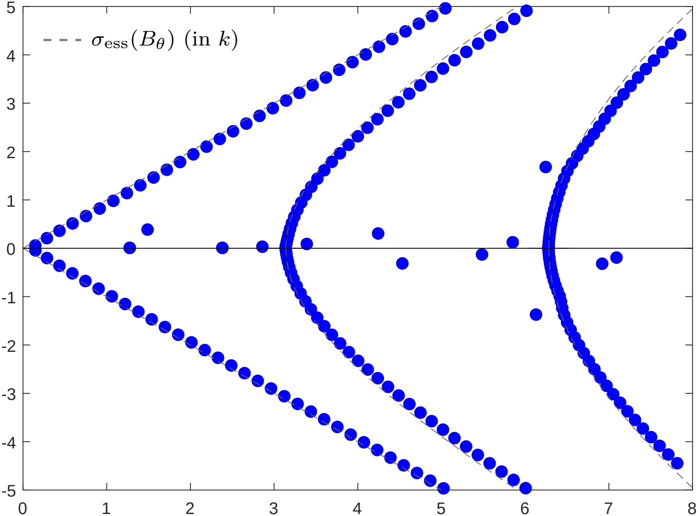
- Now the geometry is not symmetric in x nor in y:



- The operator B_θ is no longer $\mathcal{PT}$-symmetric and we expect:
  - No trapped modes
  - No invariance of the spectrum by complex conjugation.

Numerical results

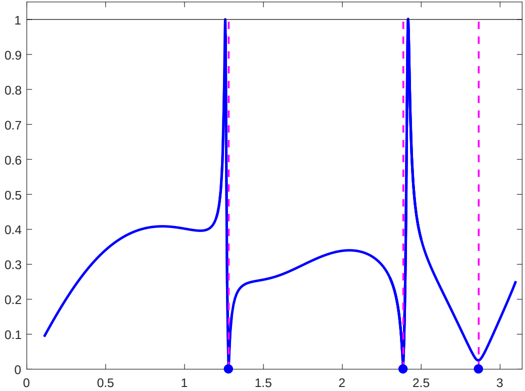
- Discretized spectrum of B_θ in k (not in k^2). We take $\theta = \pi/4$.



- Indeed, the spectrum is not symmetric w.r.t. the real axis.

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



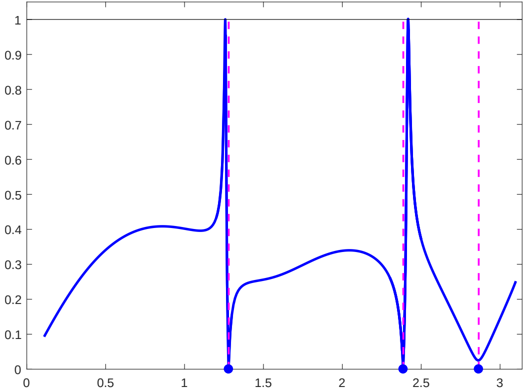
$k = 1.28 + 0.0003i$

$k = 2.3866 + 0.0005i$

$k = 2.8647 + 0.0243i$

Numerical results

- We compute $k \mapsto |R(k)|$ for $0 < k < \pi$.



$k = 1.28 + 0.0003i$

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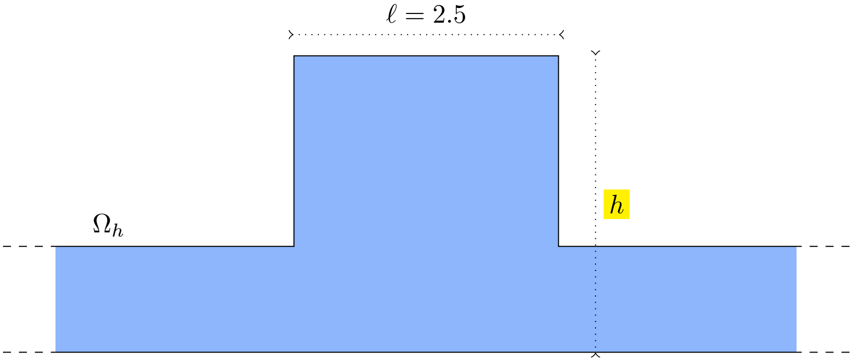
Complex eigenvalues also contain information on almost no reflection.

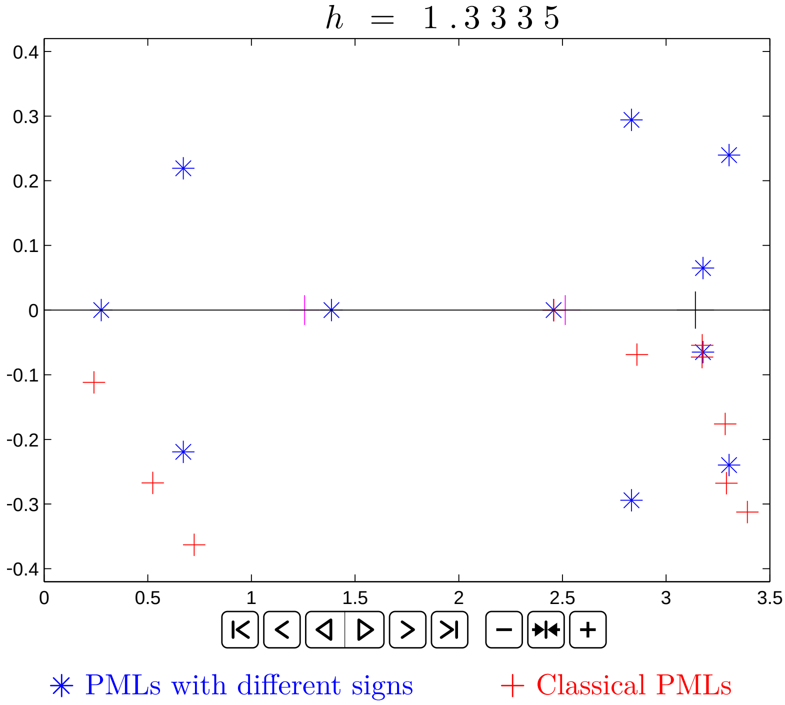
Spectra for a changing geometry

- Two series of computations: one with PMLs with different sign, one with classical PMLs. We compute the spectra for $h \in (1.3; 8)$.



- The magenta marks on the real axis correspond to $k = \pi/\ell$ & $k = 2\pi/\ell$. For $k = 2\pi/\ell$, trapped modes and $T = 1$ should occur for certain h.

- We zoom at the region $0 < \Re e\, k < \pi$.



$h = 1.3335$
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
PMLs with different signs
Classical PMLs

Conclusion

Part I

- ♠ We explained how to find simples examples of Ω where $T = 1$ for the Neumann problem.

1) This can be adapted to construct geometries supporting trapped modes for the Neumann problem.
2) However this approach does not work for the Dirichlet problem.

Part II

- ♠ Spectral approach to compute non reflecting k $(R = 0)$ for a given Ω.

1) Can we find a spectral approach to compute completely reflecting or completely invisible k?
2) Can we prove existence of non reflecting k for the $\mathcal{PT}$-symmetric pb?

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- Part I

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- Part II

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H. Hernandez-Coronado, D. Krejčiřík, P. Siegl. Perfect transmission scattering as a PT-symmetric spectral problem. Phys. Lett. A, 375(22):2149-2152, 2011.

W.R. Sweeney, C.W. Hsu, A.D. Stone. Theory of reflectionless scattering modes. Phys. Rev. A, vol. 102, 6:063511, 2020.

Conclusion of the course

What we did

Lecture 1. We presented rudiments of scattering theory in waveguides.

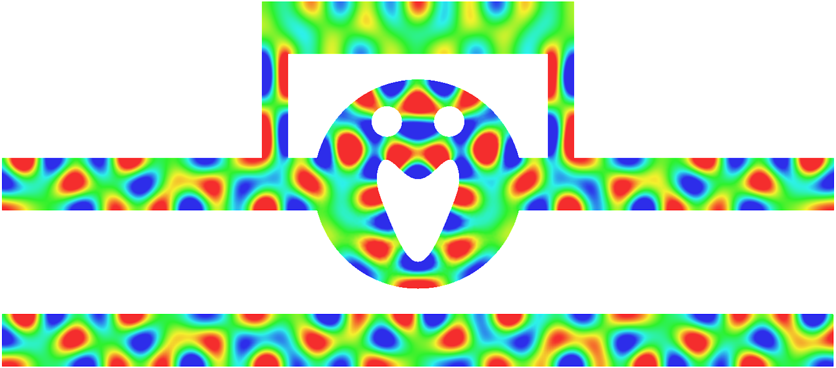
Lecture 2, 3 & 4. We used tools of asymptotic analysis and spectral theory to identify situations of invisibility:

- Construction of small amplitude invisible obstacles.

- Construction of large amplitude non reflecting obstacles using complex resonances.

- We presented a spectral problem characterizing zero reflection.

→ **To be continued...**



Thank you for your attention!